Classification of the Geodesics and Noether Conservation Laws for some Specific Kaluza-Klein Solutions of Rotating Cylindrically Symmetric Fluid Models

Fatemeh Ahangari

Department of Mathematics, Faculty of Basic Sciences, Alzahra University, Vanak, Tehran, Iran.

Abstract

In this paper, the problem of symmetries and conservation laws for some specific solutions of Kaluza-Klein field equations associated to stationary symmetric fluid models in standard Einstein theory is analyzed. For this purpose, some specific physically viable stationary Kaluza-Klein perfect fluid solutions are considered and the corresponding point generators of one parameter Lie groups of transformations that leave invariant the action integral associated to the Lagrangian, viz., Noether symmetries are computed. Mainly, a complete classification of the associated geodesic equations is presented. Moreover, all the corresponding conservation laws of the Euler-Lagrange (geodesic) equations concluded from the obtained Noether symmetries are calculated.

Key words: Kaluza-Klein model, Rotating fluids, Noether symmetries, Geodesics, Conservation laws.
PACS numbers: 11.30.-j, 04.50.Cd, 47.75.+f

1 Introduction

After Godel [2] proposed relativistic model of a rotating dust universe, the investigation of rotating fluids in the context of general relativity received notable consideration (see [4] and references therein). In 1996, Davison [1] reported a one-parameter set of solutions for a fluid admitting the equation of state $p = (2/3)\rho$, rotating about a regular axis. Considering the fact that stationary Kaluza-Klein perfect fluid models in standard Einstein theory are not available in literature, obtaining and analyzing such solutions is so worthwhile in order to investigate the effects of dimensionality on the different physical parameters. In [4], R. Tikekar and L. K. Patel have formulated the Kaluza-Klein field equations for cylindrically symmetric rotating distributions of perfect fluid. They have reported a set of physically viable solutions which is believed to be the first such Kaluza-Klein solutions and it includes the Kaluza-Klein counterpart of Davidson's solution.
In the following, according to [4], we will present a brief description of Kaluza-Klein field equations for stationary cylindrically symmetric fluid models in standard Einstein theory.
A general stationary cylindrically symmetric five dimensional spacetime is denoted by the following metric:

$$ds^2 = D^2(dt + Hd\phi)^2 - A^2dr^2 - B^2dz^2 - r^2C^2d\phi^2 - E^2d\psi^2, \tag{1.1}$$

where t is the time coordinate, r, z and ϕ are cylindrical polar coordinates, ψ represents the coordinate corresponding to the extra spatial dimension and A, B, C, D and H are functions of the radial coordinate r only. By expressing

⋆ Assistant Professor of Department of Mathematics, Alzahra University, 1993893973, Tehran, Iran.
Tel. +9821-88044040. Fax +9821-88035187.
Email: fa_ahangari@iust.ac.ir , fa.ahangari@gmail.com

with respect to pentad

$$\theta^1 = Adr, \qquad \theta^2 = Bdz, \qquad \theta^3 = rCd\phi, \qquad \theta^4 = Ed\psi, \qquad \theta^5 = D(dt + Hd\phi), \tag{1.2}$$

the metric (1.1) has the following form:

$$ds^2 = (\theta^5)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 - (\theta^4)^2. \tag{1.3}$$

If the metric (1.1) is to denote the spacetime of a stationary perfect fluid rotating about the regular axis $r = 0$, the metric coefficients will be related to the dynamical variables through the Einstein field equations which are in the pentad notation applying the system of units rendering $c = G = 1$, adopted in the form

$$\mathbf{R}_{(ab)} = -8\pi\left[(\rho + p)v_{(a)}v_{(b)} - \frac{1}{3}(\rho - p)g_{(ab)}\right], \tag{1.4}$$

Here, v_a represents components in the pentad frame of the unit time-like flow vector v^i of the fluid, which satisfies $v^i v_i = 1$. Also, ρ, p denote the matter density and the fluid pressure, respectively. It is more convenient to adapt a coordinate comoving with the observer, consequently, $v^a = (0, 0, 0, 0, 1)$, and the field equations (1.4) imply the following system of equations:

$$\mathbf{R}_{(11)} = \mathbf{R}_{(22)} = \mathbf{R}_{(33)} = \mathbf{R}_{(44)} = -\frac{8\pi}{3}(\rho - p), \tag{1.5}$$

$$\mathbf{R}_{(55)} = -\frac{16\pi}{3}(\rho + 2p), \tag{1.6}$$

$$\mathbf{R}_{(35)} = 0, \tag{1.7}$$

The field equations constitute a system of six equations relating the two physical parameters ρ and p of the fluid and the six metric coefficients A, B, C, D, E and H. The system of equations (1.5) and (1.6) leads to the following consistency conditions

$$\mathbf{R}_{(11)} = \mathbf{R}_{(22)} = \mathbf{R}_{(33)} = \mathbf{R}_{(44)} \tag{1.8}$$

Davidson [1], obtained a solution of the relativistic system of field equations for a perfect fluid in rigid rotation about a regular axis. His solution, suggests the possibility that the system of Kaluza-Klein field equations (1.5)-(1.7) can be solved by assuming the following form for the metric coefficients A, B, C, D, E and H,

$$\begin{aligned} &A = (1 + k^2r^2)^a, \qquad B = (1 + k^2r^2)^b, \qquad C = (1 + k^2r^2)^c, \\ &D = (1 + k^2r^2)^d, \qquad E = (1 + k^2r^2)^e, \end{aligned} \tag{1.9}$$

where a, b, c, d, e and k are constants. Note that expressions (1.9) ensure the regularity of the metric for all finite r. Equation $\mathbf{R}_{(22)} = \mathbf{R}_{(44)}$ in (1.8) is then satisfied if and only if

$$b = e. \tag{1.10}$$

Subsequently, (1.7) results the following two relations:

$$H = \alpha r^2, \tag{1.11}$$

$$a + c = 2b + 3d, \tag{1.12}$$

where α is the arbitrary constant of integration. Equation $\mathbf{R}_{(11)} = \mathbf{R}_{(33)}$ contained in (1.8) in view of (1.10) and (1.12) reduces to the following algebraic relation

$$2b^2 + 2b(1 + 4d) + d(1 + 2d) = 0. \tag{1.13}$$

Hence, the Kaluza-Klein field equations are equivalent to the five algebraic relations (1.10)-(1.13), relating the seven parameters a, b, c, d, e, α and k with $H(r)$ as determined by (1.11).

In [4] certain specific cases for physical relevance which follow for certain particular choices of the free parameters, are discussed. In this paper, we will comprehensively analyze the problem of symmetries and conservation laws for the following stationary Kaluza-Klein perfect fluid solution which is of special physical significance.

In the specific case, when $a = -1/2$, $b = e = c = -d = 1/4$, $\alpha^2 = k^2$, the Kaluza-Klein equations are all satisfied and the spacetime of this class of solutions has the following metric:

$$\begin{aligned} ds^2 &= (1+k^2r^2)^{-1/2}(dt+kr^2d\phi)^2 - (1+k^2r^2)^{-1}dr^2 \\ &\quad - (1+k^2r^2)^{1/2}\big(dz^2 + r^2d\phi^2 + d\psi^2\big). \end{aligned} \tag{1.14}$$

which denotes a five dimensional spacetime of a cylindrically symmetric stationary fluid with constant density and pressure related by this equation of state: $\rho + p = 0$. By setting $\Lambda = -(3/2)k^2$, the metric above denotes a five dimensional solutions of the field equations: $\mathbf{R}_{ij} = \Lambda g_{ij}$, where Λ represents the cosmological constant.

This paper is organized as follows: In the next section, we will present a complete investigation of the problem of symmetries and conservation laws for this specific Kaluza-Klein solution mentioned above. First of all, by considering the Lagrangian which is determined directly from the metric, we will compute the geodesic equations as the Euler Lagrange equations. Secondly, we obtain the point generators of the one parameter Lie groups of transformations that leave invariant the action integral corresponding to the Lagrangian (Noether symmetries). Moreover, we will calculate all the associated conservation laws of the geodesic equations, which are resulted from the obtained Noether symmetries via the celebrated Noether's theorem.

2 Computation of the Noether symmetries and classification of geodesics

The Lagrangian for the metric (1.14) is:

$$\begin{aligned} L = &\frac{\dot{t}^2}{\sqrt{1+k^2r^2}} - \frac{\dot{r}^2}{1+k^2r^2} - \sqrt{1+k^2r^2}\,\dot{z}^2 + \left(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2}\right)\dot{\phi}^2 \\ &+ \frac{2kr^2}{\sqrt{1+k^2r^2}}\,\dot{t}\dot{\phi} - \sqrt{1+k^2r^2}\,\dot{\psi}^2. \end{aligned} \tag{2.15}$$

Consequently, the corresponding simplified Euler-Lagrange equations are the geodesic equations given by:

$$\begin{cases} \ddot{t} + \dfrac{k^2r(4k^2r^2+7)}{(k^2r^2+1)(4k^2r^2+1)}\,\dot{t}\dot{r} = 0, \\ \ddot{r} - \dfrac{k^2r}{2\sqrt{1+k^2r^2}}\,\dot{t}^2 + \dfrac{2kr(k^2r^2+2)}{\sqrt{1+k^2r^2}}\,\dot{t}\dot{\phi} - \dfrac{k^2r}{1+k^2r^2}\,\dot{r}^2 - \dfrac{1}{2}\sqrt{1+k^2r^2}\,k^2r\,\dot{z}^2 \\ \quad -\dfrac{r(k^2r^2+2)}{2\sqrt{1+k^2r^2}}\,\dot{\phi}^2 - \dfrac{1}{2}\sqrt{1+k^2r^2}\,k^2r\,\dot{\psi}^2 = 0, \\ \ddot{z} + \dfrac{k^2r}{1+k^2r^2}\,\dot{r}\dot{z} = 0, \\ \ddot{\phi} - \dfrac{4k}{r(1+4k^2r^2)}\,\dot{t}\dot{r} + \dfrac{k^2r^2+2}{r(1+k^2r^2)}\,\dot{r}\dot{\phi} = 0, \\ \ddot{\psi} + \dfrac{k^2r}{1+k^2r^2}\,\dot{r}\dot{\psi} = 0. \end{cases} \tag{2.16}$$

By applying the general Lie symmetry method [3], we obtain the determining (partial differential) equations for seven unknown functions ξ, η^μ and A, where each of these is a function of six variables, i.e. s, t, r, z, ϕ and ψ. Solving these equations for the metric (1.14), it is concluded that:

The Lie group of Noether symmetries corresponding to the Kaluza-Klein solution (1.14) has a Lie algebra which is generated by the following vector fields: $\mathbf{X} = \xi\frac{\partial}{\partial s} + \eta^1\frac{\partial}{\partial t} + \eta^2\frac{\partial}{\partial r} + \eta^3\frac{\partial}{\partial z} + \eta^4\frac{\partial}{\partial \phi} + \eta^5\frac{\partial}{\partial \psi}$, where

$$
\begin{aligned}
&\xi(s,t,r,z,\phi,\psi) = c_1, \qquad \eta^1(s,t,r,z,\phi,\psi) = c_5 z + c_6\psi + c_9, \qquad \eta^2(s,t,r,z,\phi,\psi) = 0,\\
&\eta^3(s,t,r,z,\phi,\psi) = c_5 t + c_3\psi + c_4, \qquad \eta^4(s,t,r,z,\phi,\psi) = c_5 kz + c_6 k\psi + c_8,\\
&\eta^5(s,t,r,z,\phi,\psi) = -c_3 z + c_6 t + c_7, \qquad A(s,t,r,z,\phi,\psi) = c_2.
\end{aligned}
\tag{2.17}
$$

and $c_i,\ i = 1,...,9$ are arbitrary constants.

Hence, we obtain the eight dimensional Lie algebra of Noether point symmetries with the following basis:

Corollary 2.1 *Infinitesimal generators of every one parameter Lie group of Noether symmetries associated to (1.14) are as follows:*

$$
\begin{aligned}
&\mathbf{X_1} = \frac{\partial}{\partial s}, \qquad \mathbf{X_2} = \frac{\partial}{\partial t}, \qquad \mathbf{X_3} = \frac{\partial}{\partial z},\\
&\mathbf{X_4} = \frac{\partial}{\partial \phi}, \qquad \mathbf{X_5} = \frac{\partial}{\partial \psi}, \qquad \mathbf{X_6} = -\psi\frac{\partial}{\partial z} + z\frac{\partial}{\partial \psi},\\
&\mathbf{X_7} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} + kz\frac{\partial}{\partial \phi}, \qquad \mathbf{X_8} = \psi\frac{\partial}{\partial t} + k\psi\frac{\partial}{\partial \phi} + t\frac{\partial}{\partial \psi},\\
&A(s,t,r,z,\phi,\psi) = c \qquad (constant).
\end{aligned}
$$

whose nonzero commutators are:

$$
\begin{aligned}
&[X_2, X_7] = X_3, \qquad [X_2, X_8] = X_5, \qquad [X_3, X_6] = -X_5,\\
&[X_3, X_7] = X_2 + kX_4, \qquad [X_5, X_6] = X_3, \qquad [X_5, X_8] = X_2 + kX_4,\\
&[X_6, X_7] = X_8, \qquad [X_6, X_8] = -X_7, \qquad [X_7, X_8] = -X_6.
\end{aligned}
$$

where $[X_i, X_j] = X_iX_j - X_jX_i, \quad i,j = 1,...,8.$

As is well known, the theoretical Lie group method plays a significant role in obtaining exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, so by constructing an optimal system of above resulted Noether symmetries, a complete classification of the associated geodesics will be deduced.

3 Computation of the conservation laws

The main significance of Noether symmetries is clear from the celebrated Noether's theorem. According to this theorem, there is a procedure which relates the constants of the motion of a given Lagrangian system to its symmetry transformations. Lie symmetries of the system of the geodesic equations for a spacetime yield conserved quantities. On the other hand, the symmetries of a Lagrangian yield directly the conserved quantities which are of our interest. In the following, we will compute all the conserved vectors corresponding to the obtained Noether symmetries. Each of these resulted conserved quantities yields a conservation law for the system of geodesic equations (2.16).

For example, for the Noether symmetry $\mathbf{X_7} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} + kz\frac{\partial}{\partial \phi}$, we get the following conserved vector:

$$
\begin{aligned}
T^7 =& \xi L + (\eta^1 - \dot{t}\xi)\frac{\partial L}{\partial \dot{t}} + (\eta^2 - \dot{r}\xi)\frac{\partial L}{\partial \dot{r}} + (\eta^3 - \dot{z}\xi)\frac{\partial L}{\partial \dot{z}} + (\eta^4 - \dot{\varphi}\xi)\frac{\partial L}{\partial \dot{\phi}} + (\eta^5 - \dot{\psi}\xi)\frac{\partial L}{\partial \dot{\psi}} - A\\
=& z\left(\frac{2\dot{t}}{\sqrt{1+k^2r^2}} + \frac{2kr^2\dot{\phi}}{\sqrt{1+k^2r^2}}\right) - 2t\sqrt{1+k^2r^2}\ \dot{z}\\
&+kz\left(2\Big(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2}\Big)\ \dot{\phi} + \frac{2kr^2\ \dot{t}}{\sqrt{1+k^2r^2}}\right) - c
\end{aligned}
$$

Similarly, we have computed the conserved vectors corresponding to the other Noether symmetries. The results are presented in Table (1).

Table 1
Conservation laws of the system of geodesic equations (2.16)

	Noether Symmetry	Conserved Vectors
1	$\mathbf{X_1} = \partial s$	$T^1 = -\frac{\dot{t}^2}{\sqrt{1+k^2r^2}} + \frac{\dot{r}^2}{1+k^2r^2} + \sqrt{1+k^2r^2}\ \dot{z}^2 - \frac{2kr^2}{\sqrt{1+k^2r^2}}\ \dot{t}\dot{\phi}$ $-\left(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2}\right)\dot{\phi}^2 + \sqrt{1+k^2r^2}\ \dot{\psi}^2 - c$
2	$\mathbf{X_2} = \partial t$	$T^2 = \frac{2\dot{t}}{\sqrt{1+k^2r^2}} + \frac{2kr^2\ \dot{\phi}}{\sqrt{1+k^2r^2}} - c$
3	$\mathbf{X_3} = \partial z$	$T^3 = -2\sqrt{1+k^2r^2}\ \dot{z} - c$
4	$\mathbf{X_4} = \partial \varphi$	$T^4 = 2\left(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2}\right)\dot{\phi} + \frac{2kr^2\ \dot{t}}{\sqrt{1+k^2r^2}} - c$
5	$\mathbf{X_5} = \partial \psi$	$T^5 = -2\sqrt{1+k^2r^2}\ \dot{\psi} - c$
6	$\mathbf{X_6} = \psi\partial z - z\partial\psi$	$T^6 = -2\psi\sqrt{1+k^2r^2}\ \dot{z} + 2z\sqrt{1+k^2r^2}\ \dot{\psi} - c$
7	$\mathbf{X_7} = z\partial t + t\partial z + kz\partial\phi$	$T^7 = z\left(\frac{2\dot{t}}{\sqrt{1+k^2r^2}} + \frac{2kr^2\dot{\phi}}{\sqrt{1+k^2r^2}}\right) - 2t\sqrt{1+k^2r^2}\ \dot{z}$ $+kz\left(2\left(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2}\right)\ \dot{\phi} + \frac{2kr^2\ \dot{t}}{\sqrt{1+k^2r^2}}\right) - c$
8	$\mathbf{X_8} = \psi\partial t + k\psi\partial\phi + t\partial\psi$	$T^8 = \psi\left(\frac{2\dot{t}}{\sqrt{1+k^2r^2}} + \frac{2kr^2\dot{\phi}}{\sqrt{1+k^2r^2}}\right) - 2t\sqrt{1+k^2r^2}\ \dot{\psi}$ $+k\psi\left(2\left(\frac{k^2r^4}{\sqrt{1+k^2r^2}} - r^2\sqrt{1+k^2r^2}\right)\ \dot{\phi} + \frac{2kr^2\ \dot{t}}{\sqrt{1+k^2r^2}}\right) - c$

Conclusion

In this paper, we have comprehensively analyzed the problem of symmetries and conservation laws for some specific solutions of Kaluza-Klein field equations for stationary symmetric fluid models in standard Einstein theory. For this purpose, we have considered some spesific physically viable stationary Kaluza-Klein perfect fluid solutions. First of all, by considering the Lagrangian which is determined directly from the metric, we have computed the corresponding geodesic equations as the Euler Lagrange equations. Secondly, we have obtained all the corresponding Noether symmetries. Moreover, we have calculated all the associated conservation laws of the geodesic equations, which are resulted from the obtained Noether symmetries.

References

[1] W Davidson, *Class. Quant. Gravit.* **13**, 282 (1996).

[2] K Godel, *Rev. Mod. Phys.* **21**, 447 (1949).

[3] N H Ibragimov, *Elementary Lie group analysis and ordinary differential equations* (Wiely, Chichester, 1999).

[4] R Tikekar and L K Patel, *Pramana Journal of Physics.* **55**, 361 (2000).