Interplay between Entanglement and Symmetry

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Entanglement and Symmetry

- Entanglement and symmetry are two pillars of modern physics.
- In recent times, the interplay between these two fundamental concepts has become the subject of intense research activity, ranging from quantum information, quantum field theory, holography, and many-body systems.

How does entanglement decompose into different symmetry sectors in the presence of a global conserved charge (e.g. $U(1)$)?

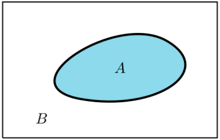
- In this talk, I will look at this problem through the lens of entanglement entropy and briefly review some of the more exciting findings for symmetry-resolved entanglement in conformal field theory.

Title

- Entanglement Measures
- Symmetry- Resolved Entanglement Measures
- Universal thermal corrections to Symmetry-Resolved Entanglement Measures
- Example

Entanglement Measures

Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$



- For bipartite pure state,

$$|\Psi\rangle \neq |\Psi\rangle_A \otimes |\Psi\rangle_B \Rightarrow |\Psi\rangle \text{ is called entangled.}$$

- Entanglement measures are Rényi and entanglement entropies. Pure state $\rho = |\Psi\rangle\langle\Psi|$, the nth Rényi entropy (RE):

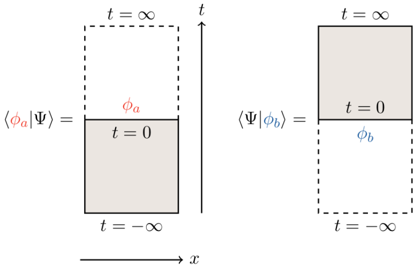
$$S_n \equiv \frac{1}{1-n} \log \mathrm{Tr}(\rho_A)^n$$

where $\rho_A = \mathrm{Tr}_B \rho$ is the reduced density matrix of subsystem A. EE is given by $S_E = -\mathrm{Tr}\rho_A \ln \rho_A = \lim_{n\to 1} S_n$.

Replica trick and QFTs

dim $\mathcal{H} = \infty$ in QFT $\Rightarrow$ Replica trick.

- Path integral representation:



States $|\phi_{a,b}\rangle$ are the boundary conditions at $t = 0$

The partition function Z is the path integral over the entire Euclidean space

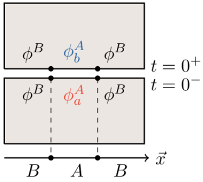
$$Z = \int [D\phi_0(t = 0, \vec{x})] \langle \Psi | \phi_0 \rangle \langle \phi_0 | \Psi \rangle \ .$$

Replica trick and QFTs

The reduced density matrix has two indices $[\rho_A]_{ab} = \langle \phi_a^A | \rho_A | \phi_b^A \rangle$ where $\phi_a^A(\vec{x} \in A)$ and $\phi_b^A(\vec{x} \in A)$ specify the boundary conditions on A at $t = 0^+$ and 0^-, respectively.

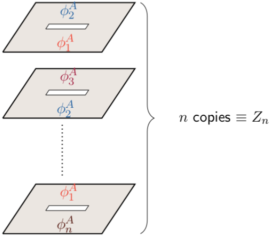
$$[\rho_A]_{ab} = \frac{1}{Z_1} \int [\mathcal{D}\phi^B(t=0, \vec{x} \in B)] \left(\langle \phi_a^A | \langle \phi^B | \right) |\Psi\rangle\langle\Psi| \left(|\phi_b^A\rangle |\phi^B\rangle \right) \ ,$$

$$= \frac{1}{Z_1} \int [\mathcal{D}\phi^B(t=0, \vec{x} \in B)]$$



Replica trick and QFTs

$$\mathrm{tr}_A \rho_A^n = \frac{1}{(Z_1)^n}$$



$$= \frac{Z_n}{(Z_1)^n}$$

The calculation of S_n reduces to computing the partition function $\mathcal{Z}_n$ on the n-sheeted Riemann surface $\mathcal{R}_n$.
Example: In $(1+1)$- dimensional CFT with central charge c,

$$S_E = \frac{c}{3} \log\left[\frac{L}{\pi} \sin\left(\frac{\pi l}{L}\right)\right] + \cdots .$$

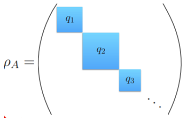
l and L are the length of the subsystem A and the total system, respectively.

Symmetry Resolved Entanglement Entropy

- Bipartite system have an internal symmetry, e.g., $U(1)$ symmetry, generated by the charge operator $\hat{Q} = \hat{Q}_A \oplus \hat{Q}_B$, where $\hat{Q}_i$ ($i = A, B$) is the charge in the subsystem i.
- Suppose ρ is the eigenstate of the symmetry generator $\hat{Q}$, i.e., $[\rho, \hat{Q}] = 0$.

$$[\rho, \hat{Q}] = 0 \rightarrow [\rho_A, \hat{Q}_A] = 0$$

ρ_A is block-diagonal according to eigenvalue q_A of charge operator $\hat{Q}_A$,



$$\rho_A = \bigoplus_{q_A} P(q_A)\rho_A(q_A)$$

where $P(q_A)$ is the probability that measurement of the charge $\hat{Q}_A$ in region A be q_A.

Symmetry Resolved Entanglement Entropy

It is interesting to understand how entanglement decomposes into the various symmetry sectors in the presence of global symmetry.

- Symmetry resolved Rényi entropies are defined as

$$S_n(q_A) \equiv \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n(q_A).$$

- Symmetry resolved entanglement entropy is defined as

$$S(q_A) = -\mathrm{Tr} \rho_A(q_A) \ln \rho_A(q_A).$$

Symmetry Resolved Entanglement Entropy

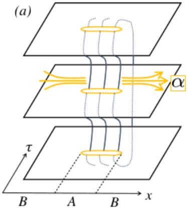
- Total entanglement entropy:

$$S_E = \sum_{q_A} P(q_A) S(q_A) - \sum_{q_A} P(q_A) \log(P(q_A)) = S^C + S^N.$$

- S^C is the configurational entropy, which measures the average of the entanglement in each charge sector.
- S^N is called number entropy, which quantify the entropy due to the fluctuations of the charge within subsystem A.

Symmetry Resolved Entanglement Entropy

- Using the path integral language, and via the insertion of an Aharonov-Bohm flux in the Riemann geometry, one could decompose entanglement measures into the contribution of individual charge sectors in the presence of global symmetry.



- Charged moments:

$$\mathcal{Z}_n(\alpha) = \mathrm{tr}\rho_A^n e^{-i\alpha\hat{Q}_A}$$

It can be seen as a partition function over the n-sheeted Riemann surface $\mathcal{R}_n$ with an inserted Aharonov-Bohm flux α .

Symmetry Resolved Entanglement Entropy

- The insertion of a flux corresponds to a twisted boundary condition, which can be implemented by some local fields $\mathcal{V}_\alpha$ acting on the boundary of subsystem A, $[u, v]$,

$$e^{i\alpha\hat{Q}_A} = \mathcal{V}_\alpha(u,0)\mathcal{V}_{-\alpha}(v,0).$$

- $\mathrm{Tr}\left(\rho_A^n e^{i\alpha\hat{Q}_A}\right)$ can be seen as a partition function $\mathcal{Z}_n(\alpha)$ over the n-sheeted Riemann surface $\mathcal{R}_n$ with twisted boundary conditions or, equivalently, as a correlation function over the n-sheeted Riemann surface $\mathcal{R}_n$ with periodic boundary conditions,

$$\mathrm{Tr}\left(\rho_A^n e^{i\alpha\hat{Q}_A}\right) = \left\langle e^{i\alpha\hat{Q}_A}\right\rangle_{\mathcal{R}_n} = \langle \mathcal{V}_\alpha \mathcal{V}_{-\alpha}\rangle_{\mathcal{R}_n}.$$

Symmetry Resolved Entanglement Entropy

- Fourier transform of charged moments $\mathcal{Z}_n(\alpha)$:

$$\mathcal{Z}_n(q_A) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\alpha}{2\pi} \mathcal{Z}_n(\alpha) e^{-i\alpha q_A},$$

- Symmetry-resolved Rényi and entanglement entropies can be obtained as :

$$S_n(q_A) = \frac{1}{1-n} \log\left[\frac{\mathcal{Z}_n(q_A)}{\mathcal{Z}_1^n(q_A)}\right].$$

$S(q_A) = \lim_{n\to 1} S_n(q_A)$.

Symmetry Resolved Entanglement Entropy

Example: compact boson:

$$Z_n(\alpha) = c_{n,\alpha}\ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)-2\frac{h_\alpha+\bar{h}_\alpha}{n}},$$

h_α and $\bar{h}_\alpha$ are the scaling dimensions of the vertex operator $\mathcal{V}_\alpha$ generating the Aharonov-Bohm flux.

$$\mathcal{Z}_n(q) \simeq \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\sqrt{\frac{n\pi}{2K\ln\ell}}\,e^{\frac{n\pi^2(q-\langle\hat{Q}_A\rangle)^2}{2K\ln\ell}}.$$

K is Luttinger parameter. The Rényi and the entanglement entropies:

$$S_n(q) = S_n - \frac{1}{2}\ln\left(\frac{2K}{\pi}\ln\ell\right) + O(\ell^0), \qquad S_{\mathrm{vN}}(q) = S_{\mathrm{vN}} - \frac{1}{2}\ln\left(\frac{2K}{\pi}\ln\ell\right) + O(\ell^0).$$

Entanglement equipartition: at leading order, the entanglement is equally distributed in the different symmetry sectors, and does not depend on the symmetry sector.

Thermal charged moments and universal corrections

What are the thermal corrections to the contribution of individual system charge sectors?

- Thermal state at low-temperature:

$$\rho = \frac{e^{-\beta H}}{tre^{-\beta H}} = \frac{|0\rangle\langle 0| + |\psi\rangle\langle\psi| e^{-2\pi\Delta_\psi \beta/L} + \cdots}{1 + e^{-2\pi\Delta_\psi \beta/L} + \cdots},$$

  where $|\psi\rangle$ is the first excited state with scaling dimension Δ_ψ. The Δ_ψ is the smallest scaling dimension of the primary operator, $\psi\,(w)$,

- The low-temperature expansion of reduced density matrix:

$$\rho_A \sim \rho_{0,A} + e^{-2\pi\Delta_\psi \beta/L}\left(\rho_{\psi,A} - \rho_{0,A}\right),$$

  where $\rho_{0,A} = \mathrm{tr}_B(|0\rangle\langle 0|)$ and $\rho_{\psi,A} = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$.

Thermal charged moments and universal corrections

Thermal charged moments

$$\mathcal{Z}_n^{(\text{th})}(\alpha) = \mathrm{Tr}\left(\rho_A^n e^{i\alpha \hat{Q}_A}\right) = \mathcal{Z}_n^{(0)}(\alpha) + n\mathcal{Z}_n^{(0)}(\alpha) e^{-2\pi\Delta_\psi \beta/L} \mathcal{F}_n(\alpha)$$

-
$$\mathcal{F}_n(\alpha) = \mathcal{M}_n(\alpha) - 1,$$

$\mathcal{M}_n$ expression

$$\mathcal{M}_n = \frac{\langle \mathcal{V}_\alpha \mathcal{V}_{-\alpha} \psi\psi \rangle_{\mathcal{R}_n}}{\langle \mathcal{V}_\alpha \mathcal{V}_{-\alpha} \rangle_{\mathcal{R}_n} \langle \psi\psi \rangle_{\mathcal{R}_1}} = \frac{1}{n^{2\Delta_\psi}} \frac{\sin^{2\Delta_\psi}(\pi x)}{\sin^{2\Delta_\psi}(\pi x/n)} G_{n,\alpha}(z, \bar{z})$$

where $x = \frac{\ell}{L}$, $z = \frac{\zeta_{12}^{(n)} \zeta_{34}^{(n)}}{\zeta_{14}^{(n)} \zeta_{23}^{(n)}}$, and $\bar{z} = \frac{\bar{\zeta}_{12}^{(n)} \bar{\zeta}_{34}^{(n)}}{\bar{\zeta}_{14}^{(n)} \bar{\zeta}_{23}^{(n)}}$.

symmetry-resolved thermal partition function

$$\mathcal{Z}_n^{(\mathrm{th})}(q_A) = \mathcal{Z}_n^{(0)}(q_A)\left[1 + gn\mathcal{F}_n(q_A)e^{-2\pi\Delta_\psi\beta/L}\right],$$

$$\mathcal{F}_n(q_A) = \frac{1}{n^{2\Delta_\psi}}\frac{\sin^{2\Delta_\psi}(\pi x)}{\sin^{2\Delta_\psi}\left(\frac{\pi x}{n}\right)}\frac{\mathcal{X}_n(q_A)}{\mathcal{Z}_n^{(0)}(q_A)} - 1,$$

and

$$\mathcal{X}_n(q_A) = \int_{-\pi}^{\pi}\frac{\mathrm{d}\alpha}{2\pi}\mathcal{Z}_n^{(0)}(\alpha)G_{n,\alpha}(z,\bar{z})e^{-i\alpha q_A}.$$

The leading correction term to the probability distribution of charge in each sector can be obtained by $P^{(\mathrm{th})}(q_A) = \mathcal{Z}_1^{(\mathrm{th})}(q_A)$.

Entanglement Measures

In general, the thermal correction to the symmetry-resolved Rényi and entanglement entropies take the following forms:

Symmetry-Resolved Entanglement Measures

$$\delta S_n(q_A) = \frac{ng}{1-n}\left[\frac{1}{n^{2\Delta_\psi}}\frac{\sin^{2\Delta_\psi}(\pi x)}{\sin^{2\Delta_\psi}\left(\frac{\pi x}{n}\right)}\frac{\mathcal{X}_n(q_A)}{\mathcal{Z}_n^{(0)}(q_A)} - \frac{\mathcal{X}_1(q_A)}{\mathcal{Z}_1^{(0)}(q_A)}\right] e^{-2\pi\Delta_\psi\beta/L},$$

$$\delta S_E(q_A) = g\left[2\Delta_\psi\left(1-\pi x\cot(\pi x)\right)\frac{\mathcal{X}_1(q_A)}{\mathcal{Z}_1^{(0)}(q_A)} + \partial_n\left(\frac{\mathcal{X}_n(q_A)}{\mathcal{Z}_n^{(0)}(q_A)}\right)\Big|_{n=1}\right] e^{-2\pi\Delta_\psi\beta/L}.$$

g denotes the degeneracy of the first excited state. Thermal corrections depend on the degeneracy of the first excited state, the charge of the sector, and the field content of the theory.

Example:compact boson

If the excited state is induced by the derivative primary operator $i\partial\phi$

$$G_{n,\alpha}(z,\bar{z}) = 1 - K\left(\frac{\alpha}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{n}\right)$$

Symmetry-Resolved Entanglement Entropy

$$\delta S_E(q_A) = \delta S_E + \mathcal{B}(q_A)e^{-2\pi\beta/L},$$

$$\mathcal{B}(q_A) = \left(3\sin^2(\pi x) - \pi(1+x)\sin(2\pi x)\right)\frac{1}{\ln(l)}$$

$$+ \left(-4\pi\sin^2(\pi x) + \pi(1+x)\sin(2\pi x)\right)\frac{\pi q_A^2}{K\ln(l)^2}.$$

Thermal corrections of the charge-sector contributions, at order $\ln(l)^{-2}$, are charge-dependent. The equipartition is broken at this order.

Conclusion

- In this talk, we have reviewed symmetry-resolved entanglement entropy and presented a formula for thermal corrections to the symmetry-resolved Rényi and entanglement entropies for general two-dimensional conformal field theories on a circle.
- Besides the size of the mass gap and the degeneracy of the first excited state, these terms depend only on the four-point function of primary fields.
- Various works have been done in different kinds of literature, including e.g. boundary conformal field theory, free and interacting integrable quantum field theories, spin systems, holographic settings, topologically ordered systems, negativity, out-of-equilibrium situations, and experimental.

Thank You