Electron-Positron Annihilation

Now we are ready to begin describing the gauge theories of the strong, weak, and electromagnetic interactions and comparing the predictions of those theories to experiment. I will begin with the electrodynamics of quarks and leptons.

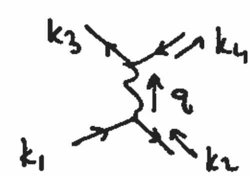
In this course, there will be many quantum field theory computations. However, my main goal will be to get the answers and understand their physical implications. In these notes, I will generally take the fastest correct route to the particular answer that I would like to discuss. However, to learn quantum field theory, you will want to learn other methods, including more straightforward methods that are trustworthy for a wider variety of problems. I recommend that you compare the derivations given in these notes to those given in other sources, for example, my textbook with Schroeder.

I would like to start with the simplest QED process, e^+e^- annihilation into $\mu^+\mu^-$. This is a purely leptonic QED process. We will now obtain the differential cross section $d\sigma/d\cos\theta$ for this reaction. In standard treatments, this is done by summing and averaging over polariations using gamma matrix traces. Here, I will use a different technique, computing the scattering amplitudes explicitly for fixed initial and final polarizations.

To specify the polarizations, I will use states of definite helicity. Helicity is the spin quantized along the direction of motion. I have already noted that, for massless particles, the helicity specifies a fundamental representation of the Lorentz group. For massive particles, the helicity can be changed by Lorentz transformations. For example, given a fermion with helicity $+\frac{1}{2}$, we can boost it to the rest frame, where it has indefinite helicity, and boost further to a frame where the helicity is $-\frac{1}{2}$. However, still, the helicity is invariant under rotations and under boosts in the direction of particle motion that do not reach the rest frame. In any case, helicity amplitudes often have a simple and intuitive structure.

So, let us compute the helicity amplitudes for $e^+e^- \to \mu^+\mu^-$ in the center of mass frame. There is one Feynman diagram in leading order. The value of this diagram is

$$i\mathcal{M} = (-ie)^2\ \bar{v}(k_2)\gamma^\mu u(k_1)\ \frac{-i}{q^2}\ \bar{u}(k_3)\gamma_\mu v(k_4)$$



I will evaluate it with the specific polarization spinors appropriate to the helicity states. In the final result, I will ignore the electron mass but keep the muon mass.

It is not hard to construct the spinors explicitly for fermions moving in the $\hat{3}$ direction. Choose the basis for the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (1, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

For $k^\mu = (E, 0, 0, k)$, $E^2 = k^2 + m^2$, the Dirac equation $(\not{k} - m)u(k) = 0$ takes the explicit form

$$\begin{pmatrix} -m & E - k\sigma^3 \\ E + k\sigma^3 & -m \end{pmatrix} u = 0$$

For $h = +\frac{1}{2}$, u is proportional to the spin up spinor. This gives the solution

$$u(k) = \begin{pmatrix} \sqrt{E-k} \\ 0 \\ \hline \sqrt{E+k} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{E-k} \\ \sqrt{E+k} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $h = -\frac{1}{2}$, u is proportional to the spin down spinor, and so

$$u(k) = \begin{pmatrix} 0 \\ \sqrt{E+k} \\ \hline 0 \\ \sqrt{E-k} \end{pmatrix} = \begin{pmatrix} \sqrt{E+k} \\ \sqrt{E-k} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Dirac equation has two solutions, and now we have found them. In the limit of zero mass, these expressions degenerate to

$$h = -\tfrac{1}{2} \qquad u = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ \hline 0 \\ 0 \end{pmatrix} \qquad\qquad h = +\tfrac{1}{2} \qquad u = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix}$$

For the antiquark, I will assign the momentum vector $k^\mu = (E, 0, 0, -k)$. The Dirac equation $(\not{k} + m)v(k) = 0$ is then.

$$\begin{pmatrix} m & E + k\sigma^3 \\ E - k\sigma^3 & m \end{pmatrix} v = 0$$

We can easily find two solutions, proportional to the spinors with spin down and spin up with respect to the axis $-\hat{3}$,

$$v(k) = \begin{pmatrix} \sqrt{E+k} \\ -\sqrt{E-k} \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad v(k) = \begin{pmatrix} \sqrt{E-k} \\ -\sqrt{E+k} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

There is a subtlety, though, about which is $h = +\frac{1}{2}$ and which is $h = -\frac{1}{2}$. To resolve this, remember that u belongs to the field that destroys the electron, while v belongs to the field that creates the positron. Alternatively, the spinor in v is the spinor of the negative-energy electron that we kick out to form the hole state that is the positron. By either logic, we use the spin-up spinor for $h = -\frac{1}{2}$ and the spin-down spinor for $h = +\frac{1}{2}$. In the massless limit,

$$h = -\tfrac{1}{2} \qquad v(k) = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ \hline 0 \\ -1 \end{pmatrix} \qquad\qquad h = +\tfrac{1}{2} \qquad v(k) = \sqrt{2E} \begin{pmatrix} -1 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix}$$

We can use these spinors to compute the initial and final fermion current expectation values. To evaluate $\overline{v}\gamma^\mu u$, note that

$$\bar{v}\gamma^\mu u = v^\dagger \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} u$$

When the fermion mass is zero, only two of the four possible expectation values are nonzero

$$\bar{v}_{h=\frac{1}{2}} \gamma^\mu u_{h=-\frac{1}{2}} = 2E \,(-1\;0)(1, -\vec{\sigma})^\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2E(0, 1, -i, 0)^\mu$$

$$\bar{v}_{h=-\frac{1}{2}} \gamma^\mu u_{h=\frac{1}{2}} = 2E \,(0\;-1)(1, \vec{\sigma})^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2E(0\,1, i, 0)^\mu$$

We can identify these expressions with the polarization vectors for a spin 1 state of left- or right-handed polarization along the $\hat{3}$ axis ($J^3 = -1$ or $+1$),

$$\varepsilon_L^\mu = \tfrac{1}{\sqrt{2}}(0, 1, -i, 0)^\mu \qquad \varepsilon_R^\mu = \tfrac{1}{\sqrt{2}}(0, 1, i, 0)^\mu$$

The third polarization vector, for $J^3 = 0$, is

$$\varepsilon_0^\mu = (0\ 0\ 0\ 1)^\mu$$

Then

$$e_L^- e_R^+ : \quad \bar{v}\gamma^\mu u = \sqrt{2}\cdot 2E \cdot \varepsilon_L^\mu$$

$$e_R^- e_L^+ : \quad \bar{v}\gamma^\mu u = -\sqrt{2}\cdot 2E \cdot \varepsilon_R^\mu$$

In the case of nonzero mass, the other two possible helicity combinations give nonzero values proportional to the fermion mass m,

$$\bar{v}_{h=-\frac12}\gamma^\mu u_{h=-\frac12} = \sqrt{E-k}\,\sqrt{E+k}\left\{ (0\,1)(1,-\vec{\sigma})^\mu \begin{pmatrix}0\\1\end{pmatrix} - (0\,1)(1,\vec{\sigma})^\mu \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

$$= +2m\,(0\,0\,0\,1)^\mu$$

$$\bar{v}_{h=+\frac12}\gamma^\mu u_{h=+\frac12} = \sqrt{E+k}\,\sqrt{E-k}\left\{ (-1\,0)(1,-\vec{\sigma})^\mu \begin{pmatrix}1\\0\end{pmatrix} - (-1\,0)(1,\vec{\sigma})^\mu \begin{pmatrix}1\\0\end{pmatrix} \right\}$$

$$= +2m\,(0\,0\,0\,1)^\mu$$

These are called *helicity-flip* amplitudes. The values of the *helicity conserving* amplitudes are unchanged from the massless case, as the result of the identity $(E+k)+(E-k) = 2E$. The final result is

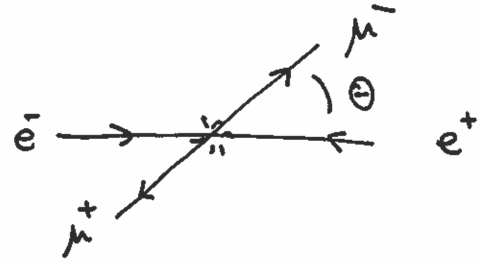
$$\bar{v}\gamma^\mu u = \begin{array}{c|cc} h_f \downarrow \ h_{\bar f} \rightarrow & -\frac12 & +\frac12 \\ \hline -\frac12 & 2m\,\varepsilon_0^\mu & \sqrt{2}\,2E\,\varepsilon_L^\mu \\ +\frac12 & \sqrt{2}\,2E\,\varepsilon_R^\mu & 2m\,\varepsilon_0^\mu \end{array}$$

The final-state current $\bar{u}\gamma^\mu v$ is just the complex conjugate of the initial-state current, so now we can compute the full amplitude

$$i\mathcal{M} = \frac{ie^2}{q^2} \quad \bar{v}(k_2)\gamma^\mu u(k_1) \quad [\bar{v}(k_4)\gamma_\lambda u(k_3)]^*$$

Then, for example, $$i\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R) = \frac{ie^2}{q^2}(2E)^2 \cdot 2 \cdot \varepsilon_L^\mu (\varepsilon_{L\lambda})^*$$

In a general final-state configuration, the outgoing muon is moving at an angle θ with respect to the incoming electron



To account for this, we must rotate the expression for the final-state current, that is, we must rotate its polarization vector. Assuming for convenience that the muon remains in the $\hat{3} - \hat{1}$ plane, the rotated vectors are

$$\varepsilon_L^\mu = \frac{1}{\sqrt{2}}(0 \quad \cos\theta \quad -i, -\sin\theta)$$

$$\varepsilon_R^\mu = \frac{1}{\sqrt{2}}(0, \cos\theta \quad i, -\sin\theta)$$

$$\varepsilon_0^\mu = (0 \quad \sin\theta \quad 0 \quad \cos\theta)$$

Then $$\varepsilon_L \cdot (\varepsilon_L(\theta))^* = -\frac{(1+\cos\theta)}{2} \qquad \varepsilon_L \cdot [\varepsilon_R(\theta)]^* = -\frac{(1-\cos\theta)}{2} \qquad \varepsilon_L \cdot (\varepsilon_0(\theta))^* = -\frac{\sin\theta}{\sqrt{2}}$$

Using these results, we find

$$i\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R) = -ie^2(1+\cos\theta)$$

$$i\mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_L) = ie^2(1-\cos\theta)$$

$$i\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_L) = i\mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_R) = -i\,\frac{e^2 m}{E}\,\sin\theta$$

Parity, which is a good symmetry of the electromagnetic interactions, flips all helicities, so the amplitudes with $L \leftrightarrow R$ in all positions are identical to these.

We can now obtain the cross sections for $e^+e^- \to \mu^+\mu^-$ using the rule

$$\int d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \int d\Pi \, |\mathcal{M}|^2$$

For $2 \to 2$ processes with massless initial states, this formula reduces to

$$d\sigma = \frac{1}{2s} \frac{1}{16\pi} \frac{2k}{E_{cm}} d\cos\theta \quad |\mathcal{M}|^2$$

where k is the momentum of the final particles in the center of mass frame and I have introduced $s = E_{\rm CM}^2$. Plugging the above expressions for the amplitudes into this formula, we find for the polarized cross sections

$$\frac{d\sigma}{d\cos\theta} = \begin{cases} \dfrac{\pi\alpha^2}{2s} \dfrac{k}{E} (1+\cos\theta)^2 & e^-_L e^+_R \to \mu^-_L \mu^+_R \\[2ex] \dfrac{\pi\alpha^2}{2s} \dfrac{k}{E} (1-\cos\theta)^2 & e^-_L e^+_R \to \mu^-_R \mu^+_L \\[2ex] \dfrac{\pi\alpha^2}{2s} \dfrac{k}{E} \dfrac{m^2}{E^2} \sin^2\theta & e^-_L e^+_R \to \mu^-_L \mu^+_L \text{ or } \mu^-_R \mu^+_R \end{cases}$$

and similarly for all helicities reversed. Now add these up and average over the 4 possibilities for initial electron and positron polarizations. This gives for the cross section from unpolarized beams

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2} \frac{\pi\alpha^2}{2s} \frac{k}{E} \left[(1+\cos\theta)^2 + (1-\cos\theta)^2 + 2 \frac{m^2}{E^2} \sin^2\theta \right]$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \frac{k}{E} \left(1 + \cos^2\theta + \frac{m^2}{E^2} \sin^2\theta \right)$$

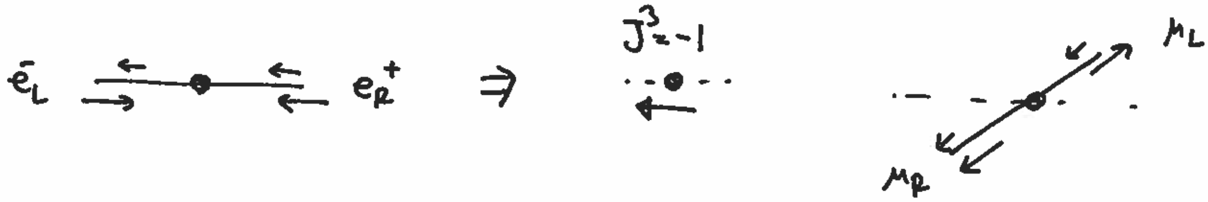
The total cross section is $\quad \sigma = \dfrac{4\pi\alpha^2}{3s} \left(1 - \dfrac{m^2}{E^2}\right)^{1/2} \left(1 + \dfrac{1}{2}\dfrac{m^2}{E^2}\right)$

This last formula—which is characteristic for pair-production of spin-$\frac{1}{2}$ particles, nicely explains the shape of the threshold cross section for $\tau^+\tau^-$ production, as shown in Figs p. 2.

The helicity formulae make the physics of muon pair production very clear. Each term in our formula for the polarization-averaged cross section is associated with a definite helicity amplitude and reflects the angular momentum transfer in that amplitude. For example, the expresssion

$$\frac{d\sigma}{d\cos\theta}(e^-_L e^+_R \to \mu^-_L \mu^+_R) = \frac{\pi\alpha^2}{2s}\frac{k}{E}(1+\cos\theta)^2$$

reflects the picture



In this component process, the muon must be forward peaked to conserve angular momentum, and angular momentum conservation requires that the backward cross section is zero.

An amazing property of these formulae is that they work not only for leptons but also for quarks. Since quarks have strong interactions and are permanently confined, it is surprising that we can even observe the quark direction of motion to test these formulae. However, at high energy, $e^+e^- \to$ hadrons events generally have the form shown in Figs p. 3. This is an event at $E_{\rm CM} = 91$ GeV observed by the SLD detector. The event consists of two back-to-back streams of hadrons, mainly pions and kaons, narrowly collimated. We call these structures *jets*. It is tempting to think of each jet as the product of one original quark or antiquark. Indeed, the angular distribution of jets is just that predicted by QED,

$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to 2\text{ jets}) \sim (1+\cos^2\theta)$$

as shown in Figs p. 4.

Let us now carefully predict the cross section for $e^+e^- \to$ hadrons using these ideas. I will assume that the energy of the reaction is very large compared to the quark mass. Then the QED formula for the $e^+e^- \to \mu^+\mu^-$ cross section would be

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

The diagram for $e^+e^- \to q_f\overline{q}_f$ would be proportional to the quark charge Q_f, so the cross section for quark pair production would be proportional to Q_f^2 for each flavor. Also, please remember that quarks carry an extra quantum number, color, that takes 3 values. Each color of quark is independently produced. In all, we predict

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3 \cdot \sum_f Q_f^2 = \text{constant}$$

where the sum is taken over flavors with mass low enough to be produced: $2m_f < E_{\text{CM}}$. For u, d, s, quarks, the right-hand side is

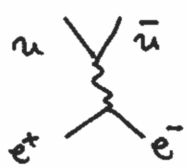
$$3 \cdot \left[\left(\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 \right] = 2$$

The data on the total cross section for $e^+e^- \to$ hadrons is shown in Figs p. 5. The green line is

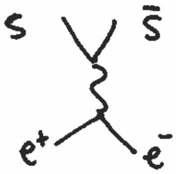
$$R = \begin{cases} 2 & 2 < E_{cm} < 3.5 \text{ GeV} \\ 3\tfrac{1}{3} & 6 < E_{cm} < 10 \text{ GeV} \\ 3\tfrac{2}{3} & 10 < E_{cm} < 50 \text{ GeV} \end{cases}$$

The last two cases correspond to the addition of heavy quarks of charge $+\frac{2}{3}$—c or charm—and $-\frac{1}{3}$—b or bottom. The green line is already an excellent representation of the data between 2 GeV and 50 GeV. The red curve add a small correction predicted in the gauge theory of strong interactions.

At energies below 2 GeV, the regularity that R should be constant is disrupted by the effects of quark binding. For example,

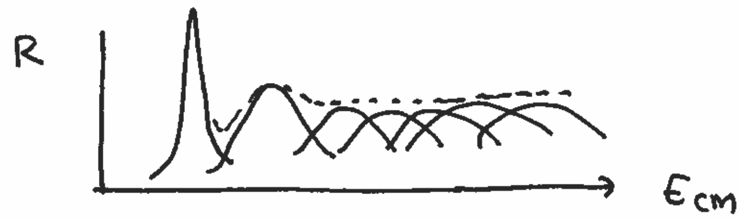


is resonant at $E_{cm} = m_\rho \;\; m_\omega$



is resonant at $E_{cm} = m_\phi$

In general, as I will discuss in detail in the next lecture, we have a resonance whenever there is a spin 1 bound state of $q\overline{q}$ with $P = -1$, $C = -1$. At higher energies, these bound states become broader (that is, shorter-lived) and merge to give a sum that represents the constant value of R



However, whenever we reach a threshold at which a new quark can appear, we might see additional resonances corresponding to the bound states of that $q\overline{q}$ pair. These resonances do appear at the thresholds for c and b; they are the narrow spikes labeled J/ψ, Υ in the figure. I will discuss the properties of these states in the next lecture.

The structure near 90 GeV is not a part of the strong interactions. It is a resonance associated with a new heavy particle Z^0. The Z^0 is a fundamental quantum of the weak interactions. I will discuss this particle in the third week of the course.