Quarkonium

In the previous lecture, I noted that impressive resonances are found in the cross section for e^+e^- annihilation to hadrons near the thresholds for $c\bar{c}$ and $b\bar{b}$ production. These resonances are part of a spectrum of atomic-like bounds states of quarks and antiquarks. The full spectrum of states is exactly what we would see in an atomic system. This is the most impressive piece of evidence that quarks and leptons really are on the same footing as elementary spin $\frac{1}{2}$ fermions.

The bound states of $c\bar{c}$ and $b\bar{b}$ are called *charmonium* and *bottomonium*, collectively, *quarkonium*. The binding is by strong interaction forces, and so it will give us clues to the nature of the strong interactions.

The closest analogy to a $q\bar{q}$ bound state is *positronium*, the system of bound states of an electron and a positron. Positronium is bound by electrical forces. The properties of positronium states can be computed to high accuracy in QED, and there is good agreement between theory and experiment. I will therefore begin the lecture by describing the spectrum of positronium as orientation for our study of $q\bar{q}$ bound states.

The spectrum of positronium is shown in Figs p. 2 and compared there to the spetrum of the Hydrogen atom. In Hydrogen, as you know well, the binding energies are given by

$$E_n = -\frac{Ry}{n^2} \qquad Ry = \frac{1}{2}\alpha^2 mc^2$$

The characteristic momentum of an electron and the size of its wavefunction is

$$p_0 = mv_0 = \alpha mc \qquad a_0 = \hbar/\alpha mc$$

In positronium, these formulae are the same, except that the electron mass is replaced by the reduced mass

$$\mu = \frac{1}{2}m$$

so the Rydberg is 2 times smaller. At higher orders of approximation, the states of both Hydrogen and positronium are split by relativistic corrections.

fine structure : $\alpha^4 mc^2 \cdot \vec{L}\cdot\vec{S}$

hyperfine structure : $\alpha^4 mc^2 \cdot \vec{S}_+\cdot\vec{S}_-$

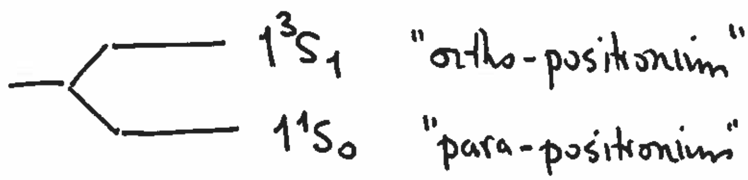
Unlike Hydrogen, the hyperfine splittings of positronium are not suppressed by the small factor m_e/m_p but rather are comparable to the fine structure splittings. Also, unlike Hydrogen, positronium has no special degeneracy between the 2S and 2P states broken only by radiative corrections through the Lamb shift.

To describe the spectrum further, I will introduce a more precise notation for the states. I will label states by

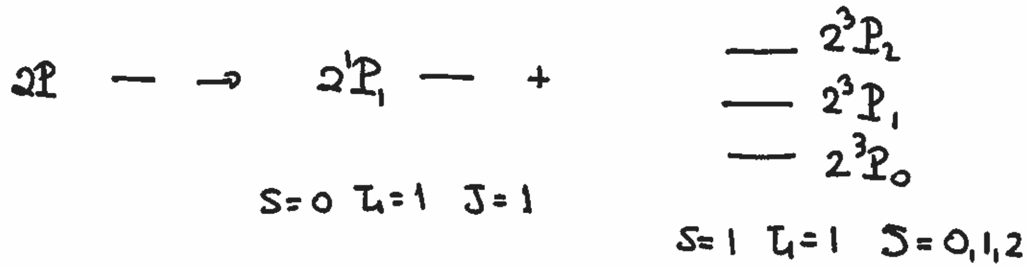
$$n^S L_J$$

where n is the principal quantum number, L denotes the orbital angular momentum (S, P, D, *etc.*), S denotes the total spin (1 for singlet, 3 for triplet), and J is the total angular momentum. In positronium, I will label the first P state as 2P, in accord with the convention for Hydrogen. However, in quarkonium, this state is called the 1P, for reasons we will see shortly.

The states of positronium are then arranged as follows: The 1S states split into two spin multiplets:



and the 2S, 3S, ..., split similarly. Among the 2P states, the spin singlet gives a single level with $J = 1$. However, for the spin triplet states, we have $S = 1$ and $L = 1$. These mix through the spin-orbit interaction to produce three levels with $J = 0, 1, 2$.



Parity and charge conjugation are symmetries of QED, and so we can use these quantum numbers to characterize the states of a particle-antiparticle system. It is a fact about the Dirac equation that a fermion and an antifermion have *opposite* parity; thus, a particle-antiparticle bound states has intrinsic parity -1. In addition, P reverses the momenta of the bound particles, but leaves the spins unchanged. In all, the P quantum number of a positronium state is

$$P = (-1)^{L+1}$$

Charge conjugation interchanges the fermion and antifermion of the same spin. More precisely, it interchanges the creation operators

$$a_k^\dagger(\uparrow) \leftrightarrow b_k^\dagger(\uparrow) \qquad a_k^\dagger(\downarrow) \leftrightarrow b_k^\dagger(\downarrow)$$

Since these operators anticommute, restoring the operators to their original order gives a factor -1 in C. In addition, interchanging the electron and positron in positronium gives a factor from reversal of the spatial wavefunction and another from reversal of the spin wavefunction. The triplet spin wavefunction is symmetric, but the singlet spin wavefunction is antisymmetric. In all

$$C = (-1)^{L+1} \cdot \begin{cases} 1 & \text{spin triplet} \\ -1 & \text{spin singlet} \end{cases}$$

From this analysis, para-positronium has C = +, while ortho-positronium has C = −. A photon has C = −, so 1-photon transitions occur between states of opposite C, for example

$$2^3P_J \to \gamma + 1^3S_1 \quad , \quad 2^3S_1 \to \gamma + 2^1S_0$$

All positronium states are unstable with respect to e^+e^- annihilation. Para-positronium can decay straightforwardly into 2 photons

$$1^1S_0 \to 2\gamma \quad \Gamma = \frac{1}{2}\alpha^5 mc^2 \qquad \tau = 1.2 \times 10^{-10}\,\text{sec}$$

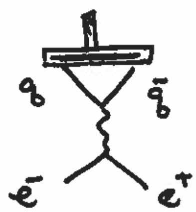
However, ortho-positronium is forbidden by C from decaying to 2 photons, Instead, it has only the more hindered decay to 3 photons. The formula for the decay rate contains an additional factor of α and also the odd phase space suppression factor $(\pi^2 - 9)$,

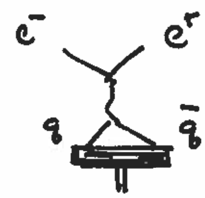
$$1^3S_1 \to 3\gamma \qquad \Gamma = \frac{2}{9\pi}(\pi^2-9)\,\alpha^6 mc^2 \qquad \tau = 1.4 \times 10^{-7}\,\text{sec}$$

Thus, when we shoot positrons into a gas and form positronium, 25% decay rapidly, and the other 75% decay with a lifetime three orders of magnitude longer.

In quarkonium, we should see all of these features repeated. First, we should discuss the resonances produced directly in e^+e^- annihilation. These states are produced through a virtual photon with $J = 1$, so they must be states with $J = 1$, P = $-$, C = $-$. Since these states are produced by a pointlike current, the wavefunctions must have a nonzero amplitude to find the q and $\bar{q}$ at relative distance $\vec{r} = 0$, thus, they must be S states. The resonances seen in e^+e^- annihilation must, then, be the n^3S_1 states of the quarkonium system.

This is all very clear purely from symmetry considerations, but we can make it clearer by explicitly computing the cross section for quarkonium production by e^+e^-. I will also compute the rate for the reverse process, the decay of a quarkonium state to e^+e^-. It is not so difficult to compute the amplitides for these processes from the diagrams





by embedding the $q\bar{q}$ in a nonrelativistic bound state wavefunction.

The matrix element for $q\bar{q} \to e^+e^-$ is

$$i\mathcal{M} = (-ie)(+iQ_f e)\ \bar{u}\gamma^\mu v\ \frac{-i}{s}\ \bar{v}\gamma_\mu u$$

From the previous lecture, $\bar{u}\gamma^\mu v = -\sqrt{2}\ 2E\ \epsilon^{\mu *}_{L,R}$

For the quarks, it will be convenient to recompute the current matrix elements in the limit in which the quarks are almost at rest. The Dirac spinors are

$$u = \sqrt{m}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \xi \qquad v = \sqrt{m}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \xi'$$

where ξ is the 2-component spinor associated with the spin of the quark and ξ' is the reverse of the spinor associated with the spin of the antiquark. Using

$$\bar{v}\gamma^\mu u = v^\dagger \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} u$$

we obtain

$$\bar{v}\gamma^\mu u = (0 \quad -m\,\xi'^\dagger\vec{\sigma}\xi\,)^\mu$$

Then, the matrix element for $q\bar{q} \to e^+e^-$ is given by

$$i\mathcal{M} = i\, Q_f e^2 \sqrt{2}\, \frac{\cdot 2E \cdot m}{(2E)^2} \, (\vec{\sigma} \cdot \vec{\epsilon}_{L,R})_{ab}\, \delta_{ij}$$

where a, b are the antiquark and quark spin indices. We should not forget to add the factor δ_{ij} indicating that a quark and antiquark can annihilate only if their colors are equal.

A nonrelativistic $q\bar{q}$ bound state has mass $M \approx 2m$. The wavefunction of such a bound state can be written as

$$|\Psi\rangle = \sqrt{2M} \int \frac{d^3k}{(2\pi)^3}\, \psi(\vec{k})\, \Sigma_{ab}\, \frac{1}{\sqrt{3}}\delta_{ij}\, \frac{1}{\sqrt{2m \cdot 2m}}\, |\bar{q}(-\vec{k}\, a\, i)\, q(\vec{k}\, b\, j)\rangle$$

In this expression, $\psi(\vec{k})$ is the momentum space Schrödinger wavefunction, the Fourier transform of the usual, real-space Schrödinger wavefunction of the bound state. The matrix Σ_{ab} represents the spin state

$$\text{singlet:} \quad \Sigma_{ab} = \frac{1}{\sqrt{2}}\delta_{ab} \qquad \text{triplet} \quad \Sigma_{ab} = \frac{1}{\sqrt{2}}(\vec{n} \cdot \vec{\sigma})_{ab} \qquad |\hat{\vec{n}}| = 1$$

The color delta function imposes that the bound state is a color singlet. Both of these objects are normalized so that they square to 1. The overall wavefunction has the relativistic normalization of states in Feynman diagrams. This is the reason for the factors of $\sqrt{2M}$ and $\sqrt{2m}$.

Notice that only the Σ_{ab} for the spin-triplet bound state has a nonzero overlap with the matrix element written above. In the spin singlet case, the overlap is proportional to $\mathrm{tr}[\Sigma\vec{\sigma} \cdot \vec{\epsilon}^*] = \mathrm{tr}[\vec{\sigma}] \cdot \vec{\epsilon}^* / \sqrt{(2)} = 0$. For the spin-triplet case, the overlap of the bound state wavefunction with the decay matrix element is

$$i\mathcal{M}(\Psi \to e^+e^-) = iQ_f e^2 \frac{2m\sqrt{2M}}{M\cdot(2m)}\sqrt{3}\int\frac{d^3k}{(2\pi)^3}\psi(\vec{k})\,\vec{n}\cdot\vec{\epsilon}^{\,*}_{L,R}$$

We recognize

$$\int\frac{d^3k}{(2\pi)^3}\psi(\vec{k}) = \psi(\vec{r}=0)$$

Then

$$i\mathcal{M} = 2ie^2 Q_f\sqrt{\frac{2}{M}}\sqrt{3}\,\psi(\vec{r}=0)\,\vec{n}\cdot\vec{\epsilon}^{\,*}_{L,R}$$

To compute the decay rate of a quarkonium state into e^+e^-, this must be averaged over the 3 quarkonium polarization states and summed over the possible final states $e^-_L e^+_R$, $e^-_R e^+_L$. That gives

$$\Gamma(\Psi \to e^+e^-) = \frac{1}{2M}\cdot\frac{1}{8\pi}\cdot\frac{1}{3}\sum_{n,\epsilon}|\mathcal{M}|^2$$

$$= \frac{16\pi\alpha^2}{M^2}Q_f^2\,|\psi(0)|^2$$

To compute the production rate of the quarkonium from e^+e^-, we must average over e^+e^- polarizations and sum over the quarkonium polarizations. This gives

$$\sigma(e^+e^- \to \Psi) = \frac{1}{2s}\,\frac{1}{4}\sum_{n,\epsilon}\int\frac{d^3k_\Psi}{(2\pi)^3}\frac{1}{2E_\Psi}(2\pi)^4\delta(\,)\;|\mathcal{M}|^2$$

This expression involves 1-body final state phase space, which is reduced as follows:

$$\int \frac{d^3k_\psi}{(2\pi)^3}\,\frac{1}{2E_\Psi}\,(2\pi)^4\,\delta^{(4)}(\;) = \int \frac{d^4k_\Psi}{(2\pi)^4}\,2\pi\,\delta(k_\psi^2-M^2)\,(2\pi)^4\,\delta^{(4)}(k_\Psi-k_1+k_2))$$

$$= 2\pi\,\delta(s-M^2)$$

Then we obtain

$$\sigma(e^+e^-\to\Psi) = 192\pi^3\alpha^2 Q_f^2\,\frac{|\psi(0)|^2}{M^3}\,\delta(s-M^2)$$

It is worth noting that, since the production and decay reactions involve the same essential matrix element, they are related by the general formula

$$\sigma(e^+e^-\to X_J) = 4\pi^2\,\frac{(2J+1)\,\Gamma(X_J\to e^+e^-)}{M}\,\delta(s-M^2)$$

which is indeed obeyed by the two results above.

In November 1974, experimenters at the SPEAR e^+e^- storage ring at SLAC found a huge enhancement of the cross section for e^+e^- annihilation at a center of mass energy of 3.1 GeV. With this announcement, the MIT group studying e^+e^- production in proton-nucleus collisions announced that they were about to publish evidence for a dramatic resonance in the final state e^+e^- mass distribution, also at 3.1 GeV. The data from these experiments is shown in Figs p. 3. These discoveries sent shock waves through the high-energy physics community and converted almost all skeptics to the idea that quarks were real. The two groups named their resonances the ψ and J, respectively. The leaders of the groups, Burt Richter and Sam Ting, soon received the Nobel Prize, but they never could agree on the name, so now this resonance is called the J/ψ.

The group at SLAC soon found a second resonance at 3.7 GeV, and several further resonances were later found. These are now understood to be the triplet S states of a system of $c\bar{c}$ bound states.

For the b quark, the first three bound state resonances were discovered at Fermilab as resonances in the mass distribution of $\mu+\mu-$ produced in proton-nucleus collisions. Later studies in e^+e^- annihilation revealed more of these $b\bar{b}$ or Upsilon (Υ) resonances.

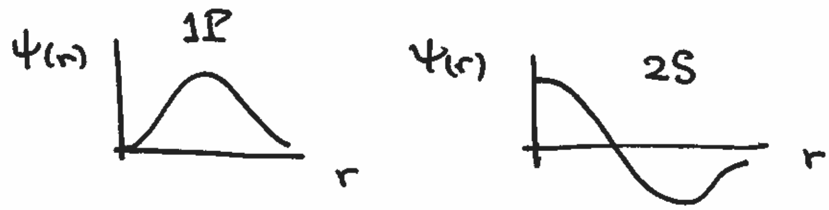
Figs p. 4 shows the form of six resonances of this Υ family as measured by the CUSB experiment at the Cornell e^+e^- collider.

We can interpret the positions of these resonances as the energy eigenstates of a nonrelativistic potential problem, with heavy quarks with masses approximately given by

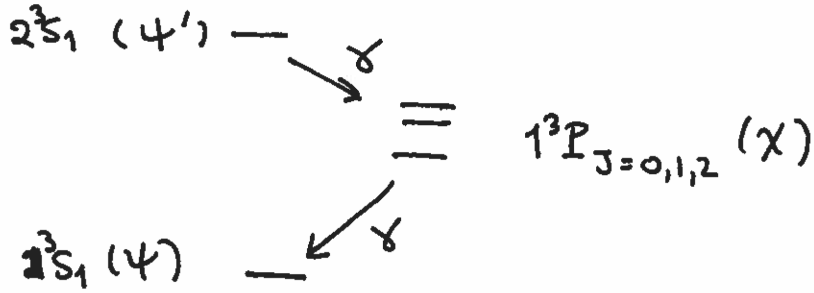
$$m_c \sim 1.5 \text{ GeV} \qquad\qquad m_b \sim 4.5 \text{ GeV}$$

about half the mass of the 1S state. If we assume that the strong interactions have the same coupling to all quark species, this is quite a bit of data to use in determining the form of the potential. Figs p. 5 shows several forms for this quark-antiquark potential suggested in the literature. Note that the form of the potential $V(r)$ is fixed very well in the region of r that corresponds to the size of the ψ and Υ states.

The potentials have a different shape from the Coulomb potential in positronium. This is expected. The Coulomb potential tends to a constant at large distances. But the potential between q and $\overline{q}$ must become very large at large distances to enforce the permanent confinement of quarks into bound states. This gives rise to an important qualitative effect. In a Coulomb potential, the 2S state is degenerate with the lowest P state. However, the 2S state is located at larger radial distance r than the P state, since the node of the P state is in the angular coordinates while the 2S state has a radial node,



Thus, a potential that is larger than Coulomb at large distances should push the 2S state above the first P state. This implies that there should exist radiative transitions



One of the first pieces of evidence for these transitions, from the SLAC experiment, is shown in Figs p. 6. The experiment sat at the ψ' resonance and looked for events of the form

$$\psi' \to \gamma + \gamma + \psi$$
$$\qquad\qquad\qquad\qquad \hookrightarrow \ell^+\ell^-$$

with one γ converting to e^+e^- in the detector, allowing a very accurate energy measurement. The energies of the γs are plotted as values of $m(\chi) = m(\psi') - E(\gamma)$. The lower energy photon gives three sharp peaks indicating the positions of the three χ states. The energy of the higher-energy photons is smeared out by Doppler shift from the recoil of the χ states. Figs p. 7 shows the inclusive γ spectrum at the ψ' observed by the Crystal Ball experiment, an excellent electromagnetic calorimeter used at SPEAR. This spectrum shows 8 transitions present in the ψ' decays, including the magnetic dipole transitions to the 1^1S_0 and 2^1S_0 states, called η_c and η_c'. The sets of three triplet P levels are also seen in the Υ system. Figs p. 8 shows the three peaks from the transitions

$$\underset{\substack{\uparrow \\ 103555\ \text{MeV}}}{\Upsilon(3S)} \to \underset{\substack{\uparrow \\ 10235,\ 10256,\ 10269\ \text{MeV}}}{\chi_b(2P)} + \gamma$$

as observed by the CLEO-II experiment at Cornell.

The full set of known charmonium and bottomonium states is shown in Figs p. 9 and Figs p. 10.

The prominence of the charmonium and bottomonium resonances raises an important problem. Why are these states so narrow, or, equivalently, so long-lived? These states should be unstable with respect to $c\bar{c}$ or $b\bar{b}$ annihilation through the strong interactions. Typical hadrons of mass 3 GeV have widths of several hundred MeV and are essentially invisible among the final states of e^+e^- annihilation. The quarkonium resonances, on the other hand, arc unusually long-lived even by the standards of light hadrons. The measured widths are

$$\Gamma(J/\psi) = 93\ \text{keV} \qquad\qquad \Gamma(\Upsilon) = 54\ \text{keV}$$

How could these values be so small?

An interesting clue is that the 1^1S_0 $c\bar{c}$ state, η_c, is also narrow, but less so,

$$\Gamma(\eta_c) = 27. \ \mathrm{MeV}$$

This is reminiscent of the situation with positronium: The spin 0 state decays relatively rapidly, to 2 photons, which the spin 1 state has a hindered decay to 3 photons. We might apply this same logic to the strong interaction decays of η_c and *psi*. If the strong interactions are mediated by spin-1 bosons, *gluons*, with C = $-$, and if it is valid to evaluate gluon effects in a perturbation series similar to that in QED, then the ψ, ortho-charmonium, should be longer-lived than the η_c, para-charmonium. Next week, we will develop a theory of strong interactions called Quantum Chromodynamics (QCD) with these properties. In QCD,

$$\Gamma({}^1S_0 \to 2g) = \frac{8\pi}{3}\alpha_s^2 \frac{|\psi(0)|^2}{m_q^2}$$

$$\Gamma({}^3S_1 \to 3g) = \frac{40}{81}(\pi^2-9)\alpha_s^3 \frac{|\psi(0)|^2}{m_q^2}$$

where $\alpha_s = g_s^2/4\pi$ measures the strength of the coupling. Very roughly, a value $\alpha_s \sim 0.1$ is needed to explain the long lifetime of the ψ relative to the η_c. Another point of comparison is given by comparing the above formula for the ψ decay rate to gluons to the formula

$$\Gamma({}^3S_1 \to e^+e^-) = 4\pi\alpha^2 Q_q^2 \frac{|\psi(0)|^2}{m_q^2}$$

derived earlier in this lecture. The value of $\Gamma(\psi \to e^+e^-)$, actually measured from the production cross section, is 5.6 keV. The relation

$$\frac{\Gamma({}^3S_1 \to 3g)}{\Gamma({}^3S_1 \to e^+e^-)} = \frac{5}{18}\left(\frac{\pi^2-9}{\pi}\right)\frac{\alpha_s^3}{\alpha^2}$$

gives the estimate $\alpha_s \sim 0.2$. I caution you that these estimates of α_s are based on the leading terms of perturbation series for which the next terms are comparable to the leading terms. I will discuss more accurate determinations of α_s next week. Nevertheless, we can see that the qualitative featues of quarkonium decays are explained by the idea that quarks coupled to the strong interactions through spin-1 particles that are weakly coupled for processes that occur at short distances or involve large momentum transfer.

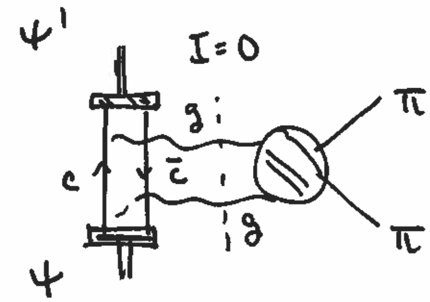
There is one more important type of quarkonium decay, the hadronic transition from higher to lower $q\overline{q}$ states. The most prominent of these is the decay

$$\psi' \to \psi + 2\pi$$

for which a typical event is shown in Figs. p. 11. The rate of this process is again highly suppressed: $\Gamma(\psi' \to \psi\pi\pi) = 160$ keV, about half of the ψ' total width. The relative rates for the two possible 2-pion states are

$$BR(\psi' \to \psi\pi^0\pi^0)\ /\ BR(\psi' \to \psi\pi^+\pi^-) = 0.5$$

indicating that the 2-pion system has $I = 0$. Models of this process also involve weakly coupled gluons that coupled to the small charge dipole of the $c\overline{c}$ bound state,



Eventually, as we ascend the spectrum of $c\overline{c}$ and $b\overline{b}$ states, we reach the point where these states can decay into pairs of mesons containing one heavy and one light quark. For c, the lightest such mesons are

$$D^+ D^0 \qquad c\bar{d},\ c\bar{u} \qquad 1870.\ /\ 1864.\ \text{MeV}$$

so the threshold is at 3728 MeV. For b, the lightest mesons are

$$B^+ B^0 \qquad u\bar{b},\ d\bar{b} \qquad 5279\ \text{MeV}$$

so the threshold is at 10558 MeV. In the Υ system, the 4S state is at 10579 MeV, just above the $B\overline{B}$ threshold. The widths of the the triplet S states are

1S	2S	3S	4S	5S	6S
54 keV	32 keV	20 keV	21 MeV	110 MeV	79 MeV

reflecting the opening of the $\Upsilon \to B\overline{B}$ decay channel. The 4S resonance is narrow, providing a large enhancement of the e^+e^- cross section, and still decays dominantly to $B\overline{B}$, thus, this is the place to go to make high-statistics studies of B meson decays.

In the ψ system, the 1D state is at about 3800 MeV, just above the $D\overline{D}$ threshold. This state mixes with the S states through tensor forces, relativistic corrections of the form

$$\left[(\vec{S}\cdot\vec{r})^2 - S^2 \right]$$

Thus, the 1^3D_1, called the ψ'', at 3773 MeV, can be produced as a resonance in e^+e^- annihilation and decays dominantly to $D\overline{D}$. This is the ideal place to go to study the D mesons.