Investigation of Chaos in the Relativistic Restricted Three-Body Problem

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Abstract

Chaos usually occurs in many complex nonlinear systems. Its detection is of importance, as it is an indication of how the system trajectories behave in future finite time spans. In this paper the method of Fast Lyapunov Indicator (FLI) is used for chaotic investigation of the planar circular relativistic restricted three-body problem (RTBP)of the Earth-Moon system.In this respect the relativistic RTBP is initially developed. Subsequently the FLI is computed for selected regions of the phase space in orderto distinguish between the regular and chaotic domains. It is found that the difference between the Newtonian and Relativistic RTBP trajectories is greater for chaotic trajectories.

Keywords: Fast Lyapunov Indicator, Relativistic Restricted Three-body Problem, Phase Space, Chaotic domain, Regular domain

1. Introduction

The Earth-Moon RelativisticRestricted Three-Body Problem is studied for chaos. There have been several chaotic investigations over the Newtonian RTBP where the authors have utilized various detection schemes such as the FLI and the Poincare sections.Villac [1] used the FLI map for preliminary spacecraft trajectory design in a multi-body environment.Sάndor [2] used short time Lyapunov in the RTBP to investigate phase space structure of the circular as well as the elliptic three-body problem. Robutel and Laskar [3] used graphical maps based on various indicators to reveal some important dynamical phenomena. While the majority of researches focus on derivation of initial conditions leading to chaotic, regular or periodic motion, most have used Newtonian gravity law and classical physics to derive the pertinent equations of motion.

Wanex [4] was the first to derive the relativistic non-Newtonian equations of motion for the restricted three-body problem.Utilizing numerical solutions of these equations, he showed the difference between the Newtonian and the relativistic models for seven specific trajectories. Chaos detection in the relativistic RTBP has not been attempted yet by any researcher. Thus, this paper discusses application of the FLI method to study the structure of the phase space for chaos detection in the Relativistic Restricted Three-body Problem.

The paper is arranged as follows: The relativistic restricted three body problem and its differences with the Newtonian model are presented in section two.Section threeintroduces the concept of the FLI map and its applications to chaos detection. Pertinent results of the relativistic model as well as conclusion and future research directions are provided in section five.

2. The Relativistic Restricted Three-Body Problem

Derivation of the relativistic restricted three body problem and its pertinent details are presented in this section.

2.1. The Definition of the Restricted Three-body Problem

The restricted three-body problem is a special case of the general motion of the three bodies moving under the gravitational influences produced by their respective masses. The dynamics of the problem are greatly simplified by considering the situation in which one of the three masses (hereafter called "test particle") is sufficiently small,such that its effect on the motion of the other two celestialbodies (hereafter called "primaries") is negligible.

The problem can be further simplified by assuming the motion of the two primary masses (such as the Earth-Moon system) to be circular about their center of mass (Barycenter). Further, the motion of the test particle in this system is constrained within the plane in which the primaries circular motion takes place, see figure 1. Subsequently, the objective of the RTBP is to determine the dynamics of the test particle as it moves under the gravitational influence of the primaries. This work is primarily focused on the idealized physical Earth-Moon system where the primaries have the masses of the Earth and moon respectively. Further the Earth-Moon distance and gravity parameters are taken from the 2000/2001 CRC handbook [5].

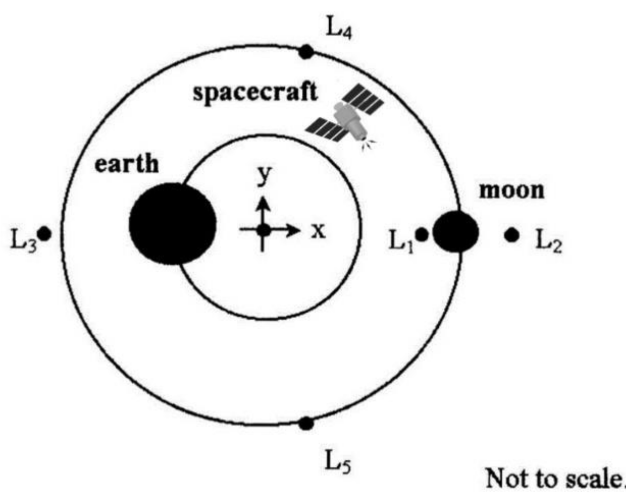


Fig. 1. Geometric description of the Circular Restricted Three Body Problem

2.2. The Relativistic Equations of Motion

In order to obtain the relativistic restricted three-body problem equations of motion, the components of the metric tensor are calculated using the post-Newtonian approximation as described by Weinberg [6]. The metric tensor components in a co-rotating frame are given in Eq.1 to 4 as follows:

$$g_{00} = c^2 - 2\left(\frac{GM_1}{r_e} + \frac{GM_2}{r_m}\right) + \frac{2}{c^2}\left(\frac{GM_1}{r_e} + \frac{GM_2}{r_m}\right)^2$$

$$\frac{+2}{c^2}\left(\frac{G^2M_1M_2}{r_e(R_1+R_2)} + \frac{G^2M_1M_2}{r_m(R_1+R_2)}\right) - \left(1 + \frac{2GM_1}{c^2r_e} + \frac{2GM_2}{c^2r_m}\right)[\omega^2(x^2+y^2)] \quad (1)$$

$$g_{01} = \left(1 + \frac{2GM_1}{c^2r_e} + \frac{2GM_2}{c^2r_m}\right)(2\omega y) \quad (2)$$

$$g_{02} = -\left(1 + \frac{2GM_1}{c^2r_e} + \frac{2GM_2}{c^2r_m}\right)(2\omega x) \quad (3)$$

$$g_{ij} = -\left(1 + \frac{2GM_1}{c^2r_e} + \frac{2GM_2}{c^2r_m}\right)\delta_{ij}(i,j=1,2) \quad (4)$$

Where c is the velocity of light, r_e and $r_m$are the distances of the test particle from the Earth and Moon respectively. R_1andR_2 are the distances of the Earth and Moon from their center of mass or the Barycenter and M_1,M_2 denote the masses of the Earth and Moon ,see Figure 2.

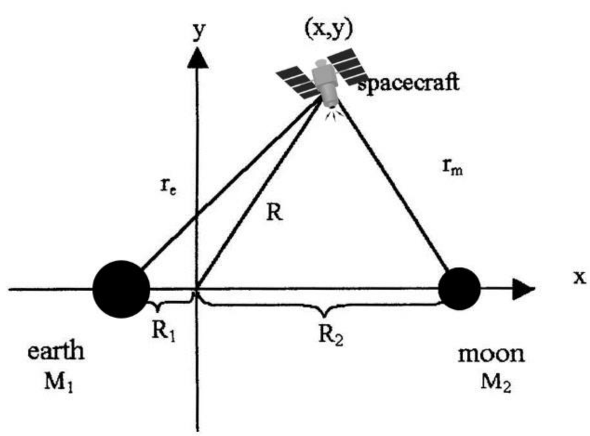


Fig. 2. Definitions of symbols given in Eq.1 to Eq.4

The Lagrangian is next obtained from the introduced metric tensors, [7]

$$L = \frac{1}{2}g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \quad (5)$$

Next the Euler-Lagrange equation will be utilized to extract the equations of motion.

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad (6)$$

Where the dot indicates differentiation with respect to proper time (τ). The equations of motion in proper time are for an observer on the test particle, and accurate to the order of $O(1/c^2)$ [4] :

$$\begin{aligned}\ddot{x} = {} & 2\omega\dot{y}\dot{t} + \omega^2 x\dot{t}^2 + \omega y\ddot{t} - G\left(\frac{M_1(x+R_1)}{r_e^3} + \frac{M_2(x-R_2)}{r_m^3}\right)\dot{t}^2 \\ & + \frac{2G}{c^2}\left\{\left(\frac{M_1[(x+R_1)\dot{x}+y\dot{y}]}{r_e^3} + \frac{M_2[(x-R_2)\dot{x}+y\dot{y}]}{r_m^3}\right)(\dot{x}-\omega y\dot{t})\right. \\ & + \left(\frac{M_1}{r_e} + \frac{M_2}{r_m}\right)(-\ddot{x} + 2\omega\dot{y}\dot{t} + \omega y\ddot{t} + \omega^2 x\dot{t}^2) \\ & + G\left[\left(\frac{M_1}{r_e} + \frac{M_2}{r_m}\right)\left(\frac{M_1(x+R_1)}{r_e^3} + \frac{M_2(x-R_2)}{r_m^3}\right)\right. \\ & \left. + \frac{M_1 M_2}{2(R_1+R_2)}\left(\frac{x+R_1}{r_e^3} + \frac{x-R_2}{r_m^3}\right)\right]\dot{t}^2 \\ & \left. - \left(\frac{M_1(x+R_1)}{2r_e^3} + \frac{M_2(x-R_2)}{2r_m^3}\right)(U^2 + \omega^2 R^2 + 2\omega\dot{t}A)\right\} \end{aligned} \quad (7)$$

$$\begin{aligned}\ddot{y} = {} & -2\omega\dot{x}\dot{t} + \omega^2 y\dot{t}^2 - \omega x\ddot{t} - G\left(\frac{M_1}{r_e^3} + \frac{M_2}{r_m^3}\right)y\dot{t}^2 \\ & + \frac{2G}{c^2}\left\{\left(\frac{M_1[(x+R_1)\dot{x}+y\dot{y}]}{r_e^3} + \frac{M_2[(x-R_2)\dot{x}+y\dot{y}]}{r_m^3}\right)(\dot{y}+\omega x\dot{t})\right. \\ & + \left(\frac{M_1}{r_e} + \frac{M_2}{r_m}\right)(-\ddot{y} - 2\omega\dot{x}\dot{t} - \omega x\ddot{t} + \omega^2 y\dot{t}^2) \\ & + G\left[\left(\frac{M_1}{r_e} + \frac{M_2}{r_m}\right)\left(\frac{M_1}{r_e^3} + \frac{M_2}{r_m^3}\right) + \frac{M_1 M_2}{2(R_1+R_2)}\left(\frac{1}{r_e^3} + \frac{1}{r_m^3}\right)\right]y\dot{t}^2 \\ & \left. - y\left(\frac{M_1}{2r_e^3} + \frac{M_2}{2r_m^3}\right)(U^2 + \omega^2 R^2\dot{t}^2 + 2\omega\dot{t}A)\right\} \end{aligned} \quad (8)$$

$$\dot{t} = \frac{k + A\omega}{c^2} + \frac{k\left(\frac{2GM_1}{r_e} + \frac{2GM_2}{r_m} + \omega^2 R^2\right)}{c^4} \quad (9)$$

Where t represents the time coordinate and K is a constant of motion. Also ;

$$U^2 = \dot{x}^2 + \dot{y}^2$$

$$A = \dot{y}x - \dot{x}y$$

$$R^2 = x^2 + y^2 \quad (10)$$

The angular frequency of the rotating frame, ω is in turn obtained for the relativistic two-body problem [8].

$$\omega = \sqrt{\frac{G(M_1+M_2)}{(R_1+R_2)^3}\left\{1 - \frac{3G(M_1+M_2)^2 - M_1 M_2}{c^2(R_1+R_2)(M_1+M_2)}\right\}} \quad (11)$$

The constant of motion arises because the coordinate time does not appear explicitly in the Lagrangian. This constant can be obtained from the initial conditions and is approximately equal to c^2.

2.3. The equations of motion in the Newtonian limit

If the derived equations of motion are derived correctly, they should be identical to the Newtonian restricted three-body problem,once extrapolated to the Newtonian limit ($c \to \infty$). It is easily shown that under this limit these equations match exactly with those of the planar Newtonian restricted three body problem given below [9]

$$\ddot{x} = 2\omega\dot{y} + \omega^2 x - G\left(\frac{M_1(x+R_1)}{r_e^3(t)} + \frac{M_2(x-R_2)}{r_m^3(t)}\right) \quad (12)$$

$$\ddot{y} = -2\omega\dot{x} + \omega^2 y - G\left(\frac{M_1}{r_e^3(t)} + \frac{M_2}{r_m^3(t)}\right)y \quad (13)$$

3. The FLI Map

There has been a growing interest in recent years for application and development of fast chaotic detection methods. Froesche' et al. (1993) [10] and Voglis and Contopoulos (1994) [11] introduced the idea of local Lyapunov characteristic numbers or stretching numbers. Contopoulos and Voglis (1996,1997) [12,13] developed the concept of helicity angles and twist angles. Froeschle' (1997) [14] introduced the method of the fast Lyapunov indicator (FLI) which is suitable even for weak chaotic situations. In addition, the rising trends of fast computers, numerical analysis techniques and graphical representation of the dynamical systems' phase space has produced many interesting new research in complex subjects of celestial mechanics and dynamical astronomy.In this respect Coffey et al. (1990) [15] used graphical representation of the integrals of the J_2 problem to obtain bifurcation information of the Erath orbiting system near critical inclination. In this study, the method of fast Lyapunov indicator (FLI) is utilized for the circular relativistic restricted three-body problem for chaotic investigation of the phase space structure in some pre-selected regions.

3.1. Fast Lyapunov Indicator (FLI)

Considering a trajectory $x(t); x(t_0) = x_0$ of a dynamical system in the form$\dot{x} = f(x,t)$, one can propagate a tangent vector v_0 along the trajectory by integrating the linearized dynamics $\acute{v} = Df\big(x(t)\big).v; v(t_0) = v_0$. At each time step, one can take the largest among the vectors, in order to define the FLI as:

$$FLI = sup_j log\left|v_j(t)\right| \quad (14)$$

While FLI tends to infinity as time increasesfor both ordered as well as chaotic orbits, however they exhibit completely different time rates allowing one to distinguish between the two cases.

3.2. Chaoticity maps

Given an indicator discriminating between a regular and chaotic motion, the representation of the indicator values over a set of initial conditions generates what can be called a “chaoticity map”. A typical example of chaoticity map generated in the case of the plane Earth-Moon relativistic restricted three body problem is provided in Fig 3. Since the map is generated via the prescribed fast Lyapunov indicator method, it is thus referred to as the FLI map. The region of the phase space selected for this map consists of initial conditions having a fixed Jacobi constant of 3.2, zero out-of-plane component ($z \,\&\, \dot{z}\ are\ zero$) and starting on the x axis (y=0). The darker highlighted regions of low FLI values correspond to mostly ordererd motion referred to as stability islands. The lighter highlighted regions consist of chaotic orbits and are therefore considered unstable.

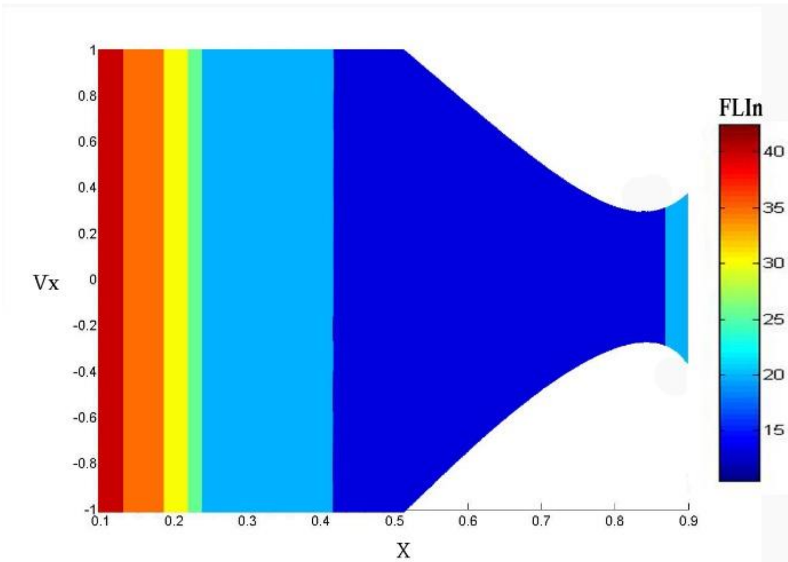


Fig. 3. The FLI map for the plane Earth-Moon system.

3.3. Using the FLI map

Note that the computation of a FLI map requiresselection of an integration time and a scheme to sample the set of initial conditions. These choice calibration phase, whereby the computation of a few trajectories are compared to each other and the integration time is selected to ensure enough differences between the regular and chaotic trajectories. This is illustrated in Figure 4 where a rather short integration time of 30 days (almost corresponding to one Moon revolution around the Earth) has been sufficient in this region of the phase space.

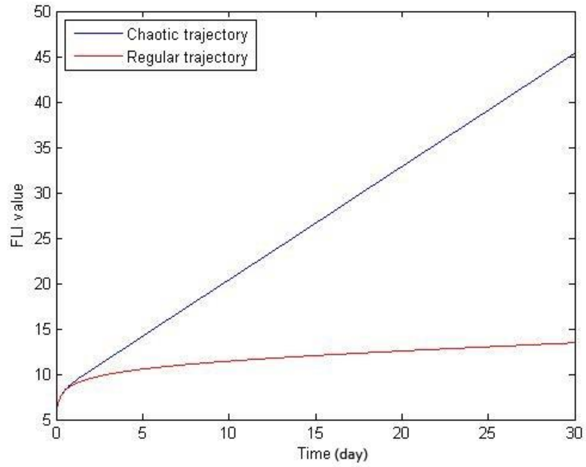


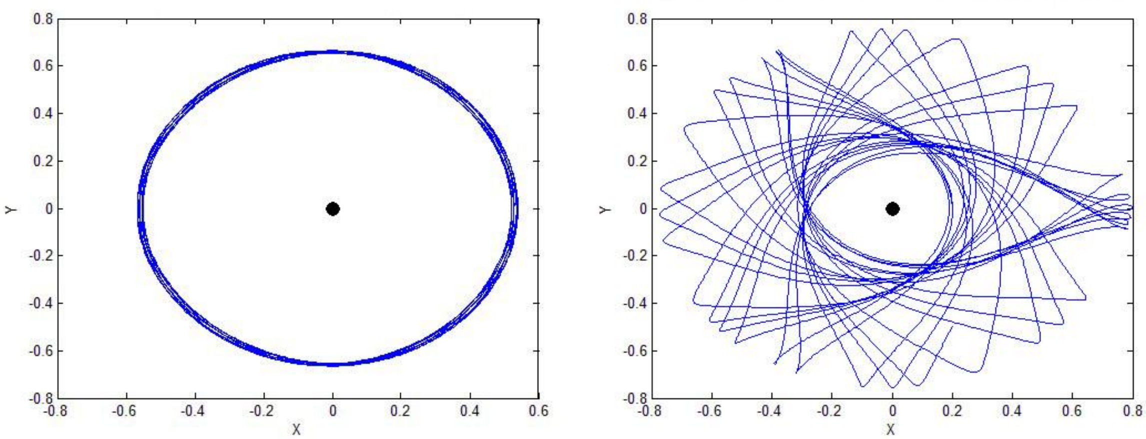
Fig. 4. Time propagation of FLI for two trajectoriesselected from the map of Fig. 3.

3.3.1. Trajectory example

Two trajectories will be used to illustrate the chaotic effect in Figure 5. The orbit in Fig.5a closes on itself and repeats indefinitely, so it is a stable. On the contrary, a chaotic non-repeating trajectory is shown in Fig.5b.

3.3.2. Comparing the Newtonian and Relativistic Trajectory

A comparison of the relativistic and Newtonian trajectories shown in Fig. 6 reveals a conspicuous deviation between the theories with identical initial conditions and thus illustrates the chaotic effect. The difference between Newtonian and general relativistic motion after 2 years is shown to be much larger for the chaotic trajectory than it is for the regular Newtonian trajectory, see Fig. 6b.



(a) (b)

Fig. 5. Phase space trajectories for two initial conditions: (a) stable orbit with position and velocity 0.54 and 0, (b) chaotic orbit with position and velocity 0.2 and 0 .

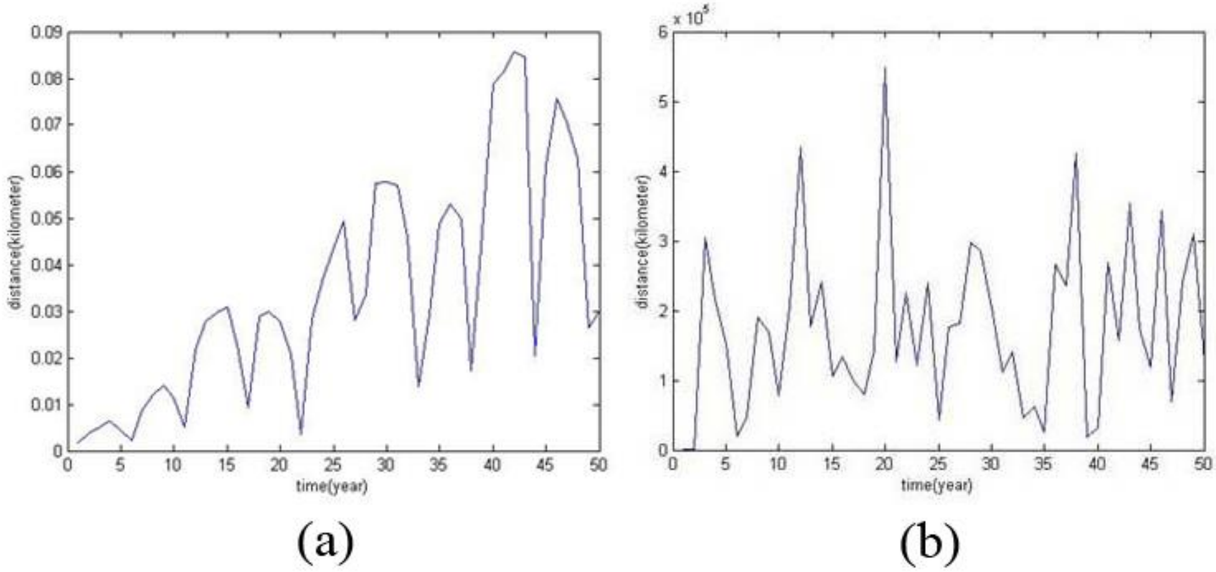


Fig. 6. Deviation between the Newtonian and relativistic trajectorieswith time for :(a) regular orbit and (b) for a chaotic trajectory.

4. **Conclusion**

The method of fast Lyapunov indicator is used to investigate the chaotic motion of thecircular relativistic restricted three-body problem. AnEarth-Moon FLI map is developed in order to pick up distinguishable conditions leading to regular as well as chaotic trajectories.

It is shown that the phase space plots can be effectively utilized by the FLI method for chaotic detection. It is also shown that there exist a significant difference the between the Newtonian and Relativistic RTBP trajectories for chaotic trajectories.

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