Final Exam of Black Hole Physics

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This exam is take-home and you have five days to return the exam sheets. **Total mark: 100**.
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1) On boosted Kerr strings. Consider the boosted Kerr string with the following metric:

$$ds^2 = -\big(1 - \frac{2Mr\cosh^2\sigma}{\rho^2}\big)dt^2 + \frac{2Mr\sinh 2\sigma}{\rho^2}\,dtdz + \big(1 + \frac{Mr\sinh^2\sigma}{\rho^2}\big)dz^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$
$$+ \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 - \frac{4Mr\cosh\sigma}{\rho^2}a\sin^2\theta dt d\phi - \frac{4Mr\sinh\sigma}{\rho^2}a\sin^2\theta dz d\phi,$$

where σ is the boost parameter, M and a are respectively mass and rotation parameters and

$$\rho^2 = r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2Mr + a^2.$$

1-1) Study the asymptotic large r behavior of the geometry.

1-2) Assume that z parameterizes a circle whose radius in the asymptotic region is R. Study the horizon topology and compute the horizon area.

1-3) What are Killing vectors of this geometry? Show that the horizon is a Killing horizon. Find the Killing vector field ξ_H which generates the horizon.

1-4) Compute the surface gravity associated with the horizon.

1-5) Verify first law of black hole thermodynamics for the family of boosted Kerr strings.

1-6) Work out the Smarr relation. (**3+5+5+4+5+3=25**).

2) Actual values of Hawking temperature and radiation. In our analysis we usually work in natural units where $\hbar = 1$, $c = 1$. We often put $G_N = 1$, too. This may obscure the physical sense of the quantities like surface gravity, Hawking temperature and Hawking radiation flux.(**3+3+4=10**)

2-1) Suppose that we have a supermassive Schwarzchild-type black hole of mass M. What is the value of M so that its surface gravity is equal to the gravity acceleration constant on the Earth $g = 9.8m/s^2$?

2-2) Consider a black hole with $M = \mu M_\odot$. What is the Hawking temperature as seen by the observer at infinity in Kelvins? Consider two cases, stellar mass black holes with $\mu \sim 10$ and supermassive black holes with $\mu \sim 10^6$.

2-3) Suppose that we have a black hole which Hawking-evaporates. What is the lower bound on the mass of black hole such that its life-time is not less than the age of Universe.

3) The generalized Second Law. Let us assume that the process of Hawking radiation is adiabatic. As the black hole Hawking-radiates it loses it mass and possibly angular momentum and charge. As a result, the horizon area is decreasing and therefore, the entropy of the black hole is decreasing. Nonetheless, the entropy carried out by the radiation $S_{rad.}$ may save the second law of thermodynamics. Using Stefan Boltzmann law and the entropy associated with radiation compute $\Delta S \equiv \delta S_{BH} + S_{rad.}$, where δS_{BH} is the change in the black hole entropy (i.e. the entropy of black hole at some time t minus the entropy of original black hole at $t = 0$). Is ΔS zero or positive? (**10**).

4) Hawking radiation and black hole thermodynamics for charged black holes. Consider a Reissner-Nordstrom black hole of mass M and charge Q, with $M > Q$. This black hole can radiate off particles of mass m and charge q and $r \equiv m/q$. (**4+11=15**)

4-1) Compute r for electron and proton.

4-2) Using black hole thermodynamic relations discuss whether this black hole can become extremal through radiating off its mass and charge, or the black hole loses its charge first to become a neutral black hole?

Note: thermal distribution of charged particles emitted by the RN black hole is given by

$$P(\omega) = \frac{1}{e^{\beta(\omega - \Phi_H q)} \pm 1}$$

where $\beta = 1/T_{BH}$, Φ_H is the horizon electric potential and ω denotes the energy (frequency) of the emitted particles as seen by the observer at infinity. The $+/-$ sign, as usual, corresponds to the fact that the emitted charged particles are fermionic/bosonic.

5) Unruh effect for massive scalar. Consider a massive real scalar field Φ of mass m whose E.o.M is

$$(\Box - m^2)\Phi = 0\,.$$

Repeat the analysis of Unruh effect for this massive scalar. That is,

5-1) Solve equations of motion and construct the full set of solutions to the E.o.M. and decompose them into positive and negative frequency modes, in both flat Minkowski coordinates and in Rindler coordinates.

5-2) Construct the Bogoliubov transformations which relates these two basis.

5-3) Construct the Minkowski and Rindler vacuum states, respectively $|0\rangle_{Mink},\ |0\rangle_{Rind.}$.

5-4) Find the Bogoliubov expansion which relates the Rindler annihilation operator $b_\omega^{Rindler}$ to the creation and annihilation operators of Minkowski $a_\omega,\ a_\omega^\dagger$.

5-5) Compute the density of states Unruh observer associates with the Minkowski vacuum, ${}_{Mink}\langle 0|b_\omega^\dagger b_\omega|0\rangle_{Mink}$. (**7+4+4+5+5=25**).

6) Unruh effect on Kerr geometry. As discussed in the lectures, Unruh effect computations readily generalizes to static, globally hyperbolic spacetimes, like Schwarzchild or RN black holes: An observer sitting at constant radial coordinate r is like an Unruh observer with constant temperature $T(r) = \frac{\kappa}{2\pi\ \sqrt{-|\xi_t|^2}}$, where ξ_t is the time-like Killing vector field which generates the horizon. Therefore, constant r surfaces are "isotemperature" surfaces. (**5+5+5=15**).

6-1) Work out the analysis for Unruh effect on Kerr geometry.

6-2) In the static geometries we introduced Hartle-Hawking vacuum state, that is the vacuum state which is well-defined everywhere outside the event horizon. Define the analogue of Hartle-Hawking state for Kerr geometry.

6-3) What are the isotemperature surfaces in the Kerr geometry?