Mid-term exam of field theory course (12/14/2015)

1) Consider a Dirac fermion ψ which under Lorentz transformation transform as $\psi(x) \to \Lambda_{1/2}\psi(\Lambda^{-1}x)$ where Λ and $\Lambda_{1/2}$ are both 4×4 matrices.

1-a) What is the form of Λ and $\Lambda_{1/2}$? What are the generators of rotations and boosts? Discuss whether you expect these generators to be Hermitian?

1-b) What is the largest subgroup of Lorentz group under which $\bar{\psi}\gamma^0\psi = \psi^\dagger\psi$ remains invariant?

1-c) Answer the same question for $\bar{\psi}\gamma^1\psi$.

1-d) Consider the following 4×4 matrix

$$M \equiv \begin{pmatrix} 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Expand M in the basis 1, γ^5, γ^μ, $\gamma^\mu\gamma^5$ and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. What are the subgroups of Lorentz transformation under which $\bar{\psi}M\psi$ and $\psi^\dagger M\psi$ are invariant?

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2) Suppose that there is a $SU(2)$ symmetry under which $(\phi_1\ \phi_2)$ transform as a doublet. That is

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \xrightarrow{SU(2)} e^{i\sum_{i=1}^{3}\sigma_i\alpha_i} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{2}$$

where σ_i are Pauli matrices and α_i are arbitrary real numbers.

2-a) Write down the most general renormalizable $SU(2)$-invariant potential for them. Write down also the Noether current(s).

2-b) Suppose that I add a term of form $\lambda|\phi_1|^2|\phi_2|^2$ to your Lagrangian. This term of course breaks the $SU(2)$ symmetry, What is the remaining symmetry? Write down its Noether current(s)?

2-c) Let us how further add a term of $\lambda'|\phi_2|^4$. What is the remaining symmetry and the corresponding Noether current(s)?

2-d) Now, I add a term $\mu^2(\phi_1\phi_2 + \phi_1^*\phi_2^*)$. What is the remaining symmetry? Write down the corresponding Noether current(s)?

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3) Consider a four-Fermi interaction of the following form between fermions ψ_1 and ψ_2:

$$[\bar{\psi}_1(A\gamma^1\gamma^5 + B\gamma^3\gamma^5)\psi_1] \cdot [\bar{\psi}_2\psi_2].$$

Suppose anti-fermion $\bar{\psi}_1$ is initially at rest and take its spin in the direction of $\hat{z}$. Fermion ψ_2 with momentum of k much smaller than m_{ψ_1} scatters off $\bar{\psi}_1$. Calculate the ratio of the probability of $\bar{\psi}_1$ scattered to a state with a spin in the direction of $\hat{x}$ to that to a state with a spin in the direction of $-\hat{x}$. Can you reach the same conclusion with angular momentum conservation?