Part I of Session 1

The Formalism of Chiral Fields and Lorentz transformation

پاییز ۸۶

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منبع

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Introduction:

As we shall see in future sessions, the MSSM is a chiral model which distinguishes between right-handed and left-handed fermions. For example, the four-component electron field in this context is considered as two Weyl fermions: e_L and e_R which respectively have superpartners $\tilde{e}_L$ and $\tilde{e}_R$. $\tilde{e}_L$ and $\tilde{e}_R$ are two different bosons.

In order to do calculation in the context of this formalism, one should master the two-component (Weyl) representation of fermions.

In the first part of this session we review the formalism. One should bear in mind that such a formalism is not new to SUSY. In fact we could work out field theory in terms of this formalism from the beginning (See

Here, I closely follow section 1 of Peskin's lecture notes. and use his notation which in different from that of Wess and Bagger's famous book [supersymmetry and supergravity].

In order to learn this formalism, it is necessary to verify all the formulation discussed here after class.

Notation and Lorentz Algebra

I emphasize that the MSSM respects Lorentz (or more generally Poincare) symmetry. Fields discussed in the MSSM (including chiral fields) are representations of the Lorentz group so we review the Lorentz algebra in this formalism.

Here, Latin letters Minkowski indices 0 1 2 3 and greek letters $(\alpha, \beta \dots)$ denote spinor indices.

Minkowski metric : $\eta^{ab} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$

$\varepsilon^{0123} = +1$ $\quad \varepsilon_{0123} = -1$

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M^{ab} denote the generators of Lorentz algebra. That is an infinitesimal Lorentz transformation is given by $\omega_{ab} M^{ab}$ where ω_{ab} are the parameters giving "the size" of the transformation.

As is well-known, scalars, spinors and vectors are different (finite) representation of the Lorentz group. Remember from your group theory course that the generators of any algebra (in this case M^{ab}) operating on any representation will transform it to a set of the same representation: That is M^{ab} acting on a scalar will yield another scalar. ~~and~~ That is while M^{ab} operating on a chiral spinor will yield another chiral spinor (<u>Not</u> a vector or scalar) and so forth.

Remember that M^{ab} operating on a scalar field can be written as

$$M^{ab} = x^a \partial^b - x^b \partial^a$$ Notice that $M^{ab} = -M^{ba}$.

It is straigthforward to verify that such operators follow the following anti-commutator relations:

$$\text{①} \quad [M^{ab}, M^{cd}] = M^{ad}\eta^{bc} - M^{ac}\eta^{bd} - M^{bd}\eta^{ac} + M^{bc}\eta^{ad}$$

Again remember from your group theory course that the generators of a group have different matrix form operating on different representation. However, on all representations they maintain the same ~~anti~~ commutator relations (the same algebra).

Thus, the relation ① which we found using the format of M^{ab} in the scalar representation holds also for M^{ab} in spinor and vector representations.

Moreover, M^{ab} in any representation is anti-symmetric with respect to $a \leftrightarrow b$.

Remember that the Lorentz transformation includes both rotation and boost.

The rotation is given by spatial elements of M^{ab}:

$J^i \triangleq \frac{i}{2} \varepsilon^{ijk} M^{jk}$ where $i, j, k \in \{1, 2, 3\}$ i.e., $J^3 = iM^{12}$

Notice that J^i can be identified with angular momentum.

From (1), we find $[J^i, J^j] = i\varepsilon^{ijk} J^k$ which is the same anticommutator relation associated with angular momentum operator.

The boost is given by $(0\ i)$ elements of M^{ab}: $K^j \triangleq i M^{0j}$ where $j \in \{1, 2, 3\}$

Exercize: Using anticommutator relation (1), show that

$[K^i, J^j] = i\varepsilon^{ijk} K^k$

$[K^i, K^j] = -i\varepsilon^{ijk} K^k$

Let us define

$$J_+^i \triangleq \frac{1}{2}(J^i + iK^i) \qquad \text{and} \qquad J_-^i = \frac{1}{2}(J^i - iK^i) .$$

It is straightforward to verify that

$$[J_+^i , J_-^k] = 0 \qquad (J_+^i)^\dagger = J_-^i , \quad \cancel{\text{and}}$$

$$[J_+^i , J_+^j] = i\varepsilon^{ijk} J_+^k \qquad \text{and} \qquad [J_-^i , J_-^j] = i\varepsilon^{ijk} J_-^k .$$

Notice that J_+^i and J_-^j have the same algebra as the angular momentum so their finite dimensional representations should be similar to those of the angular momentum.

~~In the formalism of the MSSM, the representations $(0, 0)$, $(\frac{1}{2}, 0)$~~

A given finite-dimensional representation is labelled (j_+ , j_-)

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The simplest representations are

$(0,0)$ which is a scalar;

$(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ which are chiral spinors

and $(\frac{1}{2},\frac{1}{2})$ $(1,0)$ $(0,1)$ which are four-vectors.

In the formalism of the MSSM, the representations $(0,0)$, $(\frac{1}{2},0)$ and $(1,0)$ are extensively used.

Thus, here we work out them explicitly and develop the necessary tools to handle calculations involving them.

Let us now consider an infinitesimal Lorentz transformation which is characterized by a small rotation $(\theta_1, \theta_2, \theta_3)$ and a boost (η_1, η_2, η_3).

The transformation is given by

$$\left(1 + \frac{1}{2} \omega_{ab} M^{ab}\right) = \left(1 - i\vec{\theta}\cdot\vec{J} - i\vec{\eta}\cdot\vec{K}\right) \qquad (2)$$

Remember that $\vec{J} = J_+^i + J_-^i$ and $\vec{K} = -i(J_+^i - J_-^i)$

For $(\frac{1}{2},0)$ representation we have

$J_+^i = \frac{\sigma^i}{2}$ and $J_-^i = 0$ where $\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Thus, for this representation $J^i = \frac{\sigma^i}{2}$ and $K^i = i\frac{\sigma^i}{2}$

and therefore

$$\left(1 + \frac{1}{2}\omega_{ab} M^{ab}\right) = \left(1 - i\,\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\eta}\cdot\frac{\vec{\sigma}}{2}\right) \qquad (3)$$

This is the transformation law of a left-handed Weyl fermion. For example for a boost in the direction of the spin we would have

$$e^{-\vec{\eta}\cdot\frac{\vec{\sigma}}{2}} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{2E^-} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly, the representation $(0,\frac{1}{2})$ has

$J_+^i = 0$, $J_-^i = \frac{\vec{\sigma}^i}{2}$ which implies $J^i = \frac{\sigma^i}{2}$ and $K^i = -i\frac{\sigma^i}{2}$.

Therefore,

$$1 + \frac{1}{2}\omega_{ab} M^{ab} = 1 - i\,\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\eta}\cdot\frac{\vec{\sigma}}{2} \qquad (4)$$

This is the transformation law of a right-handed Weyl fermion.

Notice that (3) and (4) are the same up to the sign in front of $\vec{\eta}\cdot\vec{\sigma}/2$)

One of the matrices extensively used in this formalism is the so-called C matrix defined as

$$C = -i\sigma^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

exercize Verify that

$$C^{-1} = -C \qquad C^T = -C \qquad C^* = C \quad \text{and} \quad C\vec{\sigma}^T = -\vec{\sigma}C$$

To make calculations simpler later on, you had better memorize these simple relations!

Using the matrix C and by a complex conjugation the right-handed and left-handed representations can transform into each other; $(\frac{1}{2},0) = (0,\frac{1}{2})^*$. This can be understood by solving the following exercize.

exercize Show that if ψ transforms as $(0,\frac{1}{2})$, $C\psi^*$ transforms as $(\frac{1}{2},0)$. In other words show that if $\psi \to (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\eta}\cdot\frac{\vec{\sigma}}{2})\psi$,

$$C\psi^* \to (1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\eta}\cdot\frac{\vec{\sigma}}{2})C\psi^*$$

Now let us discuss the Lorentz transformation of four_vector Z^a.

The generators of the Lorentz transformation in this representation have the following form

$$(M^{ab})_{mn} = \delta^a_m \delta^b_n - \delta^a_n \delta^b_m \qquad (5)$$

exercize Show that $(M^{ab})_{mn}$ given in (5) obeys

$$([M^{ab}, M^{cd}])_{mp} = (M^{ad}\eta^{bc} - M^{ac}\eta^{bd} - M^{bd}\eta^{ac} + M^{bc}\eta^{ad})_{mp}.$$

(This is the same ~~anti~~-commutator relation as Eq. (1).)

Let us define

$$(\sigma^a)_{\alpha\beta} \triangleq (1, \vec{\sigma})^a_{\alpha\beta} \qquad (\bar{\sigma}^a)_{\alpha\beta} = (1, -\vec{\sigma})^a_{\alpha\beta}$$

exercize Verify that

$$C\sigma^a = (\bar{\sigma}^a)^T C \qquad C\bar{\sigma}^a = (\sigma^a)^T C$$

Also let us define: $\sigma^{ab} = \frac{1}{4}(\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a)$ $\quad \bar{\sigma}^{ab} = \frac{1}{4}(\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a)$

exercise Show that

$$\sigma^a \bar{\sigma}^b = \eta^{ab} + 2\sigma^{ab}$$

$$\bar{\sigma}^a \sigma^b = \eta^{ab} + 2\bar{\sigma}^{ab}$$

$$\sigma^a \bar{\sigma}^b \sigma^c = \eta^{ab}\sigma^c - \eta^{ac}\sigma^b + \eta^{bc}\sigma^a + i\varepsilon^{abcd}\sigma_d$$

$$\bar{\sigma}^a \sigma^b \bar{\sigma}^c = \eta^{ab}\bar{\sigma}^c - \eta^{ac}\bar{\sigma}^b + \eta^{bc}\bar{\sigma}^a - i\varepsilon^{abcd}\bar{\sigma}_d$$

From now on, we will present scalar fields with $\phi(x)$

Spinors belonging to $(\frac{1}{2}, 0)$ with $\psi_\alpha(x)$

and right-handed spinor belonging to $(0\ \frac{1}{2})$ wit $\chi(x)$.

We will show the vector fields with $A_a(x)$.

We will treat the fields as classical objects

However, one should bear in mind that

ψ's being fermions anti-commute:

$$\psi_1 \psi_2 = -\psi_2 \psi_1$$

$$(\psi_1 \psi_2)^* = \psi_2^* \psi_1^* = -\psi_1^* \psi_2^*$$

The kinetic term for the left-handed Weyl fermion ψ can be written as

$$\boxed{\int d^4x \, \psi^\dagger i \bar{\sigma}^a \partial_a \psi \qquad (6)}$$

That is while the kinetic term for the right-hand Weyl fermion (belonging to the $(0\ \frac{1}{2})$ representation) is

$$\boxed{\int d^4x \, \chi^\dagger i \sigma^a \partial_a \chi \qquad (7)}$$

Notice that while in (6) $(\bar{\sigma})$ appears in (7) σ shows up. Any ~~Lagrangian~~ term in Lagrangian has to be Lorentz invariant and Hermitian.

> **exercize** Show that (6) and (7) are Lorentz invariant and Hermitian.

To app~~re~~eciate the elegance of this, solve the following exercize.

exercize Show that $\int d^4x \, \psi^\dagger i\sigma^a \partial_a \psi$ and $\int d^4x \, \chi^\dagger i\bar{\sigma}^a \partial_a \chi$ are <u>NOT</u> Lorentz invariant.

Thus these combinations (despite including the derivative) cannot be a kinetic term in the Lagrangian. Try other combinations for yourself

exercize Show that $\psi_1 C \psi_2$ is Lorentz invariant. Find its Hermitian Conjugate.

exercize show that if we set $\chi = C\psi^*$ then

$$\int d^4x \, \chi^\dagger i\sigma^a \partial_a \chi = \int d^4x \, \psi^* i\bar{\sigma}^a \partial_a \psi$$

exercize Show that

$$\psi_1^T C \psi_2 = \psi_2^T C \psi_1$$

Using

$$\left\{ \begin{array}{l} Tr[\bar{\sigma}^b \sigma^a] = 2\eta^{ab} \\ Tr[\bar{\sigma}^b \eta \xi^\dagger] = -\xi^\dagger \bar{\sigma}^b \eta \end{array} \right\} \Rightarrow$$

Show that $\quad \eta_\alpha \xi_\beta^\dagger = -\frac{1}{2} (\sigma^a)_{\alpha\beta} (\xi^\dagger \bar{\sigma}_a \eta)$

Using the fact $C_{\alpha\beta}$ and $(\sigma^{ab} C)_{\alpha\beta}$

[remember that $C = -i\sigma_2$, $\sigma^{ab} = \frac{1}{4}(\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a)$

and $\bar{\sigma}^{ab} = \frac{1}{4}(\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a)$]

are a complete set of matrices, prove that

$$\eta_\alpha \xi_\beta^T = \frac{1}{2} C_{\alpha\beta} (\xi^T C \eta) - \frac{1}{2} (\sigma^{ab} C)_{\alpha\beta} (\xi^T C \sigma_{ab} \eta)$$

and

$$\eta_\alpha \xi_\beta^\dagger = \frac{1}{2} C_{\alpha\beta} (\xi^\dagger C \eta^*) - \frac{1}{2} (C \bar{\sigma}^{ab})_{\alpha\beta} (\xi^\dagger \bar{\sigma}_{ab} C \eta^*)$$

Part 2 of session 1

In the first part of this lecture, we just reviewed the Weyl two-component spin $\frac{1}{2}$ representation and Lorentz group in 4 dimension and introduced some useful mathematical identities.

Now, we finally introduce the charge operator of SUSY which converts fermion and boson states to each other:

$Q\,|\text{fermion}\rangle = |\text{boson}\rangle \qquad Q\,|\text{boson}\rangle = |\text{fermion}\rangle$

This means Q carries spin $\frac{1}{2}$. In fact Q is in $(\frac{1}{2}\ 0)$ presentation.

Lorentz transformation:

$$Q \longrightarrow \left(1 - i\,\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\eta}\cdot\frac{\vec{\sigma}}{2}\right) Q \qquad \text{therefore}$$

$$Q^{\dagger} \longrightarrow \left(1 + i\,\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\eta}\cdot\frac{\vec{\sigma}}{2}\right) Q^{\dagger} \qquad (1)$$

In general, one can have more than one pair of Q and $Q^{\dagger}$:

$$Q_{\alpha}^{i} \qquad Q_{\alpha}^{\dagger i} \qquad i = 1 \cdots N \qquad \alpha = 1, 2\,.$$

Part 2 of session 1

In the MSSM, N is equal to 1. A SUSY model with $N>1$ is called "extended" SUSY

To have conserved SUSY charge, in the context of a relativistic quantum field theory, we must have

$[Q, H] = 0$

where H is the Hamiltonian.

In the following we show from very basic principles that the algebra of Q can be fixed.

Consider the following anti-commutator

$\{Q_\alpha, Q^\dagger_\beta\}$.

Since $\sigma^a = (1, \vec{\sigma})$ is a complete set of 2×2 matrices, without loss of generality we can write

$$\{Q_\alpha, Q^\dagger_\beta\} = 2\,\sigma^a_{\alpha\beta} R_a. \quad (2)$$

exercize Using the relations in (1) show that under Lorentz transformations, R_a transforms as a four-vector.

In the following we show that $[R_a, H] = 0$ and $R_a \neq 0$.

However before proving these claims, we review Jacobi and "super-Jacobi" identities which will be useful for the proof of $[R_a, H] = 0$ and elsewhere throughout this course.

Jacobi identity

Remember that at the high school we learnt the simple Jacobi identity which holds between commutators of any three matrices or operators:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \qquad (3)$$

exercize Prove the Jacobi identity shown in Eq. (3)
[Hint: just expand the commutators.]

Notice that Eq. (3) holds regardless of whether the operators are fermionic or bosonic.

However, if some of these operators are fermions, we will in practice need identities which involve both commutators and anti-commutators.

Such identities are known as 'super-Jacobi' identities. The following is the recipe to write the super-Jacobi identities

1) Write the Jacobi identity of A, B, C.
2) If there is a commutator of two fermionic operators, replace commutator with anti-commutator.
3) If the order of the fermionic operators is reversed, multiply that term by minus.

For example we can write

$$[B,\{F_1,F_2\}]+\{F_1,[F_2,B]\}-\{F_2,[B,F_1]\}=0 \qquad (4)$$

where F_1, F_2 are fermionic

exercize Prove Eq. (4).

Part 2 of session 1

exercize Using the Jacobi identity and $[H, Q] = [H, Q^\dagger] = 0$ Prove that $[R_a, H] = 0$. This means R_a is conserved

In the following we show that R_a is nonzero.

Consider an arbitrary state $|\psi\rangle$. We can write

$$\langle\psi|\{Q_\alpha, Q_\alpha^\dagger\}|\psi\rangle = \langle\psi|Q_\alpha Q_\alpha^\dagger|\psi\rangle + \langle\psi|Q_\alpha^\dagger Q_\alpha|\psi\rangle$$

$$= \| Q_\alpha^\dagger|\psi\rangle\|^2 + \| Q_\alpha|\psi\rangle\|^2$$

Thus $R_a \neq 0$ unless for any $|\psi\rangle$ $Q_\alpha|\psi\rangle = Q_\alpha^\dagger|\psi\rangle = 0$

Since we want a non-trivial symmetry at least for one $|\psi\rangle$ $Q_\alpha^\dagger, Q_\alpha|\psi\rangle \neq$ ~~so~~ so R_a must be nonzero.

So, we have proved that R_a is nonzero and is conserved.