# Exercise 2

1. Show that for $V^\nu = P_\mu S^{\mu\nu}$, where $S^{\mu\nu}$ is the spin generator and $P^\mu$ is the momentum,

$$[J_{\alpha\beta}, V_\mu] = i(\eta_{\alpha\mu}V_\beta - \eta_{\beta\mu}V_\alpha).$$

2. Show that

$$W^2 = W_\mu W^\mu = 2(P^2S^2 - 2(P.S)^2),$$

where $W_\mu$ is the Pauli-Lubanski operator and $(P_\mu S^{\mu\nu})^2 = (P.S)^2$. Therefore, for all massive and massless irreducible representation (for which $P_\mu S^{\mu\nu} = 0$) with mass $m$ and spin $s$

$$W^2 = P^2S^2 = m^2 s(s+1).$$

3. Show that $\hat{\mathcal{L}}(\phi) = \mathcal{L}(\phi) + \partial_\mu K^\mu(\phi, \partial\phi)$ and $\mathcal{L}(\phi)$ lead to the same classical equation of motion.
4. Compute the Noether current for Lorentz transformations $\mathcal{J}_{\mu\nu\alpha}$ for a free scalar field of mass $m$.
5. Given the equal time commutation relation, work out the algebra of $a_k, a_l^\dagger$(the Fourier modes of the field $\phi$).
6. Let $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$ be the generators of Poincare algebra for field $\phi$, i.e. Noether charges corresponding to translation and Lorentz symmetries.
(a) Write the explicit form of $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$ operators in terms of creation-annihilation operators $a_k, a_l^\dagger$.
(b) Using algebra of $a_k, a_l^\dagger$, show that $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$ satisfy Poincare algebra.
(c) Show that

$$\hat{P}_\mu|0\rangle = 0 \quad , \quad \hat{J}_{\mu\nu}|0\rangle = 0.$$

if $a_k|0\rangle = 0$.
(d) Show that vacuum is Poincaré invariant

$$\hat{U}(\Lambda, a)|0\rangle = |0\rangle,$$

where $\hat{U}(\Lambda, a) = e^{-i\hat{P}.a + \frac{i}{2}w_{\mu\nu}\hat{J}^{\mu\nu}}$.