Massive to Massless

by Applying a Nonlocal Field Redefinition

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Outline:

- Motivation
- Redundant parameters
- Scalar field theory ($s = 0$)
- Harmonic oscillator
- Dirac field theory ($s = \frac{1}{2}$)
- Toward higher spin theories
- Conclusion

Motivation

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- Usually, a massive theory can be reduced to a massless theory by taking the limit

$$m \to 0$$

$$\text{Massive} \quad \underset{?}{\overset{m\,\to\,0}{\rightleftharpoons}} \quad \text{Massless}$$

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$$\text{Off-shell SUSY massive HS ?} \quad \underset{\mathbf{T^{-1}}}{\overset{\mathbf{T}}{\rightleftharpoons}} \quad \text{Off-shell SUSY massless HS} \checkmark$$

- Similarly

$$\text{(A)dS} \quad \underset{?}{\overset{\Lambda\ \to\ 0}{\rightleftharpoons}} \quad \text{Flat}$$

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Redundant Parameters

Redundant parameters

Let us follow Weinberg's approach and consider the scalar field theory in the form

$$S = \int d^4x \left[-\tfrac{1}{2}\, Z \left(\partial^\mu \Phi\, \partial_\mu \Phi + m^2 \Phi^2\right) - \tfrac{1}{4!}\, g\, Z^2 \Phi^4\right] .$$

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$$\boxed{\text{parameter } x \text{ is redundant, if } \left(\frac{\delta S}{\delta x}\right)\Big|_{e.o.m} = 0}$$

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What about **nonlocal field redefinitions**?

Free Scalar Field Theory

Scalar field theory

Consider massive and massless scalar field equations (Klein-Gordon) in d dimensions

$$\left(\partial^\mu \partial_\mu - m^2\right)\Phi = 0\,, \qquad\qquad \mu = 0, 1, \ldots, d-1$$
$$\left(\partial^\mu \partial_\mu\right)\varphi = 0\,,$$

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If one supposes there is a field redefinition

$$\boxed{\Phi = \mathbf{P}\, \varphi} \tag{1}$$

and the identity

$$\boxed{\left(\partial^\mu \partial_\mu - m^2 \right) \mathbf{P} = \mathbf{Q} \left(\partial^\mu \partial_\mu \right)} \tag{2}$$

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then, the massive equation reduces to the massless one (or m becomes redundant)

$$\begin{aligned} \left(\partial^\mu\partial_\mu - m^2\right)\Phi &= 0 && \text{massive eq.} \\ \left(\partial^\mu\partial_\mu - m^2\right)\mathbf{P}\,\varphi &= 0 && (1)\downarrow \\ \mathbf{Q}\left(\partial^\mu\partial_\mu\right)\varphi &= 0 && (2)\downarrow \\ \left(\partial^\mu\partial_\mu\right)\varphi &= 0 && \text{massless eq.} \end{aligned}$$

Scalar field theory

- We find the identity

$$\boxed{\left(\partial^\mu \partial_\mu - m^2\right) \mathbf{P} = \mathbf{Q} \left(\partial^\mu \partial_\mu\right)}$$

cab be satisfied, if we introduce

$$\mathbf{P} := \sum_{n=0}^{\infty} (mx)^{2n} \frac{(-1)^n}{4^n \, n! \, (x^\mu \partial_\mu + \frac{d}{2})_n}$$

$$\mathbf{Q} := \sum_{n=0}^{\infty} (mx)^{2n} \frac{(-1)^n}{4^n \, n! \, (x^\mu \partial_\mu + \frac{d}{2} + 2)_n}$$

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in which $(\cdots)_n$ is the rising Pochhammer symbol with the argument a of any $n \in \mathbb{N}$

$$(a)_n = a\,(a+1)\,(a+2)\cdots(a+n-1)$$

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$$(a)_n = a\,(a+1)\,(a+2)\cdots(a+n-1)$$

$$\begin{aligned} (a)_0 &= 1 \\ (a)_1 &= a \\ \vdots \; & \quad \vdots \end{aligned}$$

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Note that

$$\lim_{m\to 0} \mathbf{P} = 1\,, \qquad \lim_{m\to 0} \mathbf{Q} = 1$$

Scalar field theory

To prove the identity

$$\boxed{\left(\partial^\mu \partial_\mu - m^2 \right) \mathbf{P} = \mathbf{Q} \left(\partial^\mu \partial_\mu \right)}$$

one may use

$$\left[\, \partial^\mu \partial_\mu \, , \, x^{2n} \, \right] = 4n \left(x^\mu \, \partial_\mu + \tfrac{d}{2} - n + 1 \right) x^{2n-2}$$

and

$$(a)_n = (a + n - 1) \, (a)_{n-1}$$

Scalar field theory

- Recalling

$$\Phi = \mathbf{P}\,\varphi \qquad\qquad (\,\partial^\mu\partial_\mu - m^2\,)\,\mathbf{P} = \mathbf{Q}\,(\,\partial^\mu\partial_\mu\,)$$

we find inverse operators.

Scalar field theory

- Recalling

$$\Phi = \mathbf{P}\,\varphi \qquad\qquad (\,\partial^\mu\partial_\mu - m^2\,)\,\mathbf{P} = \mathbf{Q}\,(\,\partial^\mu\partial_\mu\,)$$

we find inverse operators. Indeed, one can suppose

$$\boxed{\varphi = \mathbf{P}^{-1}\,\Phi} \tag{3}$$

and the inverse identity

$$\boxed{\mathbf{Q}^{-1}\,(\,\partial^\mu\partial_\mu - m^2\,) = (\,\partial^\mu\partial_\mu\,)\,\mathbf{P}^{-1}} \tag{4}$$

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Then, the massless equation reduces to the massive one (or m becomes restored)

$$\begin{aligned} (\,\partial^\mu\partial_\mu\,)\,\varphi &= 0 && \text{massless eq.} \\ (\,\partial^\mu\partial_\mu\,)\,\mathbf{P}^{-1}\,\Phi &= 0 && (3)\downarrow \\ \mathbf{Q}^{-1}\,(\,\partial^\mu\partial_\mu - m^2\,)\,\Phi &= 0 && (4)\downarrow \\ (\,\partial^\mu\partial_\mu - m^2\,)\,\Phi &= 0 && \text{massive eq.} \end{aligned}$$

Scalar field theory

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$$\mathbf{P}^{-1} := \sum_{n=0}^{\infty} \frac{1}{4^n\, n!\, (x^\mu\,\partial_\mu + \frac{d}{2} - n - 1)_n} \left(mx\right)^{2n}$$

$$\mathbf{Q}^{-1} := \sum_{n=0}^{\infty} \frac{1}{4^n\, n!\, (x^\mu\,\partial_\mu + \frac{d}{2} - n + 1)_n} \left(mx\right)^{2n}$$

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$$\mathbf{Q}^{-1}:=\sum_{n=0}^{\infty}\frac{1}{4^{n}\,n!\,(x^{\mu}\,\partial_{\mu}+\frac{d}{2}-n+1)_{n}}\,(\,mx\,)^{2n}$$

- One finds the Hermitian conjugation of operators reads

$$\mathbf{P}^{\dagger}=\mathbf{Q}^{-1}\qquad\qquad\mathbf{Q}^{\dagger}=\mathbf{P}^{-1}$$

Scalar field theory

At the level of the action, we consider the free massive scalar theory in d dimensions

$$
\begin{aligned}
S &= -\tfrac{1}{2}\int d^dx\,\big[\,\partial^\mu\Phi\,\partial_\mu\Phi + m^2\Phi^2\,\big] \\
&= +\tfrac{1}{2}\int d^dx\;\Phi\,\big[\,\partial^\mu\partial_\mu - m^2\,\big]\,\Phi + \text{\sout{total der.}}
\end{aligned}
$$

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Applying the field redefinition

$$\Phi = \mathbf{P}\,\varphi\,, \qquad \rightarrow \qquad \Phi = \varphi\,\mathbf{P}^\dagger$$

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Applying the field redefinition

$$\Phi = \mathbf{P}\,\varphi\,, \qquad \rightarrow \qquad \Phi = \varphi\,\mathbf{P}^\dagger$$

we find

$$S = \tfrac{1}{2}\int d^dx\ \Phi\left[\partial^\mu\partial_\mu - m^2\right]\Phi \qquad \text{massive action}$$
$$= \tfrac{1}{2}\int d^dx\ \varphi\,\mathbf{P}^\dagger\left[\partial^\mu\partial_\mu - m^2\right]\mathbf{P}\,\varphi \qquad \downarrow$$
$$= \tfrac{1}{2}\int d^dx\ \varphi\,\underbrace{\mathbf{P}^\dagger\,\mathbf{Q}}_{=1}\left[\partial^\mu\partial_\mu\right]\varphi \qquad \downarrow$$
$$= \tfrac{1}{2}\int d^dx\ \varphi\left[\partial^\mu\partial_\mu\right]\varphi \qquad \text{massless action}$$

Harmonic Oscillator

Harmonic oscillator

Consider a Newtonian massive particle whose position only depends on time and subject to the force law of the usual Hooke's law ($\omega = \sqrt{k/m}\,, \;\; m \neq 0$), and a free Newtonian massive particle subject to no force. Equations of motion read

$$\left(\frac{d^2}{dt^2} + \omega^2 \right) \mathrm{X}(t) = 0 \tag{5}$$

$$\left(\frac{d^2}{dt^2} \right) x(t) = 0 \tag{6}$$

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- The equation (5) can be reduced to (6) by applying

$$\boxed{\mathrm{X}(t) = \left(\sum_{n=0}^{\infty} (\,\omega t\,)^{2n} \; \frac{(-1)^n}{4^n \; n! \; (t\,\frac{d}{dt} + \frac{1}{2})_n} \right) x(t)}$$

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- In a reverse way, the equation (6) can be reduced to (5) by applying

$$\boxed{x(t) = \left(\sum_{n=0}^{\infty} \frac{1}{4^n \, n! \, (t \frac{d}{dt} - n - \frac{1}{2})_n} \, (\,\omega t\,)^{2n} \right) \mathrm{X}(t)}$$

Free Dirac Field Theory

Dirac field theory

Consider massive and massless Dirac field equations in d dimensions

$$(\gamma \cdot \partial + m)\Psi = 0$$

$$(\gamma \cdot \partial)\,\psi = 0$$

Dirac field theory

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$$(\,\gamma\cdot\partial\,)\,\psi=0$$

If one supposes there is a field redefinition

$$\boxed{\Psi=\mathbb{P}\,\psi} \tag{7}$$

and the identity

$$\boxed{(\,\gamma\cdot\partial+m\,)\,\mathbb{P}=\mathbb{Q}\,(\,\gamma\cdot\partial\,)} \tag{8}$$

Dirac field theory

Consider massive and massless Dirac field equations in d dimensions

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$$(\gamma \cdot \partial)\,\psi = 0$$

If one supposes there is a field redefinition

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then, the massive equation reduces to the massless one (or m becomes redundant)

$$\begin{aligned} (\gamma \cdot \partial + m)\,\Psi &= 0 &&\text{massive eq.}\\ (\gamma \cdot \partial + m)\,\mathbb{P}\,\psi &= 0 &&(7)\downarrow\\ \mathbb{Q}\,(\gamma \cdot \partial)\,\psi &= 0 &&(8)\downarrow\\ (\gamma \cdot \partial)\,\psi &= 0 &&\text{massless eq.} \end{aligned}$$

Dirac field theory

- We find the identity

$$\boxed{(\,\gamma\cdot\partial+m\,)\,\mathbb{P}=\mathbb{Q}\,(\,\gamma\cdot\partial\,)}$$

cab be satisfied, if we introduce

$$\mathbb{P}:=\sum_{k=0}^{\infty}\Big[(m\,\gamma\cdot x)^{2k}-2k\,(m\,\gamma\cdot x)^{2k-1}\Big]\frac{1}{4^k\;k!\;(x^\mu\,\partial_\mu+\frac{d}{2})_k}$$

$$\mathbb{Q}:=\sum_{k=0}^{\infty}\Big[(m\,\gamma\cdot x)^{2k}+2k\,(m\,\gamma\cdot x)^{2k-1}\Big]\frac{1}{4^k\;k!\;(x^\mu\,\partial_\mu+\frac{d}{2}+1)_k}$$

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- To prove the identity, one may use

$$\Big[\gamma\cdot\partial\;,\;(\gamma\cdot x)^{\,2k}\Big] = \;2k\,(\gamma\cdot x)^{\,2k-1}$$

$$\Big\{\gamma\cdot\partial\,,\,(\gamma\cdot x)^{\,2k-1}\Big\} = \;2\,\Big(x^\mu\,\partial_\mu + \tfrac{d}{2} - k + 1\Big)\,(\gamma\cdot x)^{\,2k-2}$$

Dirac field theory

- In a reverse way, we can reduce the massless equation to the massive one by

$$\boxed{\psi = \mathbb{P}^{-1}\,\Psi}$$

and the inverse identity

$$\boxed{\mathbb{Q}^{-1}\,(\,\gamma\cdot\partial + m\,) = (\,\gamma\cdot\partial\,)\,\mathbb{P}^{-1}}$$

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- The latter satisfies, if we introduce

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- The latter satisfies, if we introduce

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$$\mathbb{Q}^{-1} := \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k\;k!\;(x^\mu\,\partial_\mu + \frac{d}{2} - k + 1)_k}\left[(m\,\gamma\cdot x)^{2k} + 2k\,(m\,\gamma\cdot x)^{2k-1}\right]$$

- The Hermitian conjugates of operators read

$$\mathbb{P}^\dagger = -\,\gamma^0\,\mathbb{Q}^{-1}\,\gamma^0 \qquad\qquad (\,\mathbb{P}^{-1})^\dagger = -\,\gamma^0\,\mathbb{Q}\,\gamma^0$$

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Thus

$$\begin{aligned} S &= -\int d^dx\; \bar{\psi}\,(\,\gamma\cdot\partial\,)\,\psi && \text{massless Dirac action} \\ &= -\int d^dx\; \bar{\Psi}\,\mathbb{Q}\,(\,\gamma\cdot\partial\,)\,\mathbb{P}^{-1}\,\Psi && \downarrow \\ &= -\int d^dx\; \bar{\Psi}\,\underbrace{\mathbb{Q}\,\mathbb{Q}^{-1}}_{=1}(\,\gamma\cdot\partial + m\,)\,\Psi && \downarrow \\ &= -\int d^dx\; \bar{\Psi}\,(\,\gamma\cdot\partial + m\,)\,\Psi && \text{massive Dirac action} \end{aligned}$$

Toward Higher Spin Theories

Higher spins

- The massive higher spin and the massless higher spin (Fronsdal) equations read

$$\left[p^2 + m^2 - (p\cdot u + mv)\left(p\cdot\frac{\partial}{\partial u} + m\frac{\partial}{\partial v}\right) + \frac{1}{2}(p\cdot u + mv)^2\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2}\right)\right]\Phi_s = 0$$

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$$\text{spin-s fields:}\ \begin{cases}\Phi_s(x,u,v)=\sum_{r=0}^{s}\frac{1}{r!\,(s-r)!}\ u^{\mu_1}\dots u^{\mu_r}\ v^{s-r}\ \Phi_{\mu_1\dots\,\mu_r}(x)\\ \varphi_s(x,u)=\frac{1}{s!}\ u^{\mu_1}\dots u^{\mu_s}\ \varphi_{\mu_1\dots\mu_s}(x)\end{cases}$$

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(**the question is open for higher spins**)

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Thank you for your attention!