Quantum Resource Theories

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Overview

1. Resource theory of Entanglement

2. Resource theory of asymmetry and coherence

What is Entanglement?

Pure bi-partite states

$|\phi\rangle_A$ $|\theta\rangle_B$

A B

Unetangled state $$|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\theta\rangle_B$$

There is no correlation between the outcomes of measurements on *A* and *B*.

Entangled state $$|\Psi\rangle_{AB} \neq |\phi\rangle_A \otimes |\theta\rangle_B$$

Note: A pure state is entangled if and only if the two subsystems are correlated. This is not true in the case of mixed states.

Bell basis $$|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \qquad |\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

Examples of *Maximally Entangled* states

Tensor Product Decomposition

Why not?

$$|\Phi\rangle \otimes |\pm\rangle \xleftarrow{\;?\;} |\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

$$|\Psi\rangle \otimes |\pm\rangle \xleftarrow{\;?\;} |\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

Tensor product decomposition is defined based on the notion of **<u>Spatially</u> Local Subsystems**.

A little bit of History

1935 Einstein-Podolsky-Rosen paper

1964 Bell's theorem

1991 Ekert's key distribution protocol
1992 Superdense coding (Bennett-Wiesner)
1993 Teleportation (Bennett-Brassard-Crepeau-Jozsa-Peres-Wootters)
…....

Entanglement as a resource, as real as energy.

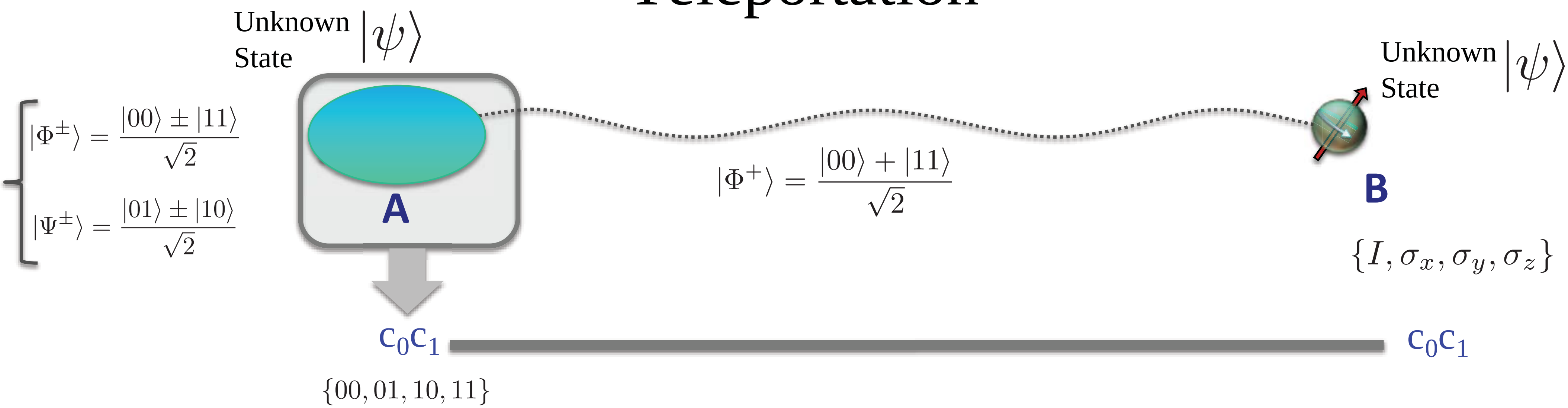
Entanglement theory should be understood as a framework to study questions about manipulating resource states for performing certain tasks, similar to the theory of thermodynamics.

How to transfer Quantum Information?

Quantum Channel

How to transfer Quantum Information?

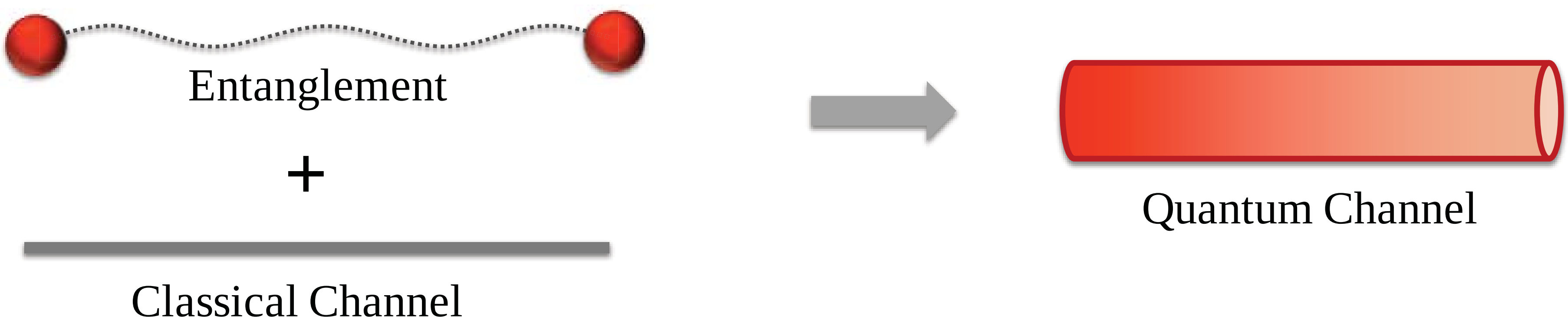
Teleportation



(i) Local Measurement on A side.
(ii) Classical Communication (two bits of information)
(iii) Local Operation on B side

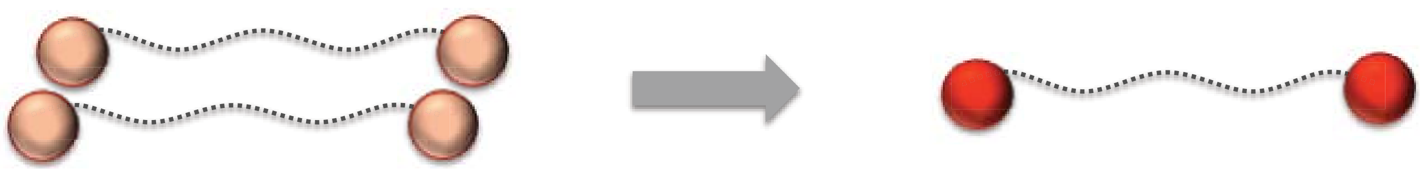
No-signaling and No-cloning are not violated.

Entanglement as a Resource



To implement teleportation we need to **consume** a maximally entangled state.

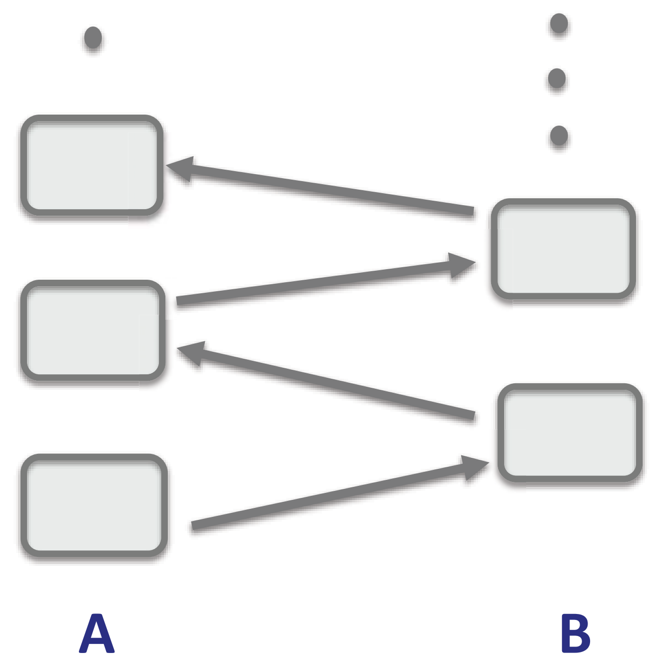
What if the state is not a maximally entangle state? What if it is mixed? What if we have multiple copies of a mixed state?



Local **O**perations and **C**lassical **C**ommunication

LOCC Paradigm

(I) Local Unitary operations on A and B
(II) Local Measurements on A and B
(III) Coupling to local ancillary systems
(IV) Discarding local subsystems
And
(V) Classical Communication



- All and the only operations we can do in the absence of a quantum channel.
- LOCC do not generate Entanglement.
- LOCC is used in teleportation.

$$\rho_{AB} \xrightarrow{\text{LOCC}} \sigma_{AB}$$

Equivalence classes of states

$$\rho_{AB} \xrightarrow{\text{LOCC}} \sigma_{AB} \qquad\qquad \sigma_{AB} \xrightarrow{\text{LOCC}} \rho_{AB}$$

The two states are equivalent resources. E.g., if one can be used as a resource for teleportation, then the other one can also be used for teleportation.

Separable states (Unentangled states): The set of states which are in the equivalence class of product states.

$$\rho_{AB} = \sum_i p_i \, \rho_A^{(i)} \otimes \sigma_B^{(i)} \quad \xleftrightarrow{\text{LOCC}} \quad |\phi\rangle_A \otimes |\theta\rangle_B$$

- Any pair of separable states can be transformed to each other via LOOC.
- Separable states may contain correlations (i.e. nonzero Mutual information).

Schmidt Decomposition

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\, |\phi_i\rangle_A \otimes |\theta_i\rangle_B$$

$$\langle\theta_j|\theta_i\rangle = \delta_{i,j} \qquad \langle\phi_j|\phi_i\rangle = \delta_{i,j}$$

$\lambda_i \geq 0 \qquad \sum_i \lambda_i = 1$ Schmidt Coefficients (Eigenvalues of the reduced states)

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|_{AB}) = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|_A$$

$$\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|_{AB}) = \sum_i \lambda_i |\theta_i\rangle\langle\theta_i|_B$$

Equivalence classes of pure states

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i^{\psi}} |\phi_i\rangle_A \otimes |\theta_i\rangle_B$$

Two pure states can be transformed to each via LOCC reversibly, if and only if they have the same Schmidt coefficients.

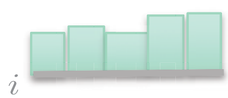
$$|\psi\rangle_{AB} \xrightarrow{\text{LOCC}} |\phi\rangle_{AB}$$
$$|\phi\rangle_{AB} \xrightarrow{\text{LOCC}} |\psi\rangle_{AB}$$
$$\Longleftrightarrow$$
$$\{\lambda_i^{\psi}\} = \{\lambda_i^{\phi}\}$$

Equivalence classes of pure states are uniquely specified by their Schmidt coefficients.

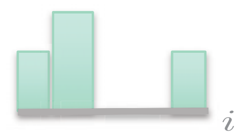
Single-copy Transformations

Nielsen's theorem

$$|\psi\rangle_{AB} \xrightarrow{\text{LOCC}} |\phi\rangle_{AB}$$



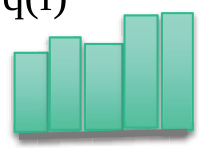
$$\{\lambda_i^{\psi}\} \xleftarrow{\text{Mixture of random permutations}} \{\lambda_i^{\phi}\}$$



This transformation can be implemented via LOCC if and only if there exists a mixture of random permutations which transforms the probability distribution $\{\lambda_i^{\phi}\}$ to the probability distribution $\{\lambda_i^{\psi}\}$.

Majorization

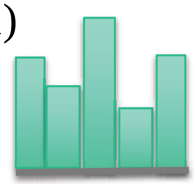
q(i)

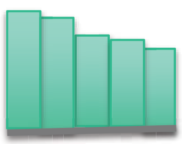


?

Mixture of random permutations

p(i)





$$q^{\downarrow}(1) \geq q^{\downarrow}(2) \geq \cdots \geq q^{\downarrow}(N)$$

$$p^{\downarrow}(1) \geq p^{\downarrow}(2) \geq \cdots \geq p^{\downarrow}(N)$$

$$q^{\downarrow}(1) \leq p^{\downarrow}(1)$$

$$q^{\downarrow}(1) + q^{\downarrow}(2) \leq p^{\downarrow}(1) + p^{\downarrow}(2)$$

$$\vdots$$

$$\vec{p} = \begin{pmatrix} p(1) \\ p(2) \\ \cdot \\ \cdot \\ \cdot \\ p(N) \end{pmatrix} \qquad \vec{q} = \begin{pmatrix} q(1) \\ q(2) \\ \cdot \\ \cdot \\ \cdot \\ q(N) \end{pmatrix}$$

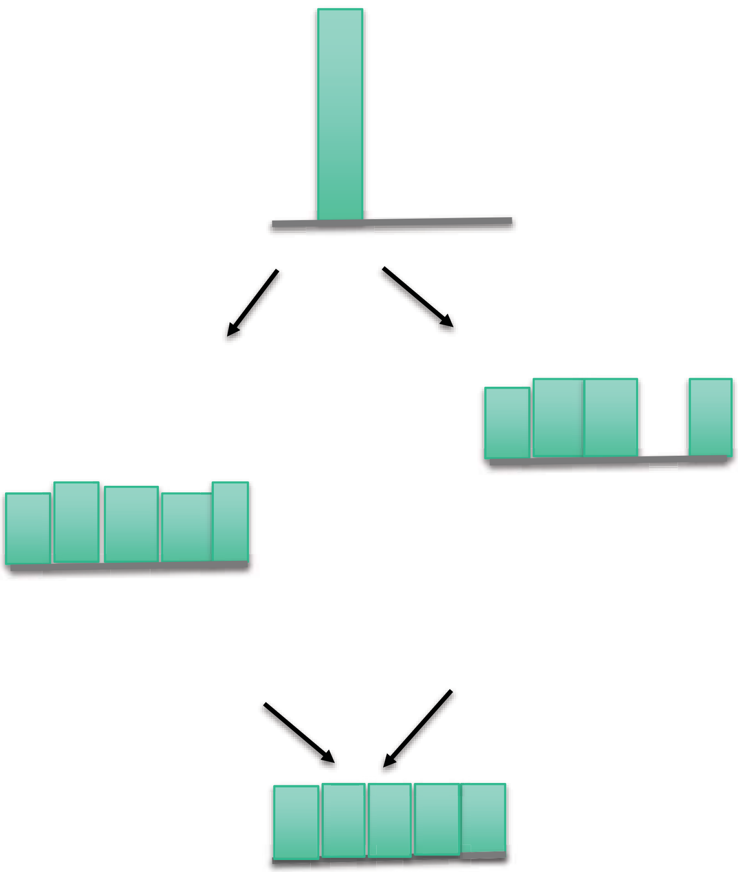
$$\vec{q} \overset{?}{=} A\vec{p}$$

For a doubly stochastic matrix A

$$A_{ij} \geq 0 \qquad \sum_j A_{ij} = 1 \qquad \sum_i A_{ij} = 1$$

$\vec{q} \prec \vec{p}$ $\quad$ $\vec{q}$ is *majorized* by $\vec{p}$.

Partial Order



Single-copy Transformations

Nielsen's theorem

$$|\psi\rangle_{AB} \xrightarrow{\text{LOCC}} |\phi\rangle_{AB}$$

This transformation can be implemented via LOCC if and only if there exists a mixture of random permutations which transforms the probability distribution $\{\lambda_i^{\phi}\}$ to the probability distribution $\{\lambda_i^{\psi}\}$.

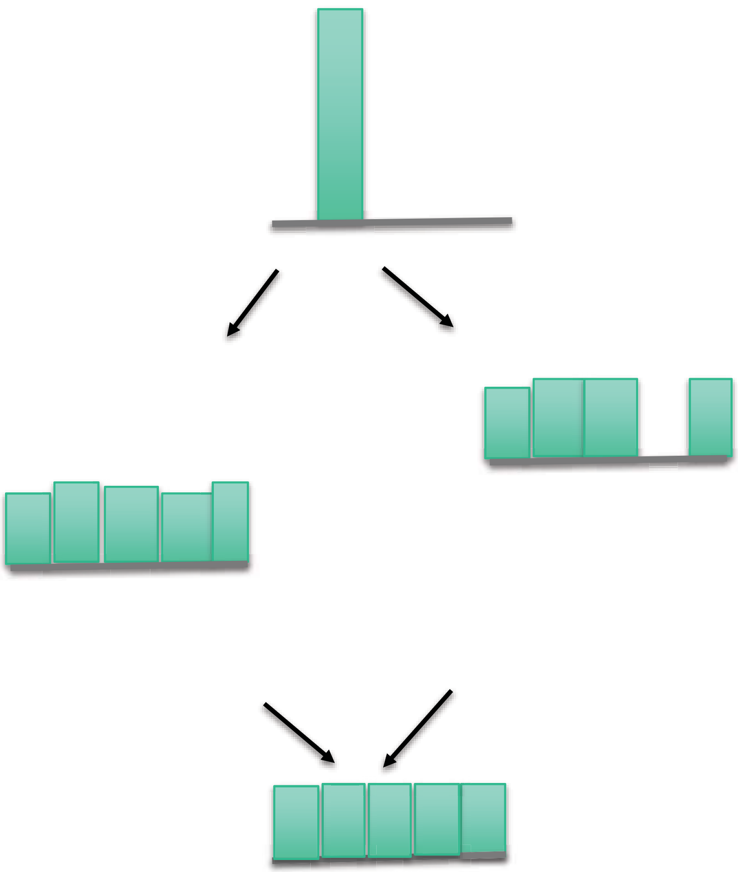
$$\boxed{|\psi\rangle_{AB} \xrightarrow{\text{LOCC}} |\phi\rangle_{AB} \quad \Longleftrightarrow \quad \{\lambda_i^{\psi}\} \prec \{\lambda_i^{\phi}\}}$$

$$\lambda^{\psi\downarrow}_1 \leq \lambda^{\phi\downarrow}_1$$

$$\lambda^{\psi\downarrow}_1 + \lambda^{\psi\downarrow}_2 \leq \lambda^{\phi\downarrow}_1 + \lambda^{\phi\downarrow}_2$$

$$\vdots$$

Partial Order



$$|\phi\rangle_A \otimes |\theta\rangle_B$$

Unentangled states

$$\sum_i \sqrt{\lambda'_i}\; |\phi'_i\rangle_A \otimes |\theta'_i\rangle_B$$

$$\sum_i \sqrt{\lambda_i}\; |\phi_i\rangle_A \otimes |\theta_i\rangle_B$$

Entangled states

$$\frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$$

Maximally Entangled states

Catalytic Transformations

$$|\psi\rangle_{AB} \xrightarrow{\mathrm{LOCC}}\!\!\!\!\!\!\!\!\!\!/\;\;\;\; |\phi\rangle_{AB} \qquad \{\lambda_i^{\psi}\} \not\prec \{\lambda_i^{\phi}\}$$

$$|\psi\rangle_{AB} \otimes |\eta\rangle_{AB} \xrightarrow{\mathrm{LOCC}} |\phi\rangle_{AB} \otimes |\eta\rangle_{AB} \qquad \{\lambda_i^{\psi}\} \times \{\lambda_j^{\eta}\} \prec \{\lambda_i^{\phi}\} \times \{\lambda_j^{\eta}\}$$

Asymptotic Transformations

$$|\psi\rangle_{AB} \xrightarrow{\cancel{\text{LOCC}}} |\Phi^+\rangle_{AB}$$

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Entanglement Purification

$$|\psi\rangle_{AB}^{\otimes n} \xrightarrow{\text{LOCC}} |\tilde{\Phi}\rangle_{AB} \approx |\Phi^+\rangle_{AB}^{\otimes m(n)}$$

$$R(|\psi\rangle_{AB} \xrightarrow{\text{LOCC}} |\Phi^+\rangle_{AB}) = \lim_{n\to\infty} \frac{m(n)}{n} = S(\rho_A)$$

Entanglement Entropy

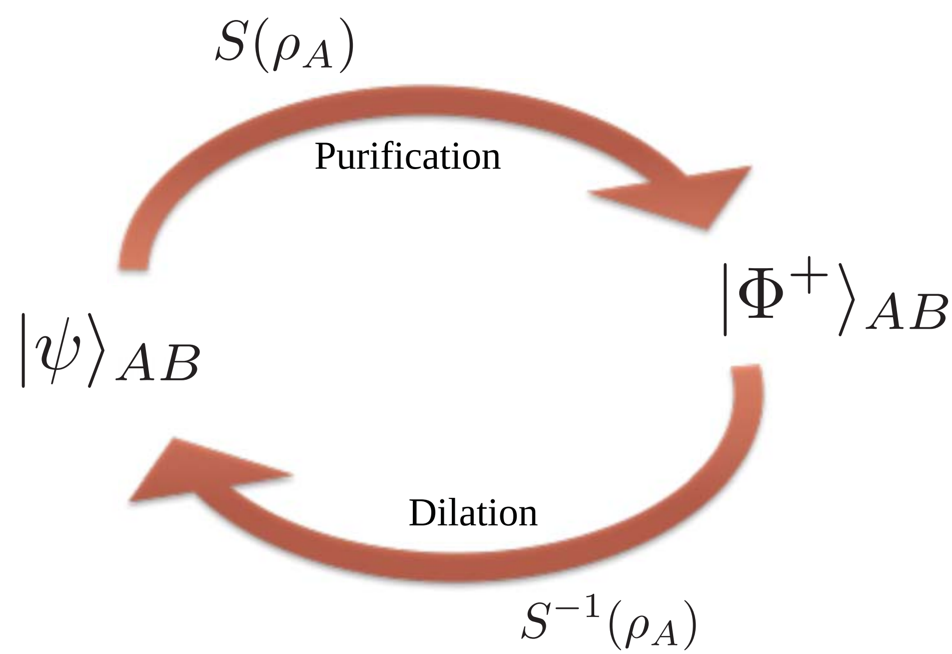
$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A) = -\sum_i \lambda_i^\psi \log \lambda_i^\psi$$

$$\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|_{AB})$$

Entanglement Dilation

$$|\Phi^+\rangle_{AB}^{\otimes m(n)} \xrightarrow{\text{LOCC}} |\tilde{\Psi}\rangle_{AB} \approx |\psi\rangle_{AB}^{\otimes n}$$

$$R(|\Phi^+\rangle_{AB} \xrightarrow{\text{LOCC}} |\psi\rangle_{AB}) = \lim_{n\to\infty} \frac{n}{m(n)} = S^{-1}(\rho_A)$$



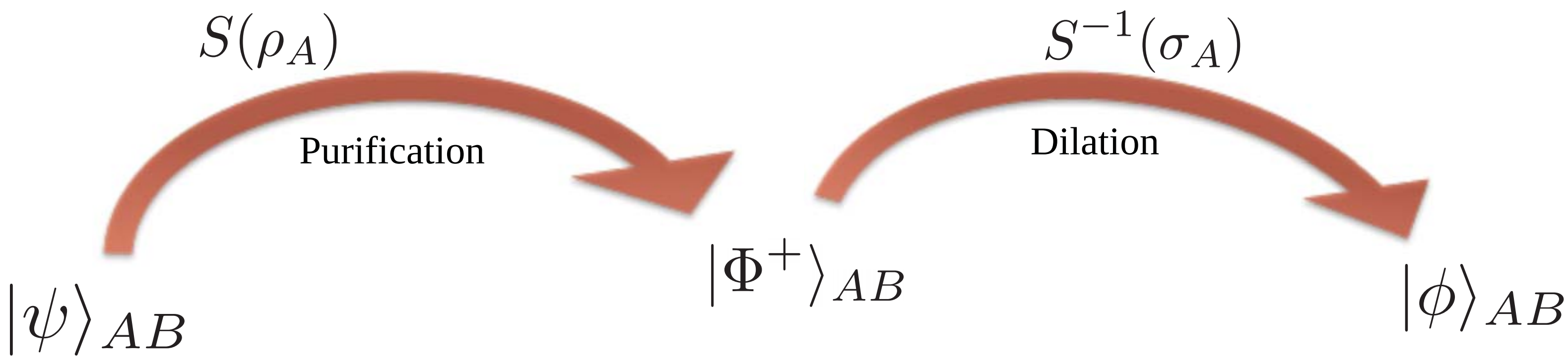
$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|_{AB})$$

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

Reversible transformations have optimal conversion rate.

Asymptotic Transformations



$$|\psi\rangle_{AB}^{\otimes n} \xrightarrow{\text{LOCC}} |\tilde{\Phi}\rangle_{AB} \approx |\phi\rangle_{AB}^{\otimes m(n)}$$

$$|\psi\rangle_{AB} \cancel{\xrightarrow{\text{LOCC}}} |\phi\rangle_{AB}$$

$$R(|\psi\rangle_{AB} \xrightarrow{\text{LOCC}} |\phi\rangle_{AB}) = \lim_{n\to\infty} \frac{m(n)}{n} = \frac{S(\rho_A)}{S(\sigma_A)}$$

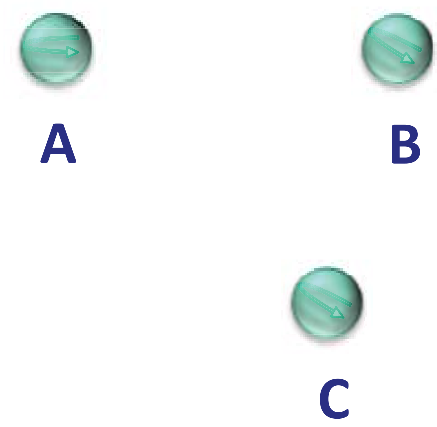
$$\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|_{AB}) \quad \sigma_A = \mathrm{tr}_B(|\phi\rangle\langle\phi|_{AB})$$

In the asymptotic regime any entangled state can be transformed to any other entangled state with a nonzero rate.

Multi-partite Entanglement

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$



They cannot be transformed to each other via LOCC, even in the asymptotic regime.

Measures of Entanglement

- A function from states to real numbers is an entanglement monotone if it is non-increasing under LOCC.

$$f(\mathcal{E}_{\mathrm{LOCC}}(\rho_{AB})) \leq (\rho_{AB})$$

In other words,

$$\rho_{AB} \xrightarrow{\mathrm{LOCC}} \sigma_{AB} \quad \Longrightarrow \quad f(\rho_{AB}) \geq f(\sigma_{AB})$$

- Therefore,

$$\begin{array}{c} \rho_{AB} \xrightarrow{\mathrm{LOCC}} \sigma_{AB} \\ \sigma_{AB} \xrightarrow{\mathrm{LOCC}} \rho_{AB} \end{array} \quad \Longrightarrow \quad f(\rho_{AB}) = f(\sigma_{AB})$$

- Any entanglement monotone takes the same value on all separable states. Therefore, by adding a constant to the monotone we can make it zero on all separable states.

ρ_{AB} is separable (unentangled) $\Longrightarrow$ $f(\rho_{AB}) = 0$

Measures of Entanglement

- Entanglement entropy is non-increasing in pure to pure state transformations.

$$|\psi\rangle_{AB} \xrightarrow{\text{LOCC}} |\phi\rangle_{AB} \quad \Longrightarrow \quad S(\rho_A) \geq S(\sigma_A)$$

$$\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|_{AB}) \quad \sigma_A = \mathrm{tr}_B(|\phi\rangle\langle\phi|_{AB})$$

- But, in general, the entropy of the reduced states of A and B can increase under LOCC.

Example: Negativity

$$\rho_{AB} = \sum_{i,j} X_i \otimes Y_j \xrightarrow{\text{Partial Transpose}} \rho_{AB}^{T_A} = \sum_{i,j} X_i^T \otimes Y_j$$

$$\rho_{AB} = \sum_i p_i \; \rho_A^{(i)} \otimes \sigma_B^{(i)} \xrightarrow{\text{Partial Transpose}} \tilde{\rho}_{AB} = \sum_i p_i \; {\rho_A^{(i)}}^T \otimes \sigma_A^{(i)}$$

$$\frac{\|\rho_{AB}^{T_A}\|_1 - 1}{2}$$

Summary

Resource Theory of Entanglement

Resource	Entanglement
Free states	Separable states
Free operations	Local Operations & Classical Communications
Physical justification of free operations*	Lack of quantum channels

* By *physical justification* we mean a restriction on experimental capabilities that yields **all and only** the set of free operations.

Note that the set of LOCC does not include all non-entangling operations.

For a general discussion about resource theories see: F. Brandão, G. Gour , *The general structure of quantum resource theories,* Phys. Rev. Lett. (2015).

Summary

Resource Theory of Entanglement

- Two pure bi-partitie states have the same entanglement properties, i.e. can be transformed to each other via LOCC reversibly, iff their **Schmidt coefficients** are the same.

- A pure bi-partite state can be transformed to another pure bi-partitie state via deterministic LOCC transformations, if and only if the Schmidt coefficients of the former state are **majorized** by the Schmidt coefficients of the latter state (Nielsen's theorem).

- Some state transformations which are not possible under LOCC, can be implemented using another entanglement state as **catalyst**.

- In the asymptotic regime, where we are given many copies of states, any pure bi-partite entangled state can be transformed to any other state via LOCC. The optimal rate of conversion is equal to the ratio of the **entanglement entropies** of the input and output states.

Quantum Entanglement, Horodecki^4, Rev. Mod. Phys. **81**, 865 (2009).

Resource Theory of Asymmetry

Resource theory of clocks, gyroscopes, and other reference frames

Resource Theory of Asymmetry

Resource	Entanglement	Asymmetry
Free states	Separable states	Symmetric states*
Free operations	Local Operations & Classical Communications	Symmetric Operations
Physical justification of free operations	Lack of quantum channels	E.g. lack of shared reference frames

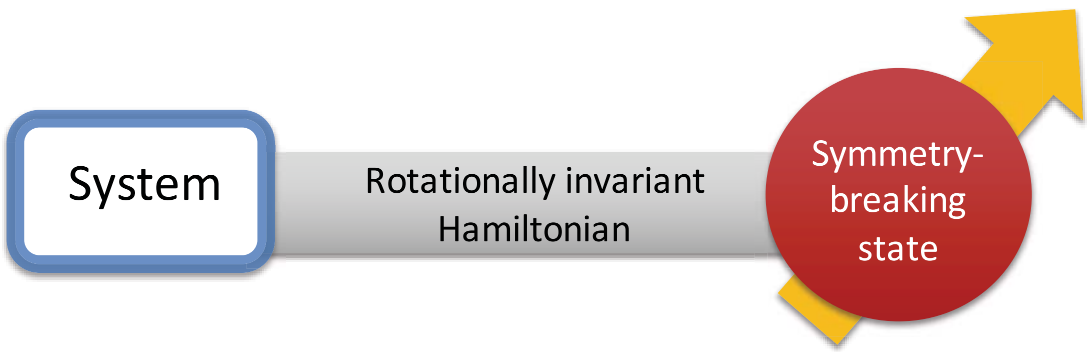
* Different symmetry groups define different resource theories of asymmetry.

Example: SO(3)

What kind of transformations can be implemented on the system if we are restricted to rotationally invariant Hamiltonians, and we can only prepare ancillas in rotationally invariant states?

Free state

Under this kind of restriction, a symmetry-breaking state is a resource.

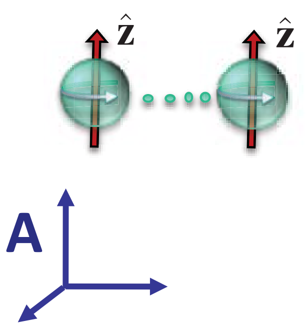


Resource state

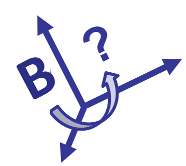
Asymmetry relative to SO(3)

A gyroscope should break rotational symmetry.

Resource theory of asymmetry for group SO(3) is useful in the study of **quantum gyroscopes**.



Quantum Gyroscope



SO(3)

Asymmetry relative to time translation

$$\frac{|E_0\rangle\langle E_0| + |E_1\rangle\langle E_1|}{2}$$ Time-independent state

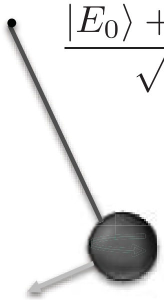
$$H|E_0\rangle = E_0|E_0\rangle$$
$$H|E_1\rangle = E_1|E_1\rangle$$

$$\frac{|E_0\rangle + |E_1\rangle}{\sqrt{2}}$$ Time-dependent state (Breaks time-translation symmetry $\{e^{-iHt} : t \in \mathbb{R}\}$)

A clock should break time-translation symmetry.

A Quantum Clock

$$\frac{|E_0\rangle + |E_1\rangle}{\sqrt{2}}$$



B

$$\frac{e^{-iE_0\delta t}|E_0\rangle + e^{-iE_1\delta t}|E_1\rangle}{\sqrt{2}}$$

Asymmetry relative to phase shift

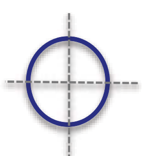
$$\frac{|n_0\rangle\langle n_0| + |n_1\rangle\langle n_1|}{2}$$ Symmetric state

$$N|n_0\rangle = n_0|n_0\rangle$$
$$N|n_1\rangle = n_1|n_1\rangle$$

$$\frac{|n_0\rangle + |n_1\rangle}{\sqrt{2}}$$ Asymmetric state (Breaks phase shift symmetry $\{e^{-iN\phi} : \phi \in (0, 2\pi]\}$)

A phase reference should break phase shift symmetry.

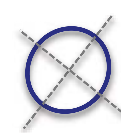
A



$$\frac{|n_0\rangle + |n_1\rangle}{\sqrt{2}}$$

B

$$\frac{e^{-in_0\delta\phi}|n_0\rangle + e^{-in_1\delta\phi}|n_1\rangle}{\sqrt{2}}$$



Asymmetry as a Resource

The relevant property of quantum states that determines their usefulness as reference frames can be understood as their *asymmetry* (symmetry-breaking) relative to a symmetry group.

Symmetry	Reference Frame
3D Rotations	Gyroscope
Time translation	Clock
Phase shift	Phase reference

Resource Theory of Asymmetry

Resource	Entanglement	Asymmetry
Free states	Separable states	Symmetric states*
Free operations	Local Operations & Classical Communications	Symmetric Operations
Physical justification of free operations	Lack of quantum channels	E.g. lack of shared reference frames

* Different symmetry groups define different resource theories of asymmetry.

Symmetric Time Evolution

Consider a symmetry group G with the unitary representation $g \in G \to U(g)$

$$
\begin{tikzcd}
\rho \arrow[r, "g"] \arrow[d, "\text{time}"'] & U(g)\rho U^{\dagger}(g) \arrow[d, "\text{time}"] \\
\mathcal{E}(\rho) \arrow[r, "g"'] & ?
\end{tikzcd}
$$

Free States	$\forall g \in G: \quad U(g)\rho U^{\dagger}(g) = \rho$	G-invariant state
Free Operations	$\forall \rho, \forall g \in G: \quad U(g)[\mathcal{E}(\rho)]U^{\dagger}(g) = \mathcal{E}[U(g)\rho U^{\dagger}(g)]$	G-covariant operation

Any such operation can be realized using symmetric unitaries and symmetric ancillary states (Steinspring's dilation theorem for symmetric operations).

Resource Theory of Asymmetry

1. What are the conditions for state interconversions? (Single-copy deterministic, stochastic, rate of asymptotic transformations, etc)
   How to quantify asymmetry?

$$\rho \xrightarrow{\text{G-cov}} \sigma$$

2. How can we simulate asymmetric operations using only symmetric operations and asymmetric states? How much resources do we need to simulate a given asymmetric operation? (similar to simulating quantum channels using entangled states via teleportation)

Asymmetry of pure states

$$\chi_\psi(g) \equiv \langle\psi|U(g)|\psi\rangle$$ Characteristic Function

Theorem: Two pure states can be reversibly transformed to each other by G-covariant operations iff their characteristic functions are equal up to a 1-d representation of group.

$$\begin{array}{c} \psi \xrightarrow{\text{G-cov}} \phi \\ \phi \xrightarrow{\text{G-cov}} \psi \end{array} \iff \forall g \in G, \quad \langle\psi|U(g)|\psi\rangle = e^{i\theta(g)}\langle\phi|U(g)|\phi\rangle$$

Using the nice properties of characteristic functions, we can answer all sorts of questions about asymptotic and single-copy interconversion of pure states.

Measures of Asymmetry

Asymmetry cannot be generated by symmetric time evolutions.

- A function from states to real numbers is an asymmetry monotone if it is non-increasing under covariant operations.

$$f(\mathcal{E}_{\text{G-cov}}(\rho)) \leq f(\rho)$$

In other words, $\rho \xrightarrow{\text{G-cov}} \sigma \quad \Rightarrow \quad f(\rho) \geq f(\sigma)$

- Therefore, $\rho \xrightarrow{\text{G-cov}} \sigma$, $\sigma \xrightarrow{\text{G-cov}} \rho \quad \Rightarrow \quad f(\rho) = f(\sigma)$

- Any asymmetry monotone takes the same value on all symmetric states. Therefore, by adding a constant to the monotone we can make it zero on all symmetric states.

ρ is symmetric $\quad \Rightarrow \quad f(\rho) = 0$

$$\rho \xrightarrow{\text{G-cov}} \sigma$$

$$\sigma \xrightarrow{\text{G-cov}} \rho$$

$$\Longrightarrow \quad f(\rho) = f(\sigma)$$

Closed system dynamics

+

Symmetry

$\Longrightarrow$

Conservation of measures of Asymmetry

How to Find measures of Asymmetry?

Question: In the case of rotational symmetry, is the (absolute value of) expectation angular momentum a measure of asymmetry?

$$|\mathrm{tr}(\rho L_z)|$$

NO, Angular momentum can be amplified.

$$\rho \xrightarrow{G-cov} \sigma \qquad\qquad |\mathrm{tr}(\rho L_z)| \ll |\mathrm{tr}(\sigma L_z)|$$

Theorem: Any measure of asymmetry which can be expressed as function of the expectation values of angular momentums or the expectation values of any function of angular momentums is a constant function.

Information Theoretic Approach

The asymmetry properties of ρ is specified by the information theoretic properties of the encoding

$$g \in G \longrightarrow U(g)\rho U^{\dagger}(g)$$

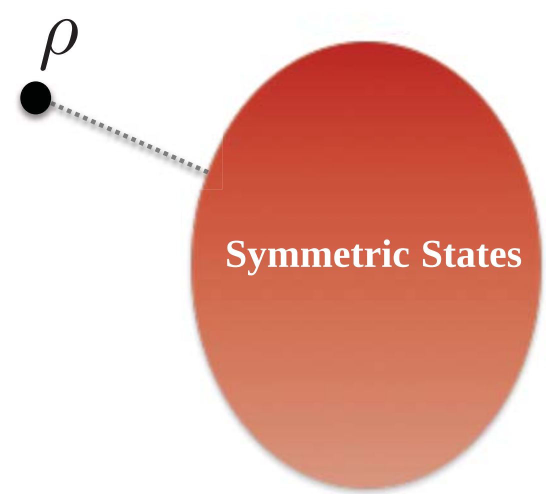
Example 1: Relative Entropy of Asymmetry

$$\Gamma(\rho) \equiv \inf_{\omega \in \mathrm{sym}(G)} S(\rho \| \omega) = S(\rho \| \mathcal{G}(\rho)) = S(\mathcal{G}(\rho)) - S(\rho)$$

$$S(\rho \| \sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma)$$

$$S(\sigma) = -\mathrm{Tr}(\sigma \log \sigma)$$

$$\mathcal{G}(\sigma) = \frac{1}{|G|} \sum_{g \in G} U(g) \sigma U^{\dagger}(g)$$



$$\Gamma(\mathcal{E}_{\text{G-cov}}(\rho)) \leq \Gamma(\rho)$$

Example 2: Wigner-Yanase-Dyson skew information

$$\Delta_{L,\alpha}(\rho) = \mathrm{Tr}(\rho L^2) - \mathrm{Tr}(\rho^\alpha L \rho^{1-\alpha} L) \qquad \alpha \in (0,1)$$

$$\Delta_{L,1/2} = -\frac{1}{2}\mathrm{Tr}([\sqrt{\rho}, L]^2)$$

L is a generator of the symmetry.

$$\Delta_{L,\alpha}(\mathcal{E}_{\text{G-cov}}(\rho)) \leq \Delta_{L,\alpha}(\rho)$$

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Applications

Finding the consequences of symmetry of dynamics

Closed system dynamics + Symmetry ⇒ Conservation laws

Noether's theorem

Symmetry	Conserved quantity
Spatial translations	Linear Momentums
3D rotations	Angular Momentums
…..	…

Deficiencies:

- These conservation laws do not capture all the consequences of symmetry.
- What about open system dynamics?
  (e.g. phase-insensitive amplifiers, thermal relaxations, ….)

Finding the consequences of symmetry of dynamics

- Measures of asymmetry put non-trivial constraints on the dynamics of open systems, based on the symmetry of dynamics.

$$f(\rho(t)) \leq f(\rho(0))$$

- They are conserved in the closed system symmetric dynamics dynamics.

$$f(\rho(t)) = f(\rho(0))$$

$$\rho = \frac{1}{2} [\, |\uparrow\rangle\langle\uparrow| \otimes |1\rangle\langle 1| + |\downarrow\rangle\langle\downarrow| \otimes |2\rangle\langle 2| \,]$$

$$\sigma = \frac{1}{2} [|\rightarrow\rangle\langle\rightarrow| \otimes |1\rangle\langle 1| + |\leftarrow\rangle\langle\leftarrow| \otimes |2\rangle\langle 2| \,]$$

1) There exists a closed system dynamics, which breaks the rotational symmetry, and maps one state to the other. In other words, this transition can happen in the absence of symmetry.

2) On the other hand, this transition is not allowed under symmetric dynamics. General measures of asymmetry take different values on these two states.

3) Noether's conserved quantities cannot see this. Angular momentum, or any function of angular momentum takes the same value for these two states.

$$\forall k: \quad tr(\rho L^k) = tr(\sigma L^k)$$

- **Noether's theorem does not capture all the consequences of symmetry.**
- **Conservation of measures of asymmetry yields independent constraints.**

IM, RW Spekkens, Extending Noether's theorem by quantifying the asymmetry of quantum states, Nature Comm. (2014).

Catalytic Transformations

$$\tau^R \xrightarrow{\text{Covariant}} \tau^R \otimes \sigma^S$$

$$\tau^R \otimes \rho^S_{\text{sym}} \xrightarrow{\text{Covariant}} \tau^R \otimes \sigma^S$$

E.g. State preparation using a clock, cloning of unknown quantum states

Theorem: There exists a G-covariant channel $\mathcal{E}^{R\to RS}$ such that

$$F(\mathcal{E}^{R\to RS}(\tau^R), \tau^R \otimes \sigma^S) \geq 2^{-\frac{\Delta\Gamma}{2}}$$

where

$$\Delta\Gamma = \Gamma(\tau^R \otimes \sigma^S) - \Gamma(\tau^R)$$

is the increase in the relative entropy of asymmetry in the desired state transition.

$$\Gamma(\rho) \equiv \inf_{\omega\in\text{sym}(G)} S(\rho\|\omega) = S(\rho\|\mathcal{G}(\rho)) = S(\mathcal{G}(\rho)) - S(\rho) \qquad F(\rho_1, \rho_2) \equiv \|\sqrt{\rho_1}\sqrt{\rho_2}\|_1$$

Follows from a recent result of Fawzi and Renner on approximate recoverability.

Conclusion

1) Resource theory of asymmetry is a framework for classifying and quantifying asymmetry of quantum states relative to a given symmetry. This framework is useful, for instance, in the context of quantum reference frames, and finding the consequences of symmetry of dynamics.

2) Noether's theorem does not capture all the consequences of symmetry of closed systems. The conservation of measures of asymmetry can give independent constraints.

Resource Theory of Asymmetry

Resource	Entanglement	Asymmetry
Free states	Separable states	Symmetric states*
Free operations	Local Operations & Classical Communications	Symmetric Operations
Physical justification of free operations	Lack of quantum channels	E.g. lack of shared reference frames

* Different symmetry groups define different resource theories of asymmetry.

G Gour, RW Spekkens, The resource theory of quantum reference frames: manipulations and monotones, NJP (2008).
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Thanks for your attention