Non-Markovian dynamics of one spin 1/2 particle embedded in layered bath

M.Mahdian, H.Mehrabpour

Department of Theoretical Physics and Astrophysics, University of Tabriz, Tabriz 51664, Iran.

E-mail: `Mahdian@tabrizu.ac.ir`

Abstract. We investigate the exact evolution of the reduced dynamics of a one qubit system as central spin coupled to a femionic layered environment with unlimited number of layers.

1. Introduction

Every quantum system encountered in the real world is an open quantum system and the theory of open quantum systems describes how a system of interest is influenced by the interaction with its environment. This interaction often leads to a loss of the quantum features of physical states and has a great impact on the dynamical behavior of the open system due to the non-unitary characteristic of the time evolution, although much care is taken experimentally to eliminate the unwanted influence of external interactions, there remains, if ever so slight, a coupling between the system of interest and the external world[1, 2]. One kind of open quantum systems study to describe the information extraction process in quantum information are the spin star systems, where a central spin -$\frac{1}{2}$ particle couples to a spin bath of N spin-$\frac{1}{2}$ particles and they have attracted a vast amount attention in the quantum community [3]-[5] because they are of significance and of interest due to their high symmetry, strong non-Markovian behavior and also as one of the best candidates of the spin-qubit quantum computation[4]-[6]. This is even more relevant when environmental influences of a non-Markovian nature, such as those due to memory-keeping and feedback-inducing system-environment mechanisms, are considered[1].

Motivated by this consideration, in this paper, we consider layered environment with a spin at the center of layers to study a generalized spin star system which can be solved exactly. It must be noted that in the model, degeneracy for coupling coefficients are considered.

2. Exact Dynamics

We consider a spin star configuration which consists of N+1 localized spin-$\frac{1}{2}$ particles. One of the spins is located at the center of the star, while the others are on concentric circles with different radii surrounding the central spin, layer by layer the difference in radius is because the coupling coefficients between layers spins and the central spin are taken differently. It must be noted that in this model degeneracy for coupling coefficients is considered, because naturally some of spins are in relation with the central spin by a constant coupling coefficient which are located in one layer. By considering such model, the most general model of single-qubit spin-Star for fermionic particles with fixed fermionic environment is made, Fig(1). However, we consider our model by the below description and explanation. The central spin σ interacts with the bath spins $\sigma^{(j)}$ via a Heisenberg XX interaction [7]represented through the Hamiltonian

$$H = 2(\sigma_+\Xi_- + \sigma_-\Xi_+), \tag{1}$$

where $\Xi_\pm$ are denoted as follows

$$\Xi_+ = \sum_{\mu=1}^{n} \alpha_\mu J_+^\mu, \tag{2}$$

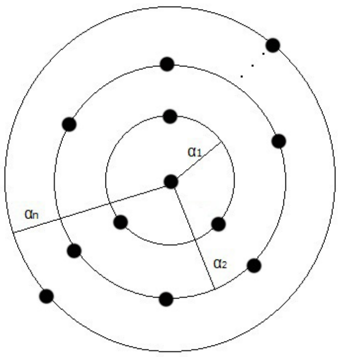

Figure 1. The figure depicts the general layered environment with one spin in the center of layers.

$$\Xi_- = \sum_{\mu=1}^{n} \alpha_\mu J_-^\mu. \tag{3}$$

Here, α coefficients specify interaction between system and environment and is dependent from distance. Also we have

$$J_\pm^\mu \equiv \sum_{j=1}^{N_\mu} \sigma_\pm^j, \tag{4}$$

here, $\mu = 1, 2, \ldots, n$ for n different layers of bath. Also, we have

$$\sigma_\pm^j \equiv \frac{1}{2}(\sigma_1^j \pm i\sigma_2^j),$$

that represents the raising and lowering operators of the jth bath spin. Equation(1) describes a very simple time independent interaction with equal coupling strength α_1 for N_1 of first bath spin and α_2 for N_2 of second bath spin to α_n for N_n of nth bath spin. It is invariant under rotation around the $z-axis$. The operator $J \equiv \frac{1}{2}\sum_{\mu=1}^{N} \sigma^\mu$ represents the total spin angular momentum of the bath (units are chosen such that $\hbar = 1$). Therefore the central spin thus couples to the collective bath angular momentum.

We introduce an orthonormal basis in the bath Hilbert space H_B consisting of states $|j_\mu, m_\mu, x\rangle$ where μ is 1 to n. These states are defined as eigenstates of $\mathbf{J}_3$ (eigenvalue m) and of $\mathbf{J}^2$ (eigenvalue j(j+1)). The index x labels the different eigenstates in the eigenspace $\boldsymbol{K}_{j,m}$ belonging to a given pair (j,m) of quantum numbers. As usual, $j_\mu \le \frac{N_\mu}{2}$ and $-j_\mu \le m \le j_\mu$ where μ is 1 to n.

$$\Upsilon(j_\mu, N_\mu) = \begin{pmatrix} N_\mu \\ \frac{N_\mu}{2} - j_\mu \end{pmatrix} - \begin{pmatrix} N_\mu \\ \frac{N_\mu}{2} - j_\mu - 1 \end{pmatrix}. \tag{5}$$

We assume that the initial state of the composite system be a product state. That is

$$\rho(0) = \rho_S(0) \otimes \rho_B(0). \tag{6}$$

We can calculate the reduced density matrix of the quantum system in the following expression.

$$\rho_S(t) = tr_B(U\rho(0)U^\dagger). \tag{7}$$

The above equation is obtained by doing partial-trace on bath and also U is the unitary operator which is defined as follows

$$U = \exp(-iHt).$$

The reduced density matrix is completely determined in terms of the Bloch vector

$$\chi(t) = \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{pmatrix} \equiv Tr(\sigma\rho_S(t)), \tag{8}$$

through the relationship

$$\rho_S(t) = \frac{1}{2}\begin{pmatrix} 1+\omega_3(t) & \omega_1(t) - i\omega_2(t) \\ \omega_1(t) - i\omega_2(t) & 1-\omega_3(t) \end{pmatrix}, \tag{9}$$

The initial state of the reduced system at t=0 is taken to be an arbitrary (possibly mixed) state

$$\rho_S(0) = \begin{pmatrix} \frac{1+\omega_3(0)}{2} & \omega_-(0) \\ \omega_+(0) & \frac{1-\omega_3(0)}{2} \end{pmatrix}, \tag{10}$$

while the spin bath is assumed to be in an unpolarized infinite temperature state:

$$\rho_B(0) = 2^{-N} I_B. \tag{11}$$

Here, I_B denotes the unit matrix in H_B and N is $N_1 + N_2 + \ldots + N_n$, and we have defined the $\omega_\pm$ as linear combinations of the components $\omega_{1,2}$ of the Bloch vector

$$\omega_\pm = \frac{\omega_1 \pm i\omega_2}{2}. \tag{12}$$

2.1. Reduced System Dynamics

In this section, we will derive the exact dynamics of the reduced density matrix $\rho_S(t)$ for our given model. We obtain the evolution of central spin with n different coupling coefficients that should be used from Eq.(7) until the solution model is exact. This yeilds

$$\rho_S(t) = tr_B\{i^l \sum_{l=0}^{\infty}\sum_{n=0}^{l} \frac{t^l}{l!}(-1)^n \begin{pmatrix} l \\ n \end{pmatrix} H^n(\rho_S(0) \otimes \frac{I_B}{2^N})H^{l-n}\}. \tag{13}$$

It can easily be verified that

$$H^{2k} = 4^k[\sigma_+\sigma_-(\Xi_-\Xi_+)^k + \sigma_-\sigma_+(\Xi_+\Xi_-)^k], \tag{14}$$

and

$$H^{2k+1} = 2 \cdot 4^k[\sigma_+\Xi_- + \sigma_-\Xi_+]. \tag{15}$$

We note that such simple expressions are obtained since a term $\sigma_3 J_3$ is missing in the interaction Hamiltonian. We substitute the last two equations into Lindblad equation[4] as follows

$$L^l\rho = i^l\sum_{n=0}^{l}(-1)^n\left(\begin{array}{c} l \\ n \end{array}\right)H^n\rho H^{l-n}, \tag{16}$$

to get the formulas

$$tr_B\{\ L^{2l+1}\rho_S(0)\otimes\ 2^{-N}I_B\} = 0, \tag{17}$$

and

$$tr_B\{\ L^{2l}\rho_S(0)\otimes\ 2^{-N}I_B\} = \sum_{l=0}^{\infty}(-4)^l\sum_{n=0}^{2l}\frac{t^{2l}}{(2l)!}\left(\begin{array}{c} 2l \\ 2n \end{array}\right)[(\frac{1+\omega_3\sigma_3}{2})\Omega_l + (\omega_+\sigma_- + \omega_-\sigma_+)\Gamma_n^{l-n}], \tag{18}$$

which hold for all l=1,2,.... Here,we have introduced the bath correlation functions

$$\Omega_l \equiv \frac{1}{2^N}tr_B\{(\Xi_+\Xi_-)^l\}, \tag{19}$$

$$\Gamma_n^{l-n} \equiv \frac{1}{2^N}tr_B\{(\Xi_+\Xi_-)^{l-n}(\Xi_-\Xi_+)^n\}, \tag{20}$$

where we have product $\Xi_\pm\Xi_\mp$ as follows

$$\Xi_\pm\Xi_\mp = (\sum_{\mu=1}^{n}\alpha_\mu J_\pm^\mu)(\sum_{\mu=1}^{n}\alpha_\mu J_\mp^\mu). \tag{21}$$

Of course, Eq.(18) is zero because it is

$$\langle j,m|J_\pm|j,m\rangle = 0.$$

We will come back to these correlation functions when we discuss approximation techniques in Sec.(4).

Using the formulas (18) and (19) in Eq.(14) we can express the components of the Bloch vector as follows,

$$\omega_\pm(t) = f_\pm(t)\omega_\pm(0), \tag{22}$$

$$\omega_3(t) = f_z(t)\omega_3(0), \tag{23}$$

where we have introduced the functions

$$f_\pm(t) \equiv tr_B\{\cos[2th_1(\alpha_1,\ldots,\alpha_n)]\cos[2th_2(\alpha_1,\ldots,\alpha_n)]\otimes 2^{-N}I_B\}, \tag{24}$$

and

$$f_z(t) \equiv tr_B\{\cos[2th_1(\alpha_1,\ldots,\alpha_n)]\otimes 2^{-N}I_B\}, \tag{25}$$

where $h_1(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $h_2(\alpha_1,\alpha_2,\ldots,\alpha_n)$ are

$$h_1(\alpha_1,\alpha_2,\ldots,\alpha_n) = \sqrt{\sum_{\mu=1}^{n}\alpha_\mu^2 J_+^\mu J_-^\mu}, \tag{26}$$

$$h_2(\alpha_1,\alpha_2,\ldots,\alpha_n) = \sqrt{\sum_{\mu}^{n}\alpha_\mu^2 J_-^\mu J_+^\mu}. \tag{27}$$

Calculating the traces over the spin bath in the eigenbasis of J_3 and $\mathbf{J}^2$ using

$$J_{\pm}J_{\mp}|j,m,x\rangle = (j \pm m)(j \mp m + 1)|j,m,x\rangle. \tag{28}$$

We find

$$f_{\pm}(t) = [\prod_{i=1}^{n} \sum_{j_i,m_i} \Upsilon(N_i,j_i)] \frac{\cos(4t\sqrt{\zeta})\cos(4t\sqrt{\eta})}{2^{\sum_{i=1}^{n} N_i}}, \tag{29}$$

and

$$f_z(t) = [\prod_{i=1}^{n} \sum_{j_i,m_i} \Upsilon(N_i,j_i)] \frac{\cos(4t\sqrt{\zeta})}{2^{\sum_{i=1}^{n} N_i}}, \tag{30}$$

here,ζ and η denoted are as

$$\zeta = \sum_{i=1}^{n} h_i^{+} \alpha_i^2,$$

and

$$\eta = \sum_{i=1}^{n} h_i^{-} \alpha_i^2,$$

and also, we have

$$h_i^{+} = h(j_i, m_i); h_i^{-} = h(j_i, -m_i),$$

where we have introduced the quantity $h(j, \pm m) = (j \pm m)(j \mp m + 1)$. Thus we have determined the exact dynamics of the reduced system.

3. References

[1] H.P.Beruer, F.Petruccione, The Theory Of Open Quantum System (Oxford University Press, Oxford, 2002).

[2] E.C.G.Sudarshan, P.M.Mathews and J.Rau, Stochastic dynamics of quantum-mechanical systems , Phys. Rev. 121, 920 (1961).

[3] A.Hutton, S.Bose, Mediated entanglement and correlations in a star network of interacting spins , Phys. Rev. A 69 042312 (2004).

[4] H.P.Breuer, D.Burgerth, F.Petruccione, Non-Markovian dynamics in a spin star system: Exact solution and approximation techniques , Phys. Rev. B 70 045323 (2004).

[5] V.Semin, I.Sinayskiy and F.Petruccione, Initial correlation in a system of a spin coupled to a spin bath through an intermediate spin , Phys. Rev. A. 86, 062114 (2012).

[6] D.Rossini, P.Facchi, R.Fazio, G.Florio, D.A.Lidar, S.Pascazio, F.Plastina, P.Zanardi, Bang-Bang control of a qubit coupled to a quantum critical spin bath , Phys. Rev. A 77(5), 052112 (2008)

[7] E.H.Lieb, T.D.Shultz, D.C.Mattis, Ann. Phys. (N.Y.) 16 407 (1961) also reprinted in The Many-Body Problem by D. C. Mattis (World Scientific, Singapore, 1993).