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Midterm Exam of GR course

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This exam is take-home and you have four days to return the exam sheets.
Total mark: 100.

1) Questions on the notion of geodesic.

1-1) Can we have a null path which has a nonzero acceleration, *i.e.* a null path which is not parallel transported along itself?

1-2) Show that if the velocity vector field u_μ is gradient of a scalar function $u_\mu = \partial_\mu \Phi$; it satisfies geodesic equation.

1-3) Given (1-2), revisit (1-1) and analyze it again.

1-4) Consider a particle in a given background metric $g_{\mu\nu}$ and electromagnetic gauge field A_μ, and let ξ^μ denote a symmetry of this background, namely, $\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi A_\mu = 0$. If the momentum of the particle is

$$P^\mu = mu^\mu + qA^\mu,$$

where m, q are respectively particles mass and electric charge and u^μ is its velocity vector field, then show that $P \cdot \xi$ is conserved on the particle's path.

2) Killings vs geodesics. When is a Killing vector the velocity vector field along a geodesic?

3) Geodesic deviation and non-free motion. Two charge particles of the same mass m and charge q are falling in a spacetime of given metric $g_{\mu\nu}$ and at $t = 0$ are at $\vec{x}_1 = 0,\ \vec{x}_2 = (\ell, 0, 0)$.

3-1) What is the acceleration of particle 2 as viewed by particle 1?
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3-2) Do we have a notion of "third law of Newton" in the general relativistic setup?

4) Chirstoffel symbol is not a tensor.

4-1) Show that under the diffeomorphism $x^\mu \to x^\mu - \xi^\mu(x)$, the Christoffel connection transforms as

$$\Gamma^\alpha_{\mu\nu} \to \Gamma^\alpha_{\mu\nu} + \delta\Gamma^\alpha_{\mu\nu}, \qquad \delta\Gamma^\alpha_{\mu\nu} = \nabla_{(\mu}\nabla_{\nu)}\xi^\alpha - R^\alpha{}_{(\mu\nu)\beta}\xi^\beta,$$

where in the above $X_{(\mu\nu)} = \frac{1}{2}(X_{\mu\nu} + X_{\nu\mu})$.

4-2) Show that $\delta\Gamma^\alpha_{\mu\nu} = 0$ if ξ^μ is a Killing vector.

5) Symmetries of Riemann curvature. Consider cases with non-vanishing torsion, the torsional connection cases.

5-1) What is the symmetry structure of the four indices on the Riemann in this case?

5-2) Count the number of independent components of Riemann curvature in torsional case.

6) On various curvature tensors.

6-1) Show that Weyl curvature tensor $W^{\mu}{}_{\nu\alpha\beta}$ is invariant under scaling $g_{\mu\nu} \to e^{2\phi(x)} g_{\mu\nu}$.

6-2) Verify and rewrite the two Bianchi identities for the Weyl tensor.

6-3) The Schouten tensor,

$$S_{\mu\nu} = \frac{1}{2}(R_{\mu\nu} - cRg_{\mu\nu}),$$

is defined such that it is invariant under rigid scaling $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$ (constant λ). Fix c which does so in any dimension d.

6-4) The Cotton tensor is defined as

$$C_{\mu\nu\alpha} = \nabla_\mu S_{\nu\alpha} - \nabla_\nu S_{\mu\alpha}.$$

Show C is traceless over any two index.

6-5) Consider the Bach tensor

$$B_{\mu\nu} = \nabla^\alpha C_{\alpha\mu\nu} + S^{\alpha\beta} W_{\alpha\mu\nu\beta}.$$

6-5-1) Show

$$B_{\mu\nu} = \nabla^\alpha \nabla^\beta W_{\alpha\mu\nu\beta} - \frac{1}{2} R^{\alpha\beta} W_{\alpha\mu\nu\beta}.$$

6-5-2) Show that

$$\nabla^\mu B_{\mu\nu} = 0, \qquad B^{\alpha}{}_{\alpha} = 0.$$

7) On ADM charges. Write the asymptotic form of a metric which has ADM momentum P^μ and ADM angular momentum $J^{\mu\nu}$.

8) On variational analysis.

8-1) Work out the equations of motion for the action

$$S = \int d^4x \sqrt{-g}\; (R^2_{\mu\nu\alpha\beta} + AR^2_{\mu\nu} + BR^2),$$

where A and B are two constants.

8-2) Choose A, B such that the above action is Weyl^2 or Gauss-Bonnet term and rewrite the equations of motion.