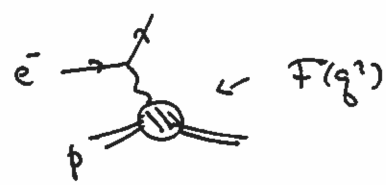
Dec.4

Deep Inelastic Scattering

In this lecture, we continue our study of the quark structure of the proton. One way to learn about this structure is to scatter electrons from a proton and try to interpret the resulting cross section.

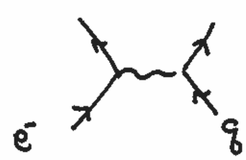
Electron scattering from a proton was studied in the 1950's by Hofstadter. He discovered that the elastic e^-p cross section falls off very quickly as the momentum transfer becomes large, with a form factor $F(q^2) \sim q^{-4}$.



For hard scattering, with momentum transfer of the order of 1 GeV and above, the dominant processes are inelastic scattering reactions, mainly $e^-p \to e^-p + n\pi$. At first, the hadronic final states can be understood as nucleon resonances such as the Δ. Even these disappear for very hard scattering.

What happens then? I would like to analyze this process using the very naive picture that the proton is a soft bag containing quarks and gluons. The electron does not see the gluons, but it can scatter electromagnetically from the quarks. In this QED reaction, the dominant process is elastic electron-quark scattering. We should obtain the QED cross section for that process; then we can compute the e^-p cross section in our simple model and compare it to experiment.

Electron-quark scattering proceeds through the diagram

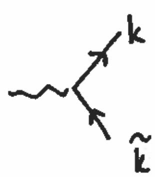


This diagram is very similar to the diagram for $e^+e^- \to q\overline{q}$, and we could easily compute it directly. However, I would like to note that the value of this diagram can be obtained more simply by recognizing that the two processes $e^-q \to e^-q$ and $e^+e^- \to q\overline{q}$ are in a certain way identical. They are related by changing the final e^- in the first process to an initial-state e^+ and an initial q in the first process to a final-state $\overline{q}$. Since the same operator creates a final e^- as destroys an initial e^+, and the same operator destroys an initial q as creates a final $\overline{q}$, the amplitudes for these two processes are derived from the same operator matrix elements. Processes connected in this way are said to be related by *crossing*.

To see how crossing works explicitly, consider the quark current matrix elements in the two processes. In $e^+e^- \to q\overline{q}$, the quark current matrix element, squared and summed over polarizations, gives

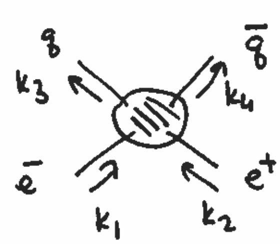
$$|\mathcal{M}|^2 \sim \mathrm{tr}\left[(\not{k}+m)\gamma^\mu(\not{\bar{k}}-m)\gamma^\nu\right]$$

In $e^-q \to e^-q$, the quark current gives

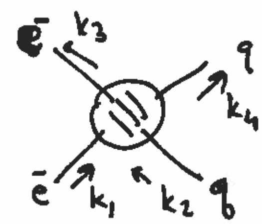


$$|\mathcal{M}|^2 \sim \mathrm{tr}\left[(\not{k}+m)\gamma^\mu(\tilde{\not{k}}+m)\gamma^\nu\right]$$

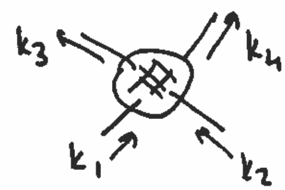
These expressions are identical after the substitution $\overline{k} = -\tilde{k}$ and multiplication by (-1). The substitution replaces a vector with a positive time component (for the final state) with one with a negative time component (for the intial state). The general rule is that we can obtain the amplitude for any process from that of a crossed process by substitution of the momentum vectors, adding an overall factor (-1) in the squared amplitude for each crossed fermion. In particular, we can obtain the matrix elements for $e^-q \to e^-q$ from those for $e^+e^- \to q\overline{q}$ by the substitutions



$$\begin{aligned} k_4 &\to -k_2 \\ k_2 &\to -k_3 \\ &\longrightarrow \\ k_3 &\to k_4 \end{aligned}$$



Crossing relations can be implemented very simply for $2 \to 2$ scattering processes by considering the basic Lorentz invariant combinations of momenta. In the process



define the *Mandelstam variables*

$$s = (k_1+k_2)^2 = (k_3+k_4)^2$$

$$t = (k_3-k_1)^2 = (k_4-k_2)^2$$

$$u = (k_4-k_1)^2 = (k_3-k_2)^2$$

I have already been using this definition of s. The two definitions of each invariant are equal by momentum conservation, $k_1 + k_2 = k_3 + k_4$.

Adding the six formulae above gives a useful identity

$$2(s+t+u) = 3(k_1^2+k_2^2+k_3^2+k_4^2) + 2[k_1\cdot k_2+k_3+k_4-k_3\cdot k_1-k_3\cdot k_2-k_4\cdot k_1-k_4\cdot k$$

$$= (k_1+k_2-k_3-k_4)^2 + 2(k_1^2+k_2^2+k_3^2+k_4^2)$$

$$= 0 + 2(m_1^2+m_2^2+m_3^2+m_4^2)$$

Thus

$$s+t+u = \sum_i m_i^2$$

For all massless particles in the center of mass frame,

$$k_1 = (E, 0, 0, E) \qquad k_2 = (E, 0, 0, -E)$$

$$k_3 = (E, E\sin\theta, 0, E\cos\theta) \qquad k_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

Then the invariants s, t, u are

$$s = 4E^2$$

$$t = -2E^2(1+\cos\theta)$$

$$u = -2E^2(1-\cos\theta)$$

These explicitly satisfy $s+t+u=0$.

In terms of s, t, u, the substitution that relates the crossed reactions $e^+e^- \to q\overline{q}$ and $e^-q \to e^-q$ becomes simply

$$s \to t \qquad t \to u \qquad u \to s$$

In the discussion above, I have derived the crossing relation for the squares of matrix elements summed over polarizations. Crossing relations are not necessarily valid for individual helicity amplitudes. However, for massless particles, where the helicity states correspond to well-defined representations of the Lorentz group, the crossing relations do hold, with an amplitude with a final e^+_R, for example, being related to the amplitude with an initial e^-_L.

We can now very simply obtain the helicity amplitudes for $e^-q \to e^-q$. We need only recall the helicity amplitudes computed on Wednesday and apply crossing. Here are the formulae we derived for $e^+e^- \to q\overline{q}$:

$$i\mathcal{M}(e^-_L e^+_R \to q_L \overline{q}_R) = ie^2 Q_f (1+\cos\theta) = -2ie^2 Q_f \frac{u}{s}$$

$$i\mathcal{M}(e^-_L e^+_R \to q_R \overline{q}_L) = -ie^2 Q_f (1-\cos\theta) = +2ie^2 Q_f \frac{t}{s}$$

Then, by crossing,

$$i\mathcal{M}(e^-_L q_L \to e^-_L q_L) = -2ie^2 Q_f \frac{s}{t}$$

$$i\mathcal{M}(e^-_L q_R \to e^-_L q_R) = +2ie^2 Q_f \frac{u}{t}$$

By parity, the same formulae apply when all helicities are reversed. Processes in which the helicity of the e^- or the q flips from the initial to the final state are forbidden in the zero mass limit.

The polarized differential cross section for $e^-_L q_L \to e^-_L q_L$ is then given by

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s} \pi\alpha^2 \cdot 4Q_f^2 \cdot \frac{s^2}{t^2}$$

For $e^-_L q_R$, we replace s^2 by u^2 in the last term. The cross section for unpolarized initial states is

$$\frac{d\sigma}{d\cos\theta}(e^- q \to e^- q) = \frac{\pi\alpha^2 Q_f^2}{s} \cdot \frac{s^2+u^2}{t^2}$$

It will be convenient to modify this formula in one more way. The differential cross section is expressed in terms of the scattering angle in the center of mass frame. In this frame

$$dt = \tfrac{1}{2}s \; d\cos\theta$$

Then we can recast the formula as

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha^2}{s^2} Q_f^2 \; \frac{s^2+u^2}{t^2}$$

and it will be valid in any frame.

We can now embed this calculation in our simple model of proton structure. To set up the model, I would like to think of the proton as colliding with the electron at high energy. The proton has a total momentum P. I will model the proton as a collection of quarks, antiquarks, and gluons, each of which moves approximately collinearly with the proton and carries a fraction ξ of its momentum

$$p = \xi P$$

I will assume that we are considering such high energies and momentum transfers that we can ignore the masses of the quarks and also that of the proton. The probability to find a quark of flavor f at momentum fraction ξ will be

$$d\xi \; f_f(\xi)$$

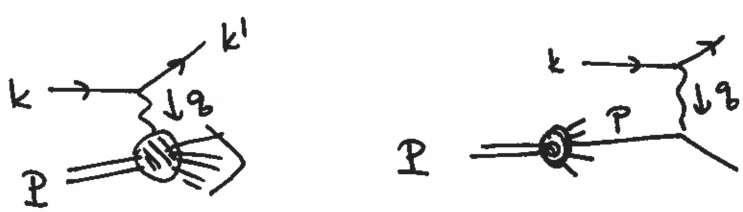
This model is called the *parton model.* It was formalized by Feynman. The constituent quarks, antiquarks, and gluons in the proton are called *partons.* The functions $f_f(\xi)$ are called *parton distribution functions.* They are derived from the square of the proton wavefunction, that is, they depend on the aspects of the strong interactions that involve strong coupling and quark confinement. However, in this model, we assume that the partons do not have large momenta transverse to the direction of the proton, as might result from a strong-interaction hard-scattering process.

In the parton model, the cross section for e^-p scattering is given by

$$\int d\sigma = \sum_f \int d\xi \; f_f(\xi) \int dt \; \frac{2\pi\alpha^2}{\hat{s}^2} Q_f^2 \; \frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$$

where $\hat{s}$, $\hat{t}$, $\hat{u}$ are the invariants for the e^-q reaction. We now need to simplify this formula so that it can be compared to experimental data. The original experiments on very hard e^-p scattering were done at SLAC. In these experiments, only the momentum of the scattered electron was measured; the final hadronic state was ignored. So, let us consider how much information we can obtain from the electron side only.

Here is the kinematics of the full e^-p scattering process and its parton model representation:



The 4-momentum measured for the final e^- is k'. The 4-momentum transfered to the proton can then be determined: $q = k - k'$. From q and the initial-state vectors, we can construct the following invariants

$$Q^2 = -q^2 > 0$$

$$x = \frac{Q^2}{2P\cdot q} \qquad y = \frac{2P\cdot q}{2P\cdot k}$$

In the lab frame where the proton is at rest, $P = (m_p, \vec{0})$. Then, in this frame, $y = q^0/k^0$, the fraction of the original energy of the electron that is transfered to the proton. Thus, $0 < y < 1$. We will see in a moment that also $0 < x < 1$. Let $s = (P+k)^2 = 2P \cdot k$ be the square of the total center of mass energy for the $e^- p$ reaction. Then $Q^2 = xys$. The full kinematics of the inelastic electron scattering can be described by the dimensionless variables x and y.

We now need to find the values of $\hat{s}$, $\hat{t}$, $\hat{u}$ in terms of experimental observables. This can be done using a beautiful bit kinematics due to Feynman. In the $e^- q$ scattering process

$$p = \xi P$$

$$p + q = \xi P + q$$

the final quark must be on its mass shell. Thus

$$0 \cong (p+q)^2 = \underbrace{p^2}_{\approx 0} + 2p\cdot q + q^2 = 2\xi P\cdot q - Q^2$$

and so

$$\xi = \frac{Q^2}{2P\cdot q} = x$$

When we measure x from the scattered electron, we are picking out the momentum fraction ξ of the quark that was scattered in that event.

Now $$\hat{s} = (p+k)^2 = 2p\cdot k = 2\xi P\cdot k = \xi \cdot s = xs$$

$$\hat{t} = -Q^2 \qquad \hat{u} = -\hat{s} - \hat{t} = -xs + xys = -xs(1-y)$$

In the expression for the cross section, $d\xi$ can be replaced by dx and dt can be replaced by $dQ^2 = d(xys) = dy \cdot xs$. Then the parton model formula for the e^-p cross section becomes

$$\int d\sigma = \sum_f \int dx\, f_f(x) \int dy\, xs \frac{2\pi\alpha^2}{(xs)^2} Q_f^2 \frac{(xs)^2(1+(1-y)^2)}{(Q^2)^2}$$

Simplifying, and adding the possible contribution of antiquark partons in the proton, we arrive at the formula

$$\frac{d\sigma}{dx\,dy} = \left[\sum_f x Q_f^2 [f_f(x) + f_{\bar{f}}(x)]\right] \cdot \frac{2\pi\alpha^2 s}{Q^4} [1+(1-y)^2]$$

This expression shows a very beautiful factorization. The expression in brackets is called $F_2(x, Q^2)$. However, we see that, in the parton model, this quantity is independent of Q^2 and is a function of x only. This behavior is called *Bjorken scaling.* It was derived by Bjorken from a more abstruse theory before the formulation of the parton model.

How well does this theory work? The cross sections for very large momentum transfer e^-p scattering—*deep inelastic scattering*—were measured at SLAC in the 1960's. Figs p. 2 shows a photograph of the apparatus. A liquid Hydrogen target, off to the left, was exposed to the 17 GeV electron beam. The blue magnets form a spectrometer for the scattered electron. The orange box to the right is an electromagnetic calorimeter that was used to distinguish e^-s from π^-s. The spectrometer is mounted on railroad tracks so that it can be rotated to different angles.

Some of the data is shown in Figs p. 3. Large Q^2 corresponds to higher energy and higher angle for the scattered electron. In the lower energy and angle regions, we see distinct nucleon resonances, but at high Q^2 these smear out into a smooth differential cross section.

Where the data is smooth ($Q^2 > 1$ GeV^2), construct

$$F_2 = \frac{[d\sigma/dx\,dy]}{\frac{2\pi\alpha^2 s}{Q^4}\left(1+(1-y)^2\right)}$$

The result is shown in Figs p 4. The data obeys Bjorken scaling! The form of $F_2(x)$ gives the distribution of quark momenta in the proton. If the proton contained only 3 quarks and some gluons, we might expect that this function would peak at $x \sim 0.2 - 0.3$. In fact, it shows many extra quarks and antiquarks at small fractions x. Feynman called these *wee partons.*

Figs p. 5 shows a modern determination of $F_2(x)$ measured at the e^-p collider HERA at DESY in Hamburg, Germany. You can see that the density of low-momentum quarks and antiquarks in the proton continues to increase at small x. Thus, we should best think of the proton as containing a net number of 3 quarks, plus many $q\bar{q}$ pairs. The quantum numbers of the proton are expressed as sum rules for the parton distribution functions

$$\int_0^1 dx\,\left(f_u(x) - f_{\bar{u}}(x)\right) = 2 \qquad \int_0^1 dx\,\left(f_d(x) - f_{\bar{d}}(x)\right) = 1$$

In addition, the statement that the partons, together, carry the full momentum of the proton is expressed by the sum rule

$$\int dx\; x\left(f_u + f_d + \cdots + f_{\bar{u}} + f_{\bar{d}} + \cdots + f_g\right) = 1$$

From the distribution functions for quarks, we learn that the quarks carry only about half of the total momentum of the proton; the gluons must account for the rest.

I will discuss the determination of parton distribution functions in more detail next week. I will also reveal that the parton distributions are not actually independent of Q^2. Instead, they evolve slowly as a function of $\log Q^2$. We will see that this effect is predicted by QCD and is confirmed by more detailed experiments.

Finally, we should ask what the hadronic final states look like in deep inelastic scattering. It was necessary to go to much higher energies than those of the SLAC experiments before this became clear. However, the experiments at HERA clearly show that the struck quark deep inelastic scattering materializes as a hadronic jet very similar to those in $e^+e^- \rightarrow q\bar{q}$. An example of a HERA event, from the H1 detector, is shown in Figs. p.6. The single track is the scattered electron, which is seen to make a shower in the electromagnetic calorimeter. The cluster of tracks going downward is the jet.

We have now seen several examples of the rule that the strong interactions, however strongly they might pull on quarks at large distances, act weakly at short distances and, equivalently, do not transfer large momenta between quarks. In the next lecture, I will explain how this odd behavior can make sense.