Dec.8

Jets

We are now very close to being able to select a unique theory of the strong interactions. This theory should be a non-Abelian gauge theory. The quarks should be in a representation of the gauge group G of this gauge theory. The only degree of freedom available is color. In fact, this explains the presence of the color quantum number of quarks as the quantum number to which gluons and the strong interactions couple. Then G should have 3-dimensional irreducible representations. The only choices are $SU(3)$ and $SO(3)$ or $SU(2)$. I will argue in a moment that hadrons should be in singlet states of G. In $SU(3)$, we can make color singlets using the invariants δ_{ij} and ϵ_{ijk}. These correspond exactly to mesons ($q\bar{q}$) and baryons (qqq). With $SO(3)$, we can also make a G-invariant by combining two 3-dimensional representations (qq); this would make a fractionally charged hadron, and no such particles are observed.

By this logic, we obtain a single candidate for the theory of strong interactions, the $SU(3)$ Yang-Mills theory with quarks as Dirac fermions in the 3-dimensional representation. This theory is called *Quantum Chromodynamics* (QCD). Its Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \sum_f \bar{q}_f(i\not{D} - m_f)q_f$$

This unique theory gives us many predictions that can be tested. I will present a number of them in the next three lectures.

We can study QCD either in the large-distance, strong coupling regime or at high energies where it is weakly coupled. It is very difficult to make quantitative calculations in QCD in the strong coupling regime. One approach is to approximate QCD by replacing continuum space-time by a discrete lattice. In this approach, invented by Wilson and Polyakov, the theory can be studied in an expansion about strong coupling ($g \to \infty$). In this expansion, it is manifest that the only states of finite energy are G singlets. If one attempts to separate a quark and an antiquark, a physical string forms that joins them, and the energy of this configuration increases linearly with the distance of separation. Using numerical simulations, it is possible to compute the hadron masses in lattice gauge theory. The first approximation gives a spectrum that resembles the quark model states discussed in the first lecture. With more detailed numerical work, it is possible to approach the limit in which the lattice spacing goes to zero, and, in that limit, the masses of the low-lying mesons and baryons are reproduced to a few percent accuracy. I apologize that I will not have time to discuss lattice gauge theory in this course. I will discuss some further strong-coupling aspects of QCD in Thursday's lecture.

Now I would like to turn to the high-energy properties of QCD. Most of this discussion will be at energies larger than 10 GeV where the five quarks u, d, s, c, b can be treated as approximately massless. In this region, the running QCD coupling constant behaves as

$$\alpha_s = \frac{g^2}{4\pi} = \frac{\alpha_s(Q_0)}{1 + \frac{b_0}{2\pi}\log\frac{Q}{Q_0}}$$

with

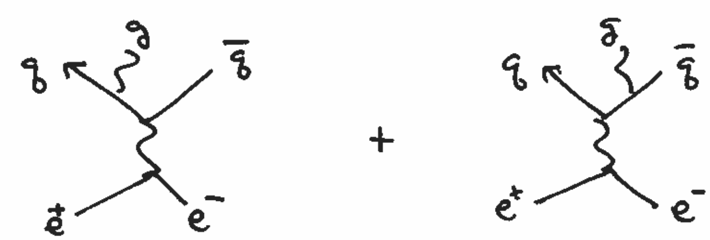
$$b_0 = \frac{11}{3}\cdot 3 - \frac{2}{3}n_f = 11 - \frac{10}{3} = \frac{23}{3}$$

For guidance, the value of α_s that will emerge from the measurements I will describe gives

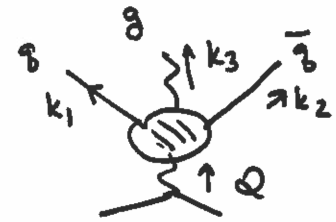
$$\alpha_s = \begin{cases} 0.18 & \text{at} \quad Q \sim 10 \text{ GeV} \\ 0.12 & \text{at} \quad Q \sim 100 \text{ GeV} \end{cases}$$

This is a weak coupling, so we should be able to see the effects of gluons directly through weak-coupling perturbation theory.

One place to look for gluons is in the final states of e^+e^- annihilation to hadrons. Quarks are charged and so they radiate photons. Quarks should be able to radiate gluons in a similar way. The rate of this process is given by the Feynman diagrams



It is not difficult to compute these diagrams. Choose the kinematics to be



with

$$Q = k_1 + k_2 + k_3 \qquad x_1 = \frac{2Q\cdot k_1}{Q^2} \qquad x_2 = \frac{2Q\cdot k_2}{Q^2} \qquad x_3 = \frac{2Q\cdot k_3}{Q^2}$$

so that

$$x_1 + x_2 + x_3 = 1 \qquad\qquad 0 < x_1, x_2, x_3 < 1$$

If E_b is the electron or positron beam energy in the center of mass frame, the energy of each parton in this frame is $E_i = x_i E_b$. Then if

$$\sigma_0 = \frac{4\pi\alpha^2}{3s} \cdot \sum_f 3Q_f^2$$

is the leading-order cross section for e^+e^- annihilation to hadrons, the cross section for gluon radiation is

$$\frac{d\sigma}{dx_1\,dx_2} = \sigma_0 \cdot \frac{2\alpha_s}{3\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

The coefficient $2\alpha_s/3\pi \approx$ 2–4%, but the denominator contains small factors, so roughly 10% of e^+e^- annihilation events should contain a visible gluon. Figs p. 2 shows a typical 2-jet event as observed by the SLD detector at 91 GeV. Figs p. 3 shows an event with 3 visible jets. Figs p. 4 shows an event with 4 jets, presumably corresponding to double gluon radiation.

To test QCD more quantitatively, we need to compare the rate of observed multi-jet events to an integral over the predicted cross section. There are two problems here: First, we need to define more precisely what we mean by a jet. Second, the cross section formula is divergent as $x_1 \to 1$ or $x_2 \to 1$, and we need to control this singularity.

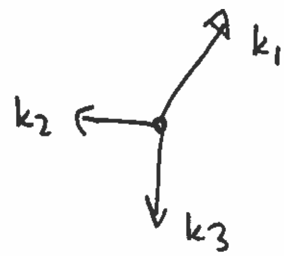
It should be no surprise that the cross section for gluon radiation is divergent. In QED, the first-order diagram for radiation predicts an infinite probability for the radiation of soft photons. A more complete analysis shown that what is really predicted is the radiation of an infinite number of soft photons. We should see similar issues in QCD.

To quantify the probability of gluon emission, we define variables that characterize the shapes of final states in e^+e^- annihilation that are computable and finite in QCD perturbation theory. Many such observables have been proposed. The single most useful one *thrust*, invented by Farhi. To compute this observable, we choose an axis $\hat{n}$ for each event that maximizes the projection of particle momenta onto that axis. Then

$$T = \max_{\hat{n}} \frac{\sum_i |\hat{n}\cdot\vec{k}_i|}{\sum_i |\vec{k}_i|}$$

This quantity can be computed at the quark-gluon level and at the hadron level. As long as the nonperturbative part of the strong interactions does not generate large momentum transfers, these two estimates should approximately coincide. From the quark-gluon side, this statement relies on the fact that thrust is *infrared-safe*, a property that will be explained below.

In a 3-parton event, viewed in the center of mass frame, the three momentum vectors sum to zero and therefore lie in a plane. In this plane, the configuration of momenta is



If the thrust axis is chosen to lie along the direction of the longest of the three vectors, then $\hat{n}\cdot\vec{k}_{max} = x_{max}E_b$, the projections of the other two vectors on $\hat{n}$ also gives x_{max}, and this is the largest possible value for this configuration. Then

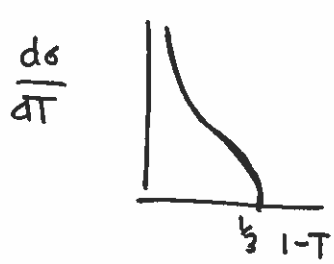
$$T = \max\{x_1, x_2, x_3\}$$

For 3-parton states, T varies from 1 for a back-to-back configuration with a very soft third parton to $\frac{2}{3}$ for a 'Mercedes' configuration in which the three vectors have equal length. Multiparticle events with partons of random momentum and orientation have $T = \frac{1}{3}$.

It is not difficult to integrate the QCD cross section for $e^+e^- \to q\bar{q}g$ over x_1 and x_2 subject to the condition that $T = x_{max}$ is fixed. Note that if $T < 1$ then neither of the two singularities can be reached. The result is

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi}\left\{ \frac{2(3T^2-3T+2)}{T(1-T)} \log\left(\frac{2T-1}{1-T}\right) - 3\,\frac{(3T-2)(2-T)}{1-T} \right\}$$

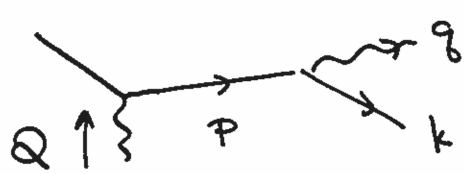
The shape of this distribution is



In real data, the thrust can never be exactly 1. In addition, multiparticle events will populate the region $(1-T) > \frac{1}{3}$ to some extent. Models that include the transition from quarks and gluons to hadrons can be used to estimate these effects. Figs p. 5 show a comparison of SLD data to two such simulation programs. The simulation results in the region $0.1 < (1-T) < 0.3$ are close to the lowest-order result from perturbative QCD, so in that region this comparison is a test of QCD.

The normalization of the QCD prediction gives us a vale of α_s. As the center of mass energy increases, the thrust distribution becomes narrower, and other measures of event shape also indicate that fewer partons are being radiated. So, the value of α_s is observed to decrease. The red points in Figs p. 6, from an analysis by Bethke, show a number of these α_s determinations as a function of the momentum transfer in the e^+e^- annihilation reaction. Notice how nicely these and the other α_s measurements shown follow the Q-dependence of the solution of the QCD renormalization group equation.

We should now look more closely at the infrared singularities in the cross section for gluon radiation. Normally, we think of infrared singularities as being associated only with soft radiation. However, we have a special situation here in which a massless particle turning into two massless particles. Label the momenta involved in this process



If $k^2 = 0$ and $q^2 = 0$, then

$$(k+q)^2 = 2k\cdot q = 2E_k E_q (1-\cos\Theta_{kq})$$

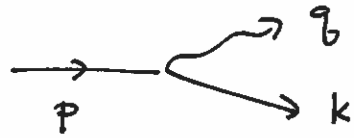
So the denominator of the Feynman diagram vanishes if the final quark and gluon are collinear, even if both partons carry substantial momentum. I will call such a process, a conversion of a massless parton to two collinear massless partons carrying fractions z and $(1-z)$ of its momentum, a *collinear splitting.*

Such collinear splittings clearly have high probability in QCD. Thus, QCD can only predict the values of event observables that are unchanged when we replace a parton by a set of two collinear partons. This is the definition of *infrared safety* referred to above. Thrust is infrared safe, since a collinear splitting converts a factor $\hat{n}\cdot\vec{k}$ into $(\hat{n}\cdot z\vec{k}+\hat{n}\cdot(1-z)\vec{k})$. An alternative measure of event shape, the *sphericity* defined by

$$S = \max_{\hat{n}} \frac{\sum_i |\hat{n}\cdot\vec{k}_i|^2}{\sum_i |\vec{k}_i|^2}$$

is not infrared safe and thus cannot be predicted in QCD.

I will now compute the probability for an approximately collinear splitting $q \to g+q$. As above, I denote the momenta as



I will assume that p is directed parallel to $\hat{3}$ and that q and k have some small transverse momentum in the $(3,1)$ plane. The 4-vectors of q and k are

$$q = \left(zE,\ k_T,\ 0,\ zE - \frac{k_T^2}{2zE} \right)$$

$$k = \left((1-z)E,\ -k_T,\ 0,\ (1-z)E - \frac{k_T^2}{2(1-z)E} \right)$$

where the last term in each line insures that the vectors are on mass shell, that is, $q^2 = k^2 = 0$, including terms of order k_T^2. By momentum conservation,

$$p = \left(E, 0, 0, E - \frac{k_T^2}{2z(1-z)E}\right)$$

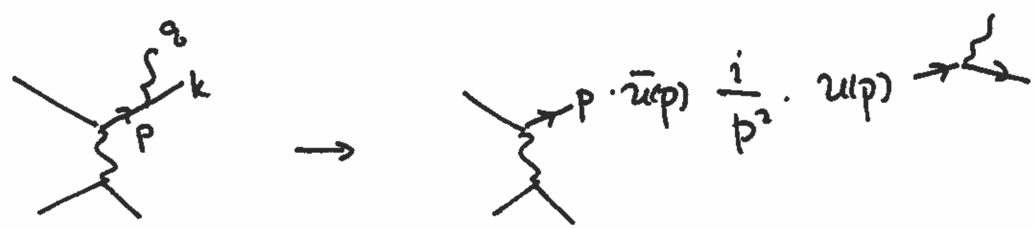
Then p will be off-shell (as expected for an intermediate line in a Feynman diagram) with

$$p^2 = \frac{k_T^2}{z(1-z)}$$

If k_T is small, we can approximately factorize the gluon emission diagram by replacing the fermion propagator by

$$\frac{i\not{p}}{p^2} = \frac{i}{p^2}\, u(p)\,\bar{u}(p)$$

Then



In the result, the diagram on the left is the lowest-order diagram for $e^+e^- \to q\bar{q}$.

The diagram on the right is a new amplitude that we need to compute. It is given by

$$i\mathcal{M} = i g\, t^a\, \bar{u}(k)\, \gamma\cdot\epsilon^*(q)\, u(p)$$

Helicity conservation implies that the helicities of p and k are the same. I will do the computation for left-handed quarks; the results for right-handed quarks will be the same by parity.

Reduce the expression above to 2-component spinors

$$i\mathcal{M} = igt^a \; u^\dagger(k)\, \bar{\sigma}\cdot\epsilon^*(q)\, u(p)$$

where

$$u(p) = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} \qquad u(k) = \sqrt{2(1-z)E}\begin{pmatrix}k_T/2(1-z)E\\1\end{pmatrix}$$

The spinor $u(k)$ is rotated by the small angle $\theta = -k_T/(1-z)E$. The emitted gluon can have either polarization. The possible polarization vectors are

$$\epsilon_L^\mu = \frac{1}{\sqrt{2}}\left(0, 1, -i, -\frac{k_T}{zE}\right) \qquad \epsilon_R^\mu = \frac{1}{\sqrt{2}}\left(0, 1, i, -\frac{k_T}{zE}\right)$$

These also reflect a small rotation in the (3,1) plane. Then

$$\bar{\sigma}\cdot\epsilon_L^* = \frac{1}{\sqrt{2}}\begin{pmatrix}-\frac{k_T}{zE} & 2\\ 0 & +\frac{k_T}{zE}\end{pmatrix} \qquad \bar{\sigma}\cdot\epsilon_R^* = \frac{1}{\sqrt{2}}\begin{pmatrix}-\frac{k_T}{zE} & 0\\ 2 & +\frac{k_T}{zE}\end{pmatrix}$$

Now we can compute the amplitude for ϵ_R,

$$i\mathcal{M} = +igt^a\, 2E\sqrt{1-z}\left(\frac{k_T}{2(1-z)E}, 1\right)\frac{1}{\sqrt{2}}\begin{pmatrix}-\frac{k_T}{zE} & 0\\ 2 & +\frac{k_T}{zE}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= +igt^a\sqrt{2}E\sqrt{1-z}\,\frac{k_T}{zE} = +igt^a\sqrt{\frac{2(1-z)}{z}}\,k_T$$

and for ϵ_L,

$$i\mathcal{M} = +igt^a\, 2E\sqrt{1-z}\, \begin{pmatrix} \frac{k_T}{2(1-z)E} & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{k_T}{zE} & 2 \\ 0 & +\frac{k_T}{zE} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= igt^a \sqrt{2}\, E\sqrt{1-z}\, \left(\frac{k_T}{zE} + \frac{k_T}{(1-z)E} \right) = igt^a \frac{\sqrt{2}}{z\sqrt{1-z})}\, k_T$$

The sum of the squares of these amplitudes is

$$\sum_\epsilon |\mathcal{M}|^2 = \frac{4}{3} g^2 \cdot \frac{2\, k_T^2}{z^2(1-z)} \left[1 + (1-z)^2 \right]$$

where I have used the relation for $SU(3)$ generators

$$t^a t^a = \frac{4}{3} \cdot \underline{1} \qquad \text{in the 3 representation}$$

Notice that the cross section for g_R emission is suppressed at large z. A soft gluon can be emitted with any helicity, but a hard gluon ($z \to 1$) must carry the spin of the emitting quark.

To compute the emission probability, we need to integrate over the final state phase space. This, in particular, involves emission over q and k. To do this, consider p as a shift of k, so $d^3k = d^3p$. However, the value of the energy of k is $E_k = (1-z)E_p$. For q, divide the integral into an integral over transverse momentum and longitudinal momentum. For the latter $dq_{\|} = E_p dz$. Then

$$\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k} \frac{d^3q}{(2\pi)^3}\frac{1}{2E_q} = \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} \cdot \frac{1}{(1-z)} \frac{2\pi k_T\, dk_T\, dz\, E_p}{(2\pi)^3\, 2z E_p}$$

The phase space integral over p can be combined with the integral over the antiquark momentum to build the complete phase space for the leading order process. Then

$$\sigma(e^+e^- \to q\bar{q}g) = \sigma_0 \cdot \int \frac{dk_T\, k_T\, dz}{8\pi^2\, z(1-z)} \left(\frac{1}{p^2}\right)^2 \cdot \sum_{\epsilon} |\mathcal{M}|^2$$

$$= \sigma_0 \int \frac{dk_T\, k_T\, dz}{8\pi^2\, z(1-z)} \quad \frac{z^2(1-z)^2}{k_T^2} \quad \frac{4}{3} g^2 \; \frac{2k_T^2}{z^2(1-z)} \left(1 + (1-z)^2\right)$$

Finally,

$$\sigma(e^+e^- \to q\bar{q}g) = \sigma_0 \int \frac{dk_T}{k_T} \int dz \quad \frac{4}{3} \frac{\alpha_s}{\pi} \left[\frac{1 + (1-z)^2}{z} \right]$$

From this formula, we find that collinear gluons are produced with a dk_T/k_T distribution in transverse momentum. This integrates up to a logarithmic infrared singularity. The longitudinal momentum distribution of the gluons is given by

$$\frac{\alpha_s}{\pi} \cdot P_{q\to g}(z) = \frac{\alpha_s}{\pi} \cdot \left[\frac{4}{3} \; \frac{1 + (1-z)^2}{z} \right]$$

The coupling constant α_s should be evaluated at the scale of k_T. $P_{q\to g}(z)$ is called the *Altarelli-Parisi splitting function.* Figs p. 7 shows the x_1, x_2, x_3 distributions of 3-jet events as measured by the SLD experiment. You can see that x_3, which typically corresponds to the gluon, reflects the shape of this function.

It is instructive to compare the formula we have just derived to the formula written earlier for the complete $e^+e^- \to q\bar{q}g$ cross section. We can take that formula and specialize to the region of a collinear splitting

$$x_1 \cong (1-z) \qquad x_3 \cong z \qquad x_2 \approx 1$$

The variable x_2 is very close to 1 in this kinematics. Its distance from 1 is measured by

$$(Q-k_2)^2 = Q^2(1-x_2) = (k_1+k_3)^2 = \frac{k_T^2}{z(1-z)}$$

so that

$$\frac{dx_2}{1-x_2} = \frac{2\,dk_T}{k_T} \qquad \text{and} \qquad dx_1 = dz$$

Then that expression has as its limit

$$\int d\sigma \;\hat{=}\; \sigma_0 \cdot \int \frac{dk_T}{k_T}\, dz \cdot \frac{4}{3}\frac{\alpha_s}{\pi} \frac{(1-z)^2+1}{z}$$

which agrees with the analysis that we have just completed.

Once a quark has radiated a collinear gluon~~s~~, it can radiate another one. Similarly, a gluon can split to a collinear $q\overline{q}$ pair or to a collinear gluon pair. It is not so hard to work out the Altarelli-Parisi splitting functions for these transitions.

$$P_{q\to g}(z) = \frac{4}{3}\frac{1+(1-z)^2}{z}$$

$$P_{q\to q}(z) = \frac{4}{3}\frac{1+z^2}{(1-z)} - A\,\delta(z-1)$$

$$P_{g\to q}(z) = P_{g\to \bar{q}}(z) = \frac{1}{2}\left[z^2+(1-z)^2\right]$$

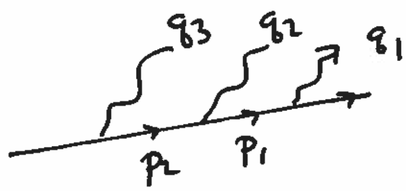
$$P_{g\to g}(z) = 3\left[\frac{1}{z(1-z)} + \frac{z^4}{z(1-z)} + \frac{(1-z)^4}{z(1-z)}\right] - B\,\delta(z-1)$$

The delta functions in these expressions remove the quark or gluon that split to a 2-parton state. For a discussion of how to treat these delta functions and the singularities as $z \to 1$, see Peskin and Schroeder, Section 17.5.

Almost collinear emissions have probability proportional to

$$\frac{\alpha_s}{\pi} \cdot \log \frac{k_T^{max}}{k_T^{min}}$$

With the logarithmic enhancement, this factor can be of order 1, and so we should sum these contributions to all orders. The dominant contribution comes from a particular region of phase space. Consider the diagram with successive emissions



The value of p_i^2 for each successive fermion propagator reflects the transverse momenta of all gluons emitted beyond that point. Thus, if q_1 has a large transverse momentum, all denominators are large. However, if q_2 has a large transverse momentum and q_1 has a smaller transverse momentum, the transverse momentum of q_1 will dominate p_1^2, the transverse momentum of q_2 will dominate p_2^2, and each emission will have a logarithmic integral over transverse momentum. This situation in which the transverse momenta are ordered from inside to outside

$$\cdots \quad k_T(q_3) > k_T(q_2) > k_T(q_1)$$

is called *strong ordering*. In that region of phase space, the emission of each successive gluon is logarithmically enhanced.

We can use strong ordering to sum up the contributions of many collinear splittings by accounting the probability of splitting systematically from small to large k_T. For a process where a quark has multiple emissions that eventually produce a gluon, let

$$dz \; f_{q \to g}(z, Q)$$

be the probability of finding this gluon at a fraction z of the momentum of the original quark, including the effects of all emissions with $k_T < Q$. We can define this object similarly for any initial and final partons, or even for the production of a final hadron from an initial quark or gluon.

$$dz\; f_{p \to p'}(z, Q) \qquad\qquad p, p' = \text{partons}$$

These functions are called *fragmentation functions.* We can now consider the change in a fragmentation function when we add a collinear splitting with transverse momentum $k_T \sim Q$. The additional contribution is

$$\Delta \int dz\, f_{p \to p'}(z, Q) = \frac{\alpha_s(Q)}{\pi} \int dz \int dw \left.\frac{dk_T}{k_T}\right|_Q \cdot P_{p \to p''}(w)\, f_{p'' \to p'}(z)$$

The integral over longitudinal variables can be rearranged by

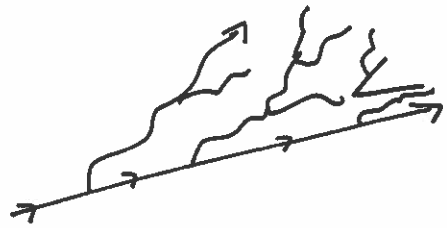
$$\int dz\, dw = \int dx \int dz\, dw\; \delta(x - zw) = \int dx \int \frac{dw}{w}$$

Finally, we can describe the change in the fragmentation function by a differential equation in the variable Q

$$\frac{d}{d\log Q} f_{p \to p'}(x, Q) = \frac{\alpha_s(Q)}{\pi} \int \frac{dw}{w} \sum_{p''} P_{p \to p''}(w)\, f_{p'' \to p'}\left(\frac{x}{w}, Q\right)$$

This is called the *Altarelli-Parisi equation.* It builds the fragmentation functions systematically in an ordered progression of emissions in k_T.

Another way to look at the development of a quark by successive collinear splittings is to model it as a Markov process, in which the Altarelli-Parisi equation gives the probability of an emission at a particular k_T and z. Carrying out this process, an initial quark is converted to a stream of approximately collinear quarks, gluons, and antiquarks. This set of particles is called a *parton shower.* The shape of the shower is set by the k_T scale of the hardest emission. The internal structure of the shower will be complex.



If α_s were constant, this structure would be scale-invariant or fractal. Since α_s runs, the shower is more diffuse at the beginning and becomes very dense at small angles. Eventually, as we reach $k_T \sim 1$ GeV, we can no longer use perturbation theory to determine the evolution of the shower, and we must turn instead to a more phenomenological approach of modelling the transition of quarks and gluons into hadons.

The shower of partons is what I have previously called a jet. In last week's lecture, it wanted to view a jet as a single quark or gluon. Now we see that a jet contains more partons the more finely we resolve its structure, in a way that is predicted by perturbative QCD. We can relate this to experiment by constructing jets in the final states of e^+e^- annihilation events with varying resolution.

Here is a way to do that, called the *JADE algorithm.* From among the final state particles in an event, choose the two i, j such that $m_{ij}^2 = (k_i + k_j)^2$ is minimal. Typically, these will be particles close together in angle. Combine these to a single particle. Repeat this operation until $m_{ij}^2 > m_{cut}^2$, then stop and identify the particles that remain at that stage as the jets. Usually, the stopping condition is expressed in terms of

$$y_{cut} = \frac{m_{cut}^2}{s}$$

For e^+e^- events at 100 GeV, $y_{cut} = 10^{-2}$ corresponds to a clustering mass of 10 GeV. As y_{cut} is decreased, we resolve the event more finely.

Figs p. 6 compares real events analyzed by the OPAL experiment at 91 GeV to the QCD prediction. At large y_{cut}, most events are 2-jet events, but as this parameter is decreased, the events are resolved into 3-, 4- and even 5-jet events. We see that the process of quark and gluon splitting predicted by QCD does a good job of describing this evolution.