Dec.7

Asymptotic Freedom

In the previous lectures, we found that experiment gives us a paradoxical description of the strong interactions. These interactions are very strong at large distances, since they permanently confine quarks into hadrons, but they are apparently weak at small distances. We saw that quarks in jets and in hadrons do not scatter one another through large momentum transfers, and that a picture with weakly coupled vector bosons is needed to explain the narrow widths of the ψ and Υ.

In quantum field theory, there is an idea that we could use to explain this paradox. Quantum field theory predicts that coupling constants evolve with distance scale, so that a coupling can be strong at one scale and weak at another. In a renormalizable quantum field theory, the evolution of a coupling constant is described by the renormalization group equation

$$\frac{dg}{d\log Q} = \beta(g)$$

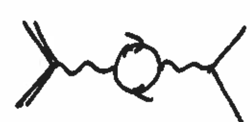
If $\beta(g) > 0$, the coupling constant increases toward the UV. If $\beta(g) < 0$, the coupling constant increases toward the IR and becomes weak in the UV. This latter behavior is the one that we need to explain the properties of the strong interactions. So, in this lecture I will investigate what kind of theory has a negative $\beta(g)$.

To begin, we will compute $\beta(g)$ in QED. We will then generalize this result to several other related theories.

It is relatively easy to analyze coupling constant evolution in QED. The exchange of a virtual photon generates the Coulomb potential

$$V(\vec{q}) = -\frac{e^2}{|\vec{q}|^2}$$

In the next order of perturbation theory, this interaction is modified by radiative corrections from the diagrams

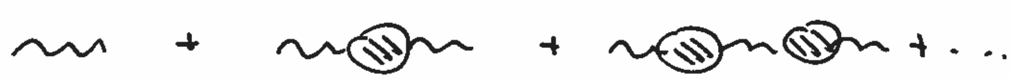


involving self-energy, vertex, and photon propagator corrections. In QED, the modification of the Coulomb potential due to the first two classes of diagrams cancels. This cancellation is part of the mechanism that insures that the electric charge is not modified differently for different particles. Experimentally, we know that the electron and proton charges obey

$$\frac{Q(e^-)}{Q(p)} = -1$$

to the accuracy of 10^{-21}. So it is important that effects on the Coulomb potential that depend on the particular structure of the electron or the proton should cancel. Technically, this cancellation is enforced by the *Ward identity* of QED.

The radiative corrections to the Coulomb potential therefore come only from the diagrams



where

is the set of one-particle-irreducible *vacuum polarization* diagrams I will notate

$$= i\Pi^{\mu\nu}(q)$$

This object is a correlation function of two electromagnetic currents $j^\mu = \overline{\psi}\gamma^\mu\psi$. Since the electromagnetic current is conserved, $\partial_\mu j^\mu = 0$, we must have $q_\mu \Pi^{\mu\nu} = 0$. By this relation and Lorentz invariance, the vacuum polarization must have the structure

$$i\Pi^{\mu\nu}(q) = i(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$$

Note that this structure contains $P^{\mu\nu} = (g^{\mu\nu} - q^\mu q^\nu / q^2)$, which is a projection operator. Using that fact, we can rewrite the series for the radiatively corrected Coulomb potential as

$$-\frac{ie^2}{q^2} g^{\mu\nu} \quad + \quad \frac{-ig^{\mu\lambda}}{q^2} e \left(i P^{\lambda\sigma} q^2 \Pi(q^2) \right) \left(\frac{-ig^{\sigma\nu} e}{q^2} \right) + \cdots$$

$$= \quad -\frac{i e^2}{q^2} \; \frac{1}{1-\Pi(q^2)} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \quad + \quad \frac{q^\mu q^\nu}{q^2} \cdot (\cdots)$$

By current conservation, the terms with $q^\mu q^\nu$ vanish when they are contracted with external electromagnetic currents. We can then identify the effective Coulomb potential as

$$V(\vec{q}) = - \frac{e^2}{1-\Pi(q^2)} \; \frac{1}{|\vec{q}|^2}$$

so that the electric charges as a function of the momentum transfer q is given by

$$e^2(q^2) = \frac{e^2}{1-\Pi(q^2)}$$

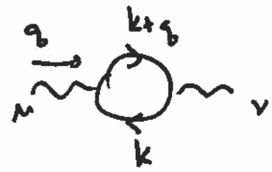
At large distances, $q \to 0$, the electric charge is

$$e^2(0) = \frac{e^2}{1-\Pi(0)}$$

This is the electric charge that is actually measured in macroscopic experiments. It may differ from the *bare charge* e^2 that is the expansion parameter in perturbation theory. In fact, it can be very different, since the quantity $\Pi(0)$, when computed directly, is logarithmically ultraviolet divergent. In renormalized perturbation theory, we subtract a constant from $\Pi(q^2)$ so that $\Pi(0) = 0$. Then the expansion parameter of QED perturbation theory is exactly the macroscopic value of the electric charge. The q^2 dependence of $\Pi(q^2)$ is still physical and causes the electric charge to evolve with the momentum scale.

We can work out the evolution of $e^2(q^2)$ by explicitly evaluating $\Pi(q^2)$ in perturbation theory. In lowest order

$$i\Pi^{\mu\nu} =$$



$$= (-ie)^2(-1)\int\frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{tr}\left[i(\not k+\not q+m)\gamma^\mu\, i(\not k+m)\,\gamma^\nu\right]}{\left((k+q)^2-m^2\right)\left(k^2-m^2\right)}$$

To compute this integral, we need some tricks. First, use Feynman's identity

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{\left[xA+(1-x)B\right]^2}$$

to combine the denominators,

$$\frac{1}{\left((k+q)^2-m^2\right)\left(k^2-m^2\right)} = \int_0^1 dx\,\frac{1}{\left[x\left((k+q)^2-m^2\right)+(1-x)\left(k^2-m^2\right)\right]^2}$$

$$= \int_0^1 dx\,\frac{1}{\left[\mathbf{k}^2+x(1-x)q^2-m^2\right]^2}$$

where

$$\mathbf{k} = k+xq \qquad k = \mathbf{k}-xq \qquad k+q = \mathbf{k}+(1-x)q$$

We can then shift the integration variable from k to $\mathbf{k}$, for which the denominator is symmetric. The numerator then becomes

$$\mathrm{tr}\,(\not k + \not q + m)\gamma^\mu(\not k + m)\gamma^\nu$$

$$= 4\left[(k+q)^\mu k^\nu + (k+q)^\nu k^\mu - g^{\mu\nu}(k+q)\cdot k + g^{\mu\nu}m^2\right]$$

$$= 4\left[2\,\mathbb{k}^\mu \mathbb{k}^\nu - g^{\mu\nu}\mathbb{k}^2 + g^{\mu\nu}[m^2 + x(1-x)q^2] - 2x(1-x)q^\mu q^\nu\right]$$

In the last line, I have discard terms linear in $\mathbf{k}$, which must integrate to zero. Then, finally,

$$i\Pi^{\mu\nu} = -8e^2\int_0^1 dx \int\frac{d^4\mathbb{k}}{(2\pi)^4}\;\frac{\mathbb{k}^\mu\mathbb{k}^\nu - \frac{1}{2}g^{\mu\nu}(\mathbb{k}^2 - m^2 - x(1-x)q^2) - x(1-x)q^\mu q^\nu}{[\mathbb{k}^2 + x(1-x)q^2 - m^2]^2}$$

The integral that is the coefficient of the $q^\mu q^\nu$ term is most straightforward to evaluate. We need

$$\int\frac{d^4\mathbb{k}}{(2\pi)^4}\;\frac{1}{[\mathbb{k}^2 + x(1-x)q^2 - m^2]^2}$$

We first Wick rotate $\mathbf{k}$ to Euclidean space, $\mathbf{k}^0 = i\mathbf{k}_E^0$ and substitute $\mathbf{k}^2 = -\mathbf{k}_E^2$. The integral is now rotationally invariant in 4-dimensional Euclidean space. The area of a unit sphere in 4 dimensions is $2\pi^2$, so

$$d^4\mathbb{k}_E = 2\pi^2\,\mathbb{k}_E^3\,d\mathbb{k}_E = \pi^2\,\mathbb{k}_E^2\,d\mathbb{k}_E^2$$

Then the integral becomes

$$i\int\frac{d\mathbb{k}_E^2\,\mathbb{k}_E^2\,\pi^2}{16\pi^4}\;\frac{1}{[\mathbb{k}_E^2 + x(1-x)(-q^2) + m^2]^2}$$

$$= \frac{i}{16\pi^2}\left[\log\left(\frac{\Lambda^2}{a(-q^2)+bm^2}\right) + \text{finite}\right]$$

The integral is logarithmically divergent, and I have chosen to evaluate it by simply cutting off the integration at $\mathbf{k}_E = \Lambda$. By dimensional analysis, the denominator inside the logarithm is some combination of q^2 and m^2.

Now we must evaluate the other terms in the vacuum polarization integral. By Lorentz invariance, all of these terms are proportional to $g^{\mu\nu}$. The integrals are quadratically ultraviolet divergent. This is a serious problem. For a quadratically divergent integral, even the boundary terms are linearly divergent. This boundary terms invalidate the shift from k to $\mathbf{k}$ and the Wick rotation. For these and other reasons, the value of the integrals is highly ambiguous. Fortunately, there is a required correct answer. The complete contribution to $\Pi^{\mu\nu}$ must be of the general form given above as a consequence of current conservation. If the quadratically divergent integral gives any other answer, then we have used a cutoff that does not respect the conservation of the electromagnetic current, and this is physically incorrect. Thus, we obtain for the value of the complete diagram

$$i\Pi^{\mu\nu} = -8ie^2 \frac{1}{16\pi^2}\int_0^1 dx\; x(1-x)\,(q^2 g^{\mu\nu} - q^\mu q^\nu)\;\log\left(\frac{\Lambda^2}{a(-q^2)+bm^2}\right)$$

Finally, to work in terms of the physical coupling constant measured at large distances, we subtract the value of $\Pi(q^2)$ at $q^2 = 0$. Then, finally,

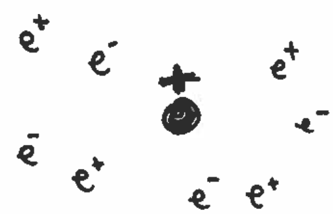
$$i\Pi^{\mu\nu} = i\frac{e^2}{12\pi^2}\,(g^{\mu\nu}q^2 - q^\mu q^\nu)\left(\log\frac{(-q^2)}{m^2} + \cdots\right)$$

in the limit $q^2 \gg m^2$. You can find a very careful computation of this diagram, one that uses a UV regulator that respects current conservation and keeps all finite terms in the final answer, in Section 7.5 of Peskin and Schroeder.

From this expression for $\Pi^{\mu\nu}$, we infer the following behavior of the QED coupling

$$e^2(Q^2) = \frac{e^2(0)}{1 - \frac{e^2(0)}{12\pi^2} \log Q^2/m^2}$$

where I have put $-q^2 = Q^2$ for spacelike momentum transfer. The value of $e^2(q^2)$ increases as Q becomes larger This is in accord with the physical picture of vacuum polarization. We start with a large electric charge at small distances. Electron-positron pairs can arise in the vacuum as quantum fluctuations. These polarize in the field of the charge and give the vacuum an effective dielectric constant. The field observed at large distances is thus smaller than the original charge, so that the strength of the Coulomb interaction decreases at large distances.



To compute the β function, differentiate

$$\frac{de}{d\log Q} = \frac{1}{2e}\frac{de^2}{d\log Q} = \frac{1}{2e(Q)} \frac{1}{\left[1 - \frac{e^2(0)}{12\pi^2}\log Q^2/m^2\right]^2} e^2(0) \cdot \frac{2e^2(0)}{12\pi^2} = \frac{e^3(Q)}{12\pi^2}$$

Then the above equation for $e^2(Q)$ satisfies the renormalization group equation

$$\frac{de}{d\log Q} = \frac{e^3}{12\pi^2} \quad \text{or} \quad \beta(e) = + \frac{e^3}{12\pi^2}$$

In the full theory of Nature, the QED coupling is affected not only by the quantum fluctuations of the electron but also by those of other fermions. For $Q^2 > 2m_\mu^2$, we find an equal contribution to the β function from the muon. As Q increases beyond 1 GeV, there are also contributions from the vacuum polarization of the quarks. In general,

$$\beta(e) = \Big[\sum_f Q_f^2 n_f \Big] \cdot \frac{e^3}{12\pi^2}$$

where f is summed over all charged fermions with $2m < Q$ and $n_f = 1$ for leptons, $n_f = 3$ for quarks. (This factor is the same one that appears in the cross section $\sigma(e^+e^- \to \text{hadrons})$.) At large distances, $\alpha = e^2/4\pi = 1/137$. Integrating the renormalization group equation to higher Q, we find

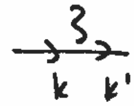
$$\alpha(30\,\text{GeV}) = \frac{1}{130} \qquad\qquad \alpha(91\,\text{GeV}) = \frac{1}{129}.$$

The first of these values is confirmed in Figs p. 2, which shows the comparison of measurements of Bhabha scattering ($e^+e^- \to e^+e^-$) to QED theory. The value of α at 91 GeV will enter the theory of the Z^0 boson that I will present next week.

In preparation for the next step, I would like to repeat the calculation of the β function for the quantum electrodynamics of a scalar field. The Lagrangian is

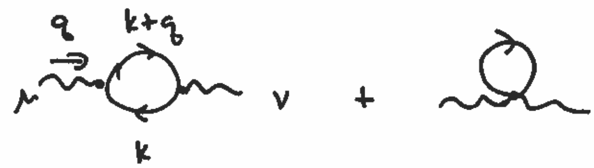
$$\mathcal{L} = D_\lambda\phi^* D^\lambda\phi - m^2\phi^*\phi = (\partial_\lambda - ieA_\lambda)\phi^*(\partial^\lambda + ieA^\lambda)\phi - m^2\phi^*\phi$$

The Feynman rules of this theory are

 $\dfrac{i}{k^2-m^2}$  $-ie(k+k')^\mu$  $2ie^2 g^{\mu\nu}$

The lowest-order vacuum polarization is given by the diagrams

$$i\Pi^{\mu\nu} =$$



The second diagram is proportional to $g^{\mu\nu}$. The first diagram is

$$(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i(2k+q)^\mu \; i(2k+q)^\nu}{((k+q)^2-m^2)(k^2-m^2)}$$

To evaluate this, use Feynman's trick to combine the denominators, shift the integral in the same way, and use $2k+q = 2\mathbf{k} + (1-2x)q$ to extract the part proportional to $q^\mu q^\nu$. The coefficient of this structure can be evaluated just as above, and the $g^{\mu\nu}$ term is then determined from current conservation. The result is

$$e^2 \frac{i}{16\pi^2} \int_0^1 dx \; (1-2x)^2 \left[\log\left(\frac{\Lambda^2}{a(q^2)+bm^2}\right) + \ldots\right] (q^\mu q^\nu - q^2 g^{\mu\nu})$$

Subtracting at $q^2 = 0$, we obtain

$$i\Pi^{\mu\nu} = i(q^2 g^{\mu\nu} - q^\mu q^\nu) \frac{e^2}{48\pi^2} \log\left(\frac{-q^2}{m^2}\right)$$

This answer can be treated in the same way as for the electron case. For scalar QED, then,

$$\beta(e) = + \frac{e^3}{48\pi^2}$$

one quarter of the result for fermions.

So ar, we are not making much progress toward our real goal. We have computed several β functions, but they are all positive. I will now try one more case, Yang-Mills theory.

The Lagrangian for Yang-Mills theory is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \bar{\Psi} i \not{D} \Psi + D_\lambda \phi^* D^\mu \phi$$

where a runs over the generators of a Lie group G, the covariant derivative is given by

$$D_\mu = \partial_\mu + ig A_\mu^a t_r^a$$

and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

The f^{abc} are the *structure constants* of G, defined from the commutation relation of the generators of G,

$$[T^a, T^b] = i f^{abc} T^c$$

The matrices t_r^a give a finite-dimensional representation of the generators of G. For example, $G = SU(N)$ has an N-dimensional representation called the *fundamental* representation. One other particularly important representation is the one to which the generators of the group belong. This is called the *adjoint* representation. In $SU(2)$ it is the 3-dimensional or spin 1 representation. In $SU(3)$, it is the 8-dimensional (octet) representation. In the adjoint representation, the representation matrices are

$$(t_G^b)_{ac} = i f^{abc}$$

A useful relation for the field strength tensor in Yang-Mills theory is that it is derivable from the commutator of covariant derivatives,

$$[D_\mu , D_\nu] = -ig F_{\mu\nu}^a t_r^a$$

You should check that this formula gives the nonlinear term in $F^a_{\mu\nu}$ correctly.

It is much trickier to compute the β function in Yang-Mills theory than in QED. The contributions to β from the vertex and self-energy diagrams do not generally cancel. In fact, they depend on the gauge in which the calculation is done. To avoid this difficult, I will use a special method here. I will check that this method gives the same result for QED as the more straightforward method used earlier in this lecture. For a direct calculation of the β function in Yang-Mills theory by more standard techniques, see Section 16.5 of Peskin and Schroeder.

First, it will be easier to keep track of the evolution of the coupling constant g if we move it to a single place. To do this, rescale the vector field of Yang-Mills theory by $gA^a_\mu \to A^a_\mu$. This removes the factors of g in the covariant derivatives and the field strength and changes the Lagragian to

$$\mathcal{L} = -\frac{1}{4g^2}(F_{\mu\nu}^a)^2 + \bar{\Psi} i \not{D} \Psi + D_\mu \phi^\dagger D^\mu \phi$$

If we now turn on a classical background field $\mathbf{A}^a_\mu$, $\mathbf{F}^a_{\mu\nu}$, the associated action will be

$$\int d^4x \, \mathcal{L} = \int d^4x \left(-\frac{1}{4g^2}(\mathbb{F}_{\mu\nu}^a)^2\right)$$

plus quantum corrections. Note that this is a physical, gauge-invariant expression. If the quantum corrections have the effect of changing the coefficient of the $(\mathbf{F}_{\mu\nu})^2$ term in this action, we can interpret that as a renormalization of the coupling constant g^2. I will now set up a method for calculating the first quantum correction that is completely gauge-invariant with respect to the external field.

First, expand A^a_μ about $\mathbf{A}^a_\mu$.

$$A^a_\mu = \mathbb{A}^a_\mu + g\, a^a_\mu$$

For $F^a_{\mu\nu}$,

$$F^a_{\mu\nu} = \mathbb{F}^a_{\mu\nu} + g\partial_\mu a^a_\nu - g\partial_\nu a^a_\mu + g f^{abc}\mathbb{A}^b_\mu a^c_\nu + g f^{abc} a^b_\mu \mathbb{A}^c_\nu + g^2 f^{abc} a^b_\mu a^c_\nu$$

We can interpret the terms with $f^{abc}A^b_\mu$ as part of the covariant derivative in the adjoint representation. This this expression becomes

$$F^a_{\mu\nu} = \mathbb{F}^a_{\mu\nu} + g(\mathbb{D}_\mu a^a_\nu - \mathbb{D}_\nu a^a_\mu) + g^2 f^{abc} a^b_\mu a^c_\nu$$

When we square this, we can use integration by parts to simplify the expression. In particular, after integration by parts, the terms linear in a^a_μ multiply

$$\mathbb{D}^\mu \mathbb{F}^a_{\mu\nu} = 0$$

by the equation of motion for the classical Yang-Mills field. Then

$$(F^a_{\mu\nu})^2 = (\mathbb{F}^a_{\mu\nu})^2 + 2g^2 a^a_\mu \left[-\mathbb{D}^2 g^{\mu\nu} + \mathbb{D}^\nu \mathbb{D}^\mu\right]_{ac} a^c_\nu + 2\mathbb{F}^a_{\mu\nu} g^2 f^{abc} a^b_\mu a^c_\nu$$

In the middle term, it will be useful to interchange the order of $\mathbf{D}^\nu\mathbf{D}^\mu$. This can be done using the commutator

$$[\mathbb{D}^\nu, \mathbb{D}^\mu] = +i\, \mathbb{F}^{\mu\nu a} t_G^a$$

The result is that we obtain a new term identical to the third term in $(F_{\mu\nu})^2$. In all,

$$-\frac{1}{4g^2}(F_{\mu\nu}^a)^2 = -\frac{1}{4g^2}(\mathbb{F}_{\mu\nu}^a)^2 - \frac{1}{2} a_\mu^a [-\mathbb{D}^2 g^{\mu\nu} + \mathbb{D}^\mu \mathbb{D}^\nu]_{ac} a_\nu^c$$
$$+ \frac{1}{2}\cdot 2 a_\mu^a f^{abc} \mathbb{F}^{\mu\nu b} a_\nu^c$$

Finally, we must gauge-fix this expression in order to integrate over a_μ^a. A useful gauge-fixing condition is

$$\mathbb{D}_\mu a^{\mu a} = 0$$

The easiest prescription is to go to the analogue of Feynman gauge with this gauge-fixing. The the term $\mathbf{D}^\mu\mathbf{D}^\nu$ disappears, and the ghost action also has a simple form. Note that we are fixing the gauge for the fluctuating field a_μ^a only, while keeping full gauge-invariance with respect to the background field. The final action, keeping terms to quadratic order in all fluctuating fields, is

$$\mathcal{L} = -\frac{1}{4g^2}(\mathbb{F}_{\mu\nu}^a)^2 - \frac{1}{2} a_\mu^a \left[-\mathbb{D}^2_{ac} g^{\mu\nu} - 2 f^{abc} \mathbb{F}_{\mu\nu}^b\right] a_\nu^c$$
$$+ \bar{c}^a(-\mathbb{D}^2)_{ac} c^c + \bar{\Psi} i \not{\mathbb{D}} \Psi + \phi^*(-\mathbb{D}^2)\phi$$

Before we compute Feynman diagrams with this action, I would like to do one more rearrangement. If we integrate over one Dirac fermion, we will obtain

$$\left[\det i\not{\mathbb{D}} \right]$$

To make the calculation more symmetrical between the fermions and the bosons, I would like to rewrite this as

$$\left[\det (i\not{\mathbb{D}})^2 \right]^{\frac{1}{2}}$$

where

$$(i\not{\mathbb{D}})^2 = -\gamma^\mu \mathbb{D}_\mu \gamma^\nu \mathbb{D}_\nu$$

$$= -\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\mathbb{D}_\mu \mathbb{D}_\nu - \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]\mathbb{D}_\mu \mathbb{D}_\nu$$

and

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \qquad S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$$

$S^{\mu\nu}$ is the matrix representation of the angular momentum algebra on a Dirac spinor. $S^{\mu\nu}$ is antisymmetric, so we can replace the covariant derivatives in this term by their commutator and obtain

$$(i\not{\mathbb{D}})^2 = -\mathbb{D}^2 - 2i\, S^{\mu\nu} \cdot \tfrac{1}{2} i \mathbb{F}^b_{\mu\nu} t^b_r$$

$$= -\mathbb{D}^2 + 2 \cdot \tfrac{1}{2} \mathbb{F}^b_{\mu\nu} S^{\mu\nu} t^b_r$$

This structure is reminiscent of the structure of the gauge action. We can make the connection explicit by noting that the representation of angular momentum on a vector representation is

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta\right)$$

The the vector action can be rewritten as

$$-\frac{1}{2} A^a_\mu \left[-\mathbb{D}^2_{ac}\, g^{\mu\nu} + 2\cdot\frac{1}{2}\, \mathbb{F}^b_{\mu\nu}\, g^{\mu\nu} (t^b_G)_{ac}\right] A^c_\nu$$

again interpreting if^{abc} as the representation matrix of G in the adjoint representation.

The final form of the action for the fluctuating fields is

$$\mathcal{L} = -\frac{1}{4}(\mathbb{F}^a_{\mu\nu})^2 - \frac{1}{2} A^a_\mu\, \Delta^{\mu\nu}_{G,1\,ac} A^c_\nu + \bar{c}^a\, \Delta_{G,0\,ac}\, c^c$$

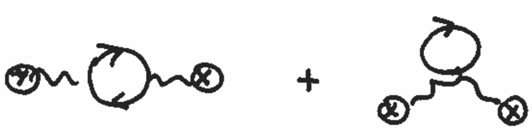
where

$$+ \bar{\psi}\, \Delta_{r,\frac{1}{2}}\, \psi + \phi^* \Delta_{r,0}\, \phi$$

$$\Delta_{rJ} = -\mathbb{D}^2 + 2\,\frac{1}{2}\, \mathbb{F}^b_{\mu\nu}\, g^{\mu\nu}\, t^b_r$$

in which r denotes the G and J denotes the angular momentum representation. We must also remember to integrate over half of the true number of fermions to compensate for using the square of $i\not{D}$.

Now we can compute Feynman diagrams with this action. I would like to work out, in particular the terms that involve two powers of the background field $\mathbf{A}^a_\mu$. First, compute the diagrams involving the scalar field. These are



The value of these diagrams is just that of the scalar field vacuum polarization computed above, with the two factors of $\mathbf{A}^a_\mu t^a_r$ attached, and with an extra factor $\frac{1}{2}$ from the symmetry factor because these two external points are not distinguished. The value of the diagrams is then

$$\frac{1}{2}\int\frac{d^4q}{(2\pi)^4} A^a_\lambda(-q) A^b_\nu(q) \cdot i(q^2g^{\mu\nu} - q^\mu q^\nu) \frac{1}{48\pi^2} \operatorname{tr}[t^a_r t^b_r] \cdot \log(-q^2)$$

$$= \quad + \frac{i}{4}\int\frac{d^4q}{(2\pi)^4} F^a_{\mu\nu}(-q) F^{\mu\nu a}(q) \quad \frac{C(r)}{48\pi^2} \log(-q^2)$$

The group theory factor can be evaluated by defining $C(r)$ by

$$\operatorname{tr}[t^a_r t^b_r] = C(r)\,\delta^{ab}$$

Taking this correction into the exponential, we see that it generates a shift

$$\frac{1}{4g^2} \to \frac{1}{4g^2}\left(1 - \frac{C(r)g^2}{48\pi^2}\log(-q^2)\right)$$

This replaces the fixed coupling constant g^2 by

$$g^2(q^2) = \frac{g^2}{1 - \frac{C(r)g^2}{48\pi^2}\log(-q^2)}$$

Setting $C(r) = 1$, $g = e$, this exactly the result for the q^2-dependence of the coupling that we obtained in scalar QED.

Next, evaluate the diagrams with fermions. In our original notation, the fermions would give simply the QED vacuum polarization diagram. In our new notation, the fermions contribute three diagrams

The last of these involves two vertices from the $\mathbf{F}_{\mu\nu}S^{\mu\nu}$ interaction. The value of this diagram (for one complex scalar in the loop) is

$$\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\,\mathbb{F}^a_{\mu\nu}(-q)\,\mathbb{F}^b_{\lambda\sigma}(q)\;\frac{i}{16\pi^2}\cdot tr[t^a_r t^b_r]\; tr[\mathcal{J}^{\mu\nu}\mathcal{J}^{\lambda\sigma}]\left(-\log(-q^2)\right)$$

$$= \frac{i}{4}\int\frac{d^4q}{(2\pi)^4}\,\mathbb{F}^a_{\mu\nu}(-q)\,\mathbb{F}^{\mu\nu a}(q)\cdot\left(-\frac{4C(r)C(J)}{16\pi^2}\right)\log(-q^2)$$

where $C(J)$ is defined by

$$tr[\mathcal{J}^{\mu\nu}\mathcal{J}^{\lambda\sigma}] = [g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda}]\;C(J)$$

The value of $C(J)$ is

$$C(J) = \begin{cases} 0 & \text{scalar} \\ 1 & \text{Dirac spinor} \\ 2 & \text{4-vector} \end{cases}$$

assembling the pieces, we find that the fermion diagrams contribute

$$\frac{i}{4}\int\frac{d^4q}{(2\pi)^4}\,\mathbb{F}^c_{\mu\nu}(-q)\,\mathbb{F}^{\mu\nu a}(q)\cdot\frac{1}{2}\cdot(-1)\left[\frac{4}{48\pi^2} - \frac{4C(J)}{16\pi^2}\right]C(r)\;\log(-q^2)$$

where I have included a factor (-1) for the fermion loop, a factor of 4 for the first two diagrams for 4 components of the Dirac field, and a factor $\frac{1}{2}$ to compensate the fact that we have squared the Dirac operator. Reducing the coefficient, we have

$$\frac{i}{4}\int\frac{d^4q}{(2\pi)^4}\, F^a_{\mu\nu}(-q)F^{\mu\nu a}(q)\cdot\left(+\frac{1}{12\pi^2}C(r)\log(-q^2)\right)$$

Setting $C(r)=1$, $g=e$, we find the QED results for fermions.

We can now easily write the contribution from the the Yang-Mills gauge bosons. The same three diagrams appear as in the fermion case. The coefficient from these diagrams is

$$\frac{1}{2}\cdot\left[\frac{4}{48\pi^2}-\frac{4C(J)}{16\pi^2}\right]C(G) = -\frac{5}{24\pi^2}C(G)$$

I include a factor $\frac{1}{2}$ because the Yang-Mills fields are real, not complex. The ghosts gives the same contribution as one complex scalar in the adjoint representation, with a fermion minus sign.

$$(-1)\quad\frac{1}{48\pi^2}\,C(G)$$

Adding all contributions, we find that the coupling constant of Yang-Mills theory is shifted by

$$\frac{1}{4g^2}\to\frac{1}{4g^2}\left(1-g^2\left[\frac{n_b C(r)}{48\pi^2}+\frac{n_f C(r)}{12\pi^2}-\frac{(10+1)}{48\pi^2}C(G)\right]\times\log(-q^2)\right)$$

This corresponds to a β function in Yang-Mills theory

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C(G) - \frac{4}{3}n_f C(r) - \frac{1}{3}n_b C(r)\right]$$

where n_f is the number of Dirac fermions and n_b is the number of complex bosons in the G representation r.

For example, if $G = SU(N)$ and the fermions and bosons are in the fundamental representation

$$C(G) = N \qquad C(r) = \frac{1}{2}$$

then

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}N - \frac{2}{3}n_f - \frac{1}{6}n_b\right]$$

If N is sufficiently large compared to n_f, n_b, the β fuction is negative. Then the coupling in the theory will become weak at short distances and strong at large distances, according to

$$g^2(Q^2) = \frac{g^2(Q_0^2)}{1 + \frac{g^2}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right)\log Q^2/Q_0^2}$$

This is the marvelous property of *asymptotic freedom.* It was discovered as a property of Yang-Mills theory by 't Hooft, Gross, Politzer, and Wilczek.

It is odd that Yang-Mills theory with a non-Abelian group can give the reverse of normal dielectric behavior. It is possible to get some feel for this by studying the non-Abelian version of Coulomb's law

$$\vec{D} \cdot \vec{E}^a = g \rho^a$$

For example, work with $G = SU(2)$ and write this equation with a point source in the color 1 direction as

$$\vec{\nabla} \cdot \vec{E}^a = g\, \delta(\vec{x})\, \delta^{a1} + g\, \epsilon^{abc} \vec{A}^b \cdot \vec{E}^c$$

Then a vacuum fluctuation of $\vec{A}^a$ in the color 2 or 3 direction and in any random orientation will generate a vacuum polarization in the 1 direction that points the wrong way. This is illustrated in Figs p. 3.

In the next lecture, I will turn this observation about Yang-Mills theory into a theory of strong interactions.