Exercise 3 — Supersymmetry for Superstrings

1. SUSY generalities
   (suggested reading : Vafa et. al., *Mirror symmetry*, Ch 10)

Supersymmetry often leads to: I) stability of quantum systems, and II) control over the non-perturbative regime of quantum systems.

As for item I, in string theory supersymmetry allows removing the bosonic string tachyon in Minkowski background. (Question: do you remember the masses of the open and closed bosonic string tachyons in Minkowski background?) In other words, the superstring spectrum, unlike the bosonic spectrum, does not signal a perturbative instability in the system.

The fact that supersymmetry (SUSY) removes the tachyon (after GSO, of course), is in accord with the general wisdom that supersymmetry often stabilizes quantum systems.

- Write down the $\mathcal{N} = (1, 1)$ SUSY algebra (in terms of Hilbert-space operators) in 1+0, 1+1, and 1+3 dimensional quantum field theories. [Hint: ask Google.] Make sure you understand the role of $\dagger$. Show that in all cases the SUSY algebra implies a positive semi-definite energy. [Hint: write the expectation value of the Hamiltonian in terms of the supercharge operators.]

- Write down the SUSY algebra in 1+1 dimensional conformal field theory (CFT). This CFT could be the one living on the worldsheet of superstrings. Does the algebra imply a positivity constraint on the spectrum?

Task: read about Witten’s SUSY-inspired proof of the positive energy theorem in classical GR.

As for item II, in string theory supersymmetry allows understanding various strong-weak dualities between the five superstring theories and M-theory. In particular, a strong-weak supersymmetric duality known as the AdS/CFT correspondence has allowed a non-perturbative formulation of superstring theory on asymptotically AdS spacetimes.
The power that SUSY provides for control over non-perturbative regimes of quantum systems is demonstrated by the Witten index in SUSY quantum mechanics.

- In the simplified setting of 1+0 dimensional QFT (viz quantum mechanics), use the SUSY algebra to argue that states with non-zero energy come in bose-fermi pairs.
- Use the information from the previous problem to argue that the quantum mechanical Witten index $\text{Tr}(-1)^F$ is invariant under smooth changes of couplings in SUSY quantum systems under sufficiently nice conditions.

Task: read about Witten’s supersymmetric index in SUSY QFTs in various dimensions, and how it allows extracting non-perturbative information under sufficiently nice conditions. See Witten’s classic paper Constraints on supersymmetry breaking.

Task: study the super-particle in Homework Problems 4.1–4.3 of Becker-Becker-Schwarz.

Task: make sure you understand the Lagrangian and the Hamiltonian formulations of supersymmetry, separately.

2. Worldsheet SUSY
(suggested reading: GSW, and section 4 of 0201253)

To approach the worldsheet supersymmetric conformal field theory (SCFT) of the superstring via a Lagrangian, we need to study 2d fermion fields.

- Task: read about fermions in diverse dimensions. (See section 4.1 of 0201253 for example.)
- In which dimensions can we have Majorana-Weyl fermions? How many on-shell degrees of freedom do they have? How many on-shell degrees of freedom do Majorana fermions have in 2, 4, and 10 dimensions?
- (This is a warmup problem for familiarity with 2d fermions; we won’t use its result.) Given a pair of 2d Majorana fermions $\psi$ and $\chi$, show that

$$\psi_A \bar{\chi}_B = -\frac{1}{2} \left( \bar{\chi} \psi \delta_{AB} + \bar{\chi} \gamma_a \psi (\gamma^a)_{AB} + \bar{\chi} \gamma_3 \psi (\gamma_3)_{AB} \right), \quad (1)$$
where $\gamma_3 = \gamma_0 \gamma_1$. [Hint: one way of doing it is checking the matrix components of the two sides, using the explicit 2d gamma matrices, and the relation $\bar{\psi} := \psi^\dagger (i \gamma^0)$.]

- Derive the equations of motion for the worldsheet action (see subsection 4.6 of 0201253 for a more general action, with NS-NS background fields turned on)

$$\int d^2 \sigma (\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu),$$ (2)

and recognize the supersymmetric partnerships in them. Make sure you are comfortable working with 2d Majorana spinors $\psi$, as well as 2d Majorana-Weyl spinors $\psi_-, \psi_+$, especially in analyzing the above action and its resulting equations of motion.

- Show that the above action is supersymmetric on shell. Then use Noether’s procedure for supersymmetry to derive the conserved supercurrents.

- In order to reduce the computational load of checking supersymmetry in Lagrangians, one often works in the “superspace”, parameterized by $(\sigma^\alpha, \theta)$. On superspace live “superfields” $Y^\mu (\sigma^\alpha, \theta)$. A superfield can be expanded as

$$Y^\mu (\sigma^\alpha, \theta) = X^\mu (\sigma^\alpha) + \bar{\theta} \psi^\mu (\sigma^\alpha) + \frac{1}{2} \bar{\theta} \theta F^\mu (\sigma^\alpha),$$ (3)

where $F$ is an auxiliary field which helps realizing the SUSY algebra off-shell. Question: why there are no linear-in-$\theta$ terms in the above expansion? [Hint: show that for a Majorana fermion $\psi$ we have $\bar{\psi} \theta = \bar{\theta} \psi$.] See pages 113–118 of Becker-Becker-Schwarz for more on superspace.

3. **Super-Virasoro, the constraint equations, and GSO** (suggested reading: Becker-Becker-Schwarz, GSW, and Johnson’s *D-branes* book)

In the bosonic string theory, the equation of motion for the worldsheet metric implies conformal invariance, which allows choosing a light-cone gauge yielding a manifestly positive-norm spectrum. (Question: do you remember what happens with “spurious states”?) In the superstring theory, a supersymmetric conformal (or superconformal) invariance arises, which again allows a light-cone gauge analysis.
• Write down the super-Virasoro algebra arising in the NS sector of the superstring worldsheet. Check that its global part closes (this is called the SU(1,1|1) algebra). Write down the corresponding “physical state conditions”.

• Write down the super-Virasoro algebra arising in the R sector of the superstring worldsheet. Check that there is no closed global part in there that contains $F_0$ (the zero-mode of the supercurrent). Write down the corresponding “physical state conditions” (aka “constraint equations”).

• Check that the Ramond ground state is degenerate, and the states in it form a spacetime fermion with 32 components. Construct the 32-dimensional representation by acting on a chosen ground state with 5 spinor creation operators formed from the Ramond fermion zero mode. (Do not use Gamma matrices here; they are not Hilbert space operators!) [The answer is given in Johnson’s book, pp. 158,159.] Show that the $F_0$ equation (aka the Dirac-Ramond equation) removes half of the degrees of freedom, leaving 16 components. Another half is of course thrown away by the GSO projection, leaving 8 components (which are, depending on the GSO chirality, in $8_c$ or $8_s$ of the spacetime little group SO(8)).

• Find the tachyonic and massless spectrum of the closed superstring before the GSO projection.

• Show that the product of two vector representations of SO($N$) decomposes to a scalar, an anti-symmetric tensor, and a traceless symmetric tensor. Use the decomposition of $8_v \times 8_v$ to deduce that the massless NS-NS sector of the closed superstring has a dilaton, a Kalb-Ramond field $B_{\mu\nu}$, and a metric $G_{\mu\nu}$.

• Argue that the decomposition of $8_s \times 8_s$ and $8_c \times 8_c$ are the same as that of $8_v \times 8_v$. Show that the decomposition for the product of $8_c \times 8_s$ reads $8 + 56$. [The required machinery is described in section 4.5 of 0506011.]

• Use the GSO projection along with the decompositions discussed above, to find the type IIA and IIB closed superstring massless spectra. These have a low-energy description as IIA and IIB supergravity theories.

• Show that the spectra of IIA and IIB supergravity theories become the same upon dimensional reduction on a spatial circle. [Hint: use Hodge duality for the bosonic parts; for the spinorial parts, argue that the differing chiralities in ten dimensions are not important after reduction to 9d where there is no chirality (see section 1.7.
for more on the reduction of fermions).]

4. Supergravity, p-branes, and D-branes
(suggested reading: Freedman-van Proeyen and Maldacena’s thesis)

Let us focus on IIB supergravity. One of its equations of motion (omitting the derivative
terms in the dilaton for simplicity)
\[
R_{\mu\nu} = \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}^{\rho\sigma} + e^{2\phi} \left( F_{1\mu} F_{1\nu} + \frac{1}{4} \tilde{F}_{3\mu\rho\sigma} \tilde{F}_{3\mu}^{\rho\sigma} + \frac{1}{24} \tilde{F}_{5\mu\rho\sigma\tau} \tilde{F}_{5\nu}^{\rho\sigma\tau} \right).
\]

(4)

- Show that the 3-brane solution with \( B = A_2 = C = \phi = 0 \), and
\[
ds^2 = H^{-1/2}(y) dx^2 + H^{1/2}(y) dy^2,
F_5 = *_6 dH,
H = 1 + \frac{L^4}{y^4},
\]
solves the IIB equation of motion; the coordinates \( x_i \) (\( i = 0, 1, 2, 3 \)) are along the
brane, and the \( y_j \) are perpendicular to it; the Hodge operator *\(_6\) dualizes forms
on the space of \( y_j \)s. This is an extremal black brane solution with “Schwarzschild
radius” \( L \).

- Use the D3-brane tension equation to argue that if the above \( p \)-brane solution is
sourced by \( N \) D3-branes, it would have \( L^4 \propto Ng_s\alpha'^2 \). So the horizon radius is
too small for the solution to represent a useful blackhole. (Note that increasing
\( N \) doesn’t help, because then the effective coupling near the horizon will be \( Ng_s \),
so the theory will be strongly coupled and hence not very useful still.)

- Use the gravitino supersymmetry variation in IIB supergravity to show that the
SUSY generators \( \varepsilon_R^k \) (\( k = 1, 2 \)) of the IIB theory should satisfy
\[
\Gamma^0 \cdots \Gamma^3 \varepsilon_R^1 = \varepsilon_R^2,
\]
for the 3-brane solution to be supersymmetric; so one spinor is determined in terms
of the other. The 3-brane background thus breaks half of the supersymmetries of
the theory (or is “half BPS”). [The result can also be found in Eq. (25.128) of the
2nd edition of Ortin’s Gravity and Strings; note the typo in Eq. (25.127) there,
in which “odd” and “even” should be swapped.]
The equations that are derived from the vanishing constraint on the SUSY variations of the supergravity (SUGRA) fermionic fields are often called Killing spinor equations. For the brane backgrounds of interest to us, no Killing spinor equation arises from the dilatino field; only the gravitino field yields non-trivial Killing spinor constraints. The equation for the $p$-brane background with general $p$ (odd, for IIB, and between 1 and 9 of course) reads [again you can find it in Ortin’s book; note that $\Gamma_0 \cdots \Gamma_p$ used there is the same as $\Gamma_0 \cdots \Gamma_p$ (why?); also the hats in there only mean that the objects are ten dimensional]

\[ \Gamma_0 \cdots \Gamma_p \varepsilon_R^1 = \varepsilon_R^2. \]

\[ \vdots \]

- Show that a D$p$-D$(p + 4)$ system (such as a D1-D5 system) preserves 1/4 of the IIB supersymmetries (or is “quarter BPS”), but a D$p$-D$(p + 2)$ system (such as D1-D3) breaks all the supersymmetries. [Hint: write (7) for $p$ and $p + 2$; derive $\Gamma_0 \cdots \Gamma_p \varepsilon_R^2 = \Gamma_0 \cdots \Gamma_{p+2} \varepsilon_R^2$; multiply both sides by $\Gamma_0 \cdots \Gamma_p$ and use the fact that $(\Gamma_0 \cdots \Gamma_p)^2 = (-1)^{(p-1)/2}$ (can you see why?) to arrive at $\varepsilon_R^2 = (-1)^{(p-1)/2} \Gamma_{p+1} \Gamma_{p+2} \varepsilon_R^2$. Finally use $(\Gamma_{p+1} \Gamma_{p+2})^2 = -1$ to show that the Killing spinor equation can only be true if $\varepsilon_R^2 = 0$, meaning that SUSY is completely broken. Similarly, use $(\Gamma_{p+1} \Gamma_{p+4})^2 = +1$ (why?) to show that the Killing spinor equation for the D$p$-D$(p + 4)$ system has non-trivial solutions, and more precisely that the D$(p + 4)$ Killing spinor equation breaks the SUSY preserved by D$p$ down to one-quarter.]

- The $p$-brane solution corresponding to the D1-D5 system, placed on the background $R^2 \times R^4 \times T^4$, has the explicit metric, dilaton, and 3-form flux of IIB supergravity (we follow the notation in arXiv:hep-th/9702050)

\[ e^{-2\phi} = f_5/f_1, \]

\[ ds^2 = f_1^{-1/2} f_5^{-1/2} dx_1^2 + f_1^{1/2} f_5^{1/2} (dr^2 + r^2 d\Omega_3^2) + f_1^{1/2} f_5^{-1/2} dx_{M_4}^2, \]

\[ H_3 = 2r_5^2 \epsilon_3 + 2r_1^2 e^{-2\phi} \ast_{10} \epsilon_7, \]

\[ f_i := 1 + r_i^2/r^2 \quad i = 1, 5, \]

where $dx_1^2 = -dt^2 + dx^2$ is the metric on the $R^2$, with $x$ the coordinate along the D1-brane, and $r$ parameterizing the radial coordinate on the $R^4$. The forms $\epsilon_3$ and $\epsilon_7$ are the volume forms of the three-cycle and the seven-cycle at $r = 1$ inside the $R^4 \times T^4$; the cycles can be alternatively described as the unique three-cycle.
\( C_3 \) and seven-cycle \( C_7 \) inside \( S^3 \times T^4 \). Show that the IIB supergravity equation (4) is satisfied by this solution. D-brane tension analysis shows that \( r_5^2 = g \alpha' Q_5 \) and \( r_1^2 = g \alpha' Q_1 / v \), where \( v = \text{vol}(T^4) / (2\pi)^4 \alpha'^2 \), and \( Q_1, Q_5 \) are respectively the number of D1 and D5 branes. Again, the black p-brane Schwarzschild radius is small unless \( Q_1 \) and \( Q_5 \) are very large. But this time, unlike the 3-brane case, we get large blackholes with nice semi-classical description (see Maldacena’s phd thesis for instance.)

- Show that the near horizon geometry of the 3-brane solution in (5) contains an \( \text{AdS}_5 \) factor. Show that the near horizon geometry of the p-brane solution of the D1-D5 system contains an \( \text{AdS}_3 \) factor. These are two of the most important tips of the AdS/CFT iceberg.