Anderson Localization, Renormalization Group, Transfer Matrix

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plan

- Anderson Localization
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    - Stochastic Differential Equation (SDE)
    - Effective description of SDE
    - Renormalization Group
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- Transfer Matrix
    - Products of Random Matrices
    - Furstenberg Theorem
    - Osceledec Theorem
    - Virster Theorem
    - Numerical computation of LE
    - Example

Anderson Localization: Continuous and Discrete models

RG

Continuous $$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = E\psi$$

Weak disorder

TM

Tight binding Hamiltonian $$H = \sum_i \varepsilon_i \hat{a}_i^\dagger \hat{a}_i + t \sum_{\langle ij \rangle} \hat{a}_i^\dagger \hat{a}_j$$

Intermediate and strong disorder

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Stochastic Differential Equation (SDE)

Additive noise: $$\mathcal{L}\phi(\mathbf{x},t) = F[\phi(\mathbf{x},t)] + \eta(\mathbf{x},t)$$

$\mathcal{L} = \frac{\partial}{\partial t} - \nu\nabla^2$ Diffusion Equation

$\mathcal{L} = \frac{\partial^2}{\partial t^2} - \nabla^2$ Wave Equation

$\mathcal{L} = \frac{\partial}{\partial t}$ Langevin Equation

$F[\phi] = \frac{\lambda}{2}(\nabla\phi)^2$ KPZ equation

$F[\phi] = \text{Polynomial}$ Reaction Diffusion

Multiplicative noise: $$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = E\psi$$

Effective description of SDE

Expectation values of fields: $\langle Q(\phi)\rangle \equiv \int [\mathcal{D}\eta]\; P[\eta]\; Q(\phi_{sol}(\mathbf{x},t \mid \eta))$

Some identities:

$$\begin{aligned}\phi_{sol}(\mathbf{x},t \mid \eta) &\equiv \int [\mathcal{D}\phi]\; \phi\; \delta[\phi - \phi_{sol}(\mathbf{x},t \mid \eta)] \\ &= \int [\mathcal{D}\phi]\; \phi\; \delta[\mathcal{L}\phi - F[\phi] - \eta]\; \sqrt{JJ^\dagger}\end{aligned}$$

$$\begin{aligned}x_* &= \int dx\; x\; \delta(x - x_*) \\ &= \int dx\; x\; \delta(f(x))\; |\, f'(x)\, | \\ &= \int dx\; x\; \delta(f(x))\sqrt{f'(x)[f'(x)]^*}\end{aligned}$$

$$J \equiv \det\left(\mathcal{L} - \frac{\delta F}{\delta \phi}\right)$$

Auxiliary field

$$\delta[\mathcal{L}\phi - F[\phi] - \eta] = \int [\mathcal{D}\tilde{\phi}] \exp\{-i \int d^d x dt \tilde{\phi}(\mathbf{x},t)\{\mathcal{L}\phi(\mathbf{x},t) - F[\phi(\mathbf{x},t)] - \eta(\mathbf{x},t)\}\}$$

$$J = \det\left(\mathcal{L} - \frac{\delta F}{\delta \phi}\right) = \int [\mathcal{D}\chi^*][\mathcal{D}\chi] \exp\left\{-\int d^d x dt \, \chi^*(\mathbf{x},t) \left(\mathcal{L} - \frac{\delta F}{\delta \phi}\right) \chi(\mathbf{x},t)\right\}$$

$$I = \int \exp\{-\sum_{i,j=1}^{n} \chi_i^* A_{ij} \chi_j\} \prod_{i=1}^{n} d\chi_i^* d\chi_i = \det(A)$$

Fadeev-Popov ghost fields

$$\langle Q(\phi_{sol}(\mathbf{x},t \mid \eta))\rangle = \int [\mathcal{D}\phi][\mathcal{D}\tilde{\phi}][\mathcal{D}\chi^*][\mathcal{D}\chi][\mathcal{D}\eta] P[\eta] \, Q(\phi)$$
$$\times e^{-\int d^d\mathbf{x}' dt' \{ i\tilde{\phi}(\mathbf{x}',t')\{\mathcal{L}\phi(\mathbf{x}',t') - F[\phi(\mathbf{x}',t')] - \eta(\mathbf{x}',t')\} + \chi^*(\mathbf{x}',t')\left(\mathcal{L} - \frac{\delta F}{\delta \phi}\right)\chi(\mathbf{x}',t')\}}$$

Integration over noise → $e^{-\int d^d\mathbf{x} d^d\mathbf{x}' dt dt' \tilde{\phi}(\mathbf{x},t) C(\mathbf{x},t;\mathbf{x}',t') \tilde{\phi}(\mathbf{x}',t')}$

$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = 2 \, C(\mathbf{x},t;\mathbf{x}',t')$$

$$S = S_0 + S_I$$

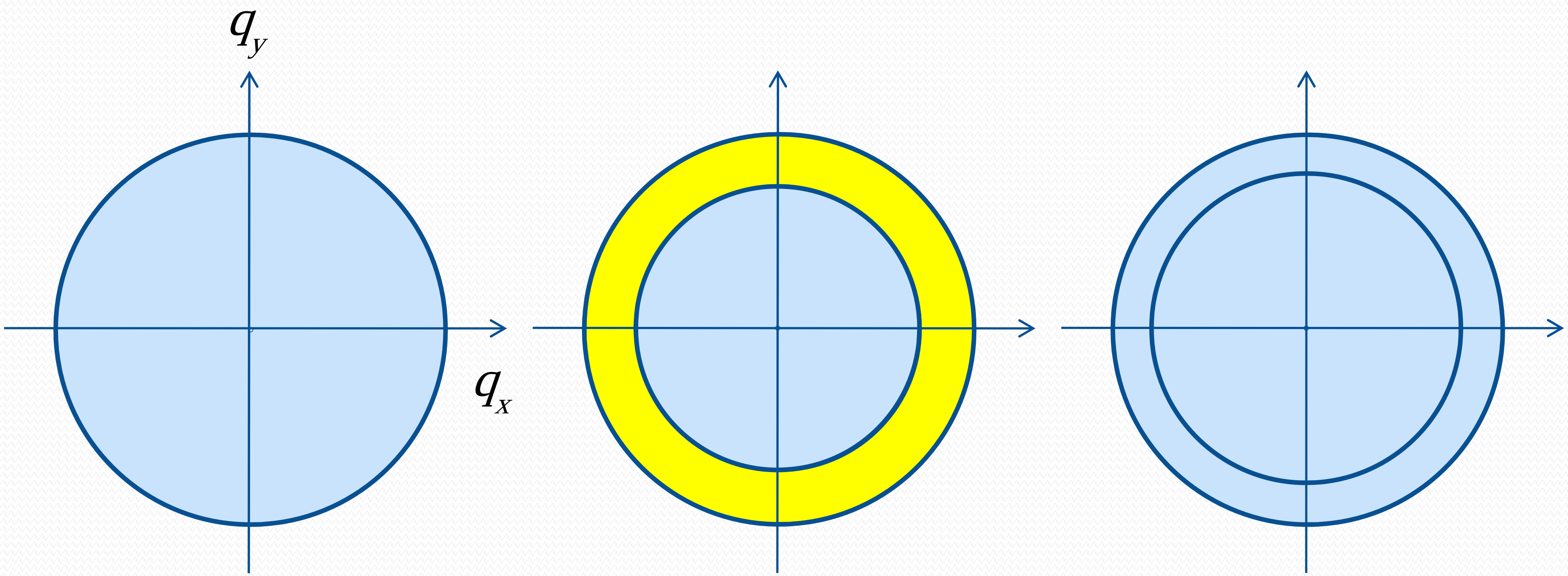
$$S_0 = -\int d^d\mathbf{x}dt \left\{ i\tilde{\phi}(\mathbf{x},t)\mathcal{L}\phi(\mathbf{x},t) + \chi^*(\mathbf{x},t)\mathcal{L}\chi(\mathbf{x},t) \right\} - \int d^d\mathbf{x}d^d\mathbf{x}'dtdt'\tilde{\phi}(\mathbf{x},t)C(\mathbf{x},t;\mathbf{x}',t')\tilde{\phi}(\mathbf{x}',t')$$

$$S_I = -\int d^d\mathbf{x}dt \left\{ i\tilde{\phi}(\mathbf{x},t)F[\phi(\mathbf{x},t)] + \chi^*(\mathbf{x},t)\frac{\delta F}{\delta\phi}\chi(\mathbf{x},t) \right\}$$

↓ Coupling function or constant

⇨ Fourier transform of fields in time and space

Renormalization Group



Momentum space | Summing over high momentum degrees of freedom | Rescaling

Example: Elastic Waves

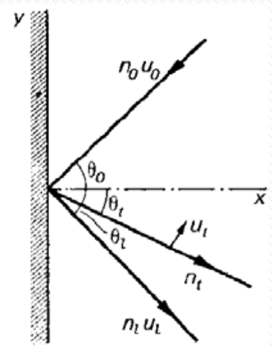
$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$ Strain tensor

$$\sigma_{ij} = 2\mu(x)u_{ij} + \lambda(x)u_{kk}\delta_{ij}$$ Stress tensor

$$m\frac{\partial^2 u_i}{\partial t^2} = \partial_j \sigma_{ij}$$ Elastic Waves

Simplest Elastic Model for waves in solid supporting both transverse and longitudinal modes

They are complex due to mode conversion

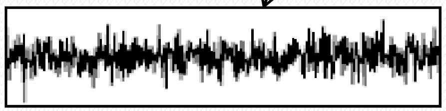


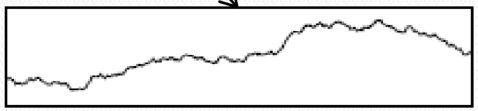
Disorder: Random Lame Coefficients

$$\mu(\mathbf{x}) = \lambda(\mathbf{x}) = \lambda_0 + \eta(\mathbf{x})$$

$$\langle \eta(\mathbf{x}) \rangle = 0$$

$$\langle \eta(\mathbf{x})\eta(\mathbf{x}') \rangle = 2K(\mathbf{x}-\mathbf{x}') = 2D_0\delta^2(\mathbf{x}-\mathbf{x}') + 2D_\rho|\mathbf{x}-\mathbf{x}'|^{2\rho-2}$$





M. Sahimi, S.E. Tajer 2005

$$P[\eta(\mathbf{x})] = \frac{1}{\mathcal{N}} \exp\left\{ -\int d\mathbf{x} d\mathbf{x}' \eta(\mathbf{x}) D(\mathbf{x}-\mathbf{x}') \eta(\mathbf{x}') \right\}$$

$$\int d\mathbf{x}'' K(\mathbf{x}-\mathbf{x}'') D(\mathbf{x}''-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$$

$$S_e = S_0 + S_I$$

$$S_0 = \int_k \sum_{a=R,I} \left[i\tilde{\mathbf{u}}^a(-k) \cdot \mathcal{L}_0^d \mathbf{u}^a(k) + \boldsymbol{\chi}^{a*}(-k) \cdot \mathcal{L}_0^d \boldsymbol{\chi}^a(k) \right]$$

$$\begin{aligned} S_I &= \int_{k,p_i} \left[\sum_{a=R,I} i\tilde{\mathbf{u}}^a(p_1) \cdot \mathcal{L}_I \mathbf{u}^a(p_2) + \boldsymbol{\chi}^{a*}(p_1) \cdot \mathcal{L}_I \boldsymbol{\chi}^a(p_2) \right] \\ &\times \left[g_0 \delta(\sum_i p_i) + g_\rho k^{-2\rho} \delta(p_1 + p_2 - k) \delta(p_3 + p_4 + k) \right] \\ &\times \left[\sum_{a=R,I} i\tilde{\mathbf{u}}^a(p_3) \cdot \mathcal{L}_I \mathbf{u}^a(p_4) + \boldsymbol{\chi}^{a*}(p_3) \cdot \mathcal{L}_I \boldsymbol{\chi}^a(p_4) \right] , \end{aligned}$$

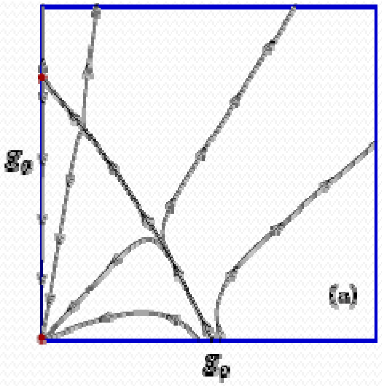
$$\mathcal{L}_0^d = \begin{pmatrix} -k^2 + \omega^2/\lambda_0 & 0 \\ 0 & -3k^2 + \omega^2/\lambda_0 \end{pmatrix} , \quad \mathcal{L}_I = \begin{pmatrix} A(p_1,p_2) & -C(p_1,p_2) \\ C(p_1,p_2) & B(p_1,p_2) \end{pmatrix}$$

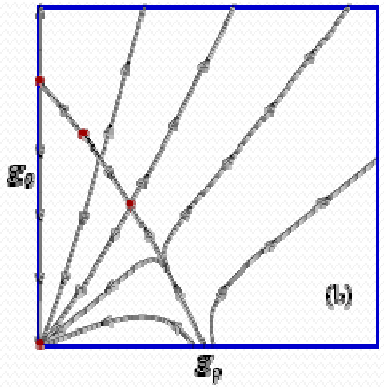
$$g_0 = D_0/\lambda_0^2 \qquad g_\rho = D_\rho/\lambda_0^2$$

RG flows

There is transition in 2D but at weak disorder strength

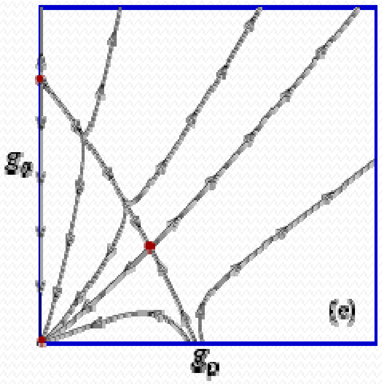
$\rho < 0.14$

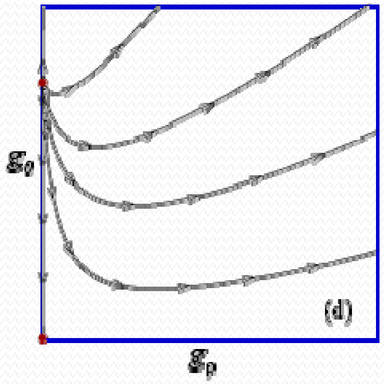




$0.14 < \rho < 0.18$

$0.18 < \rho < 1$





$\rho > 1$

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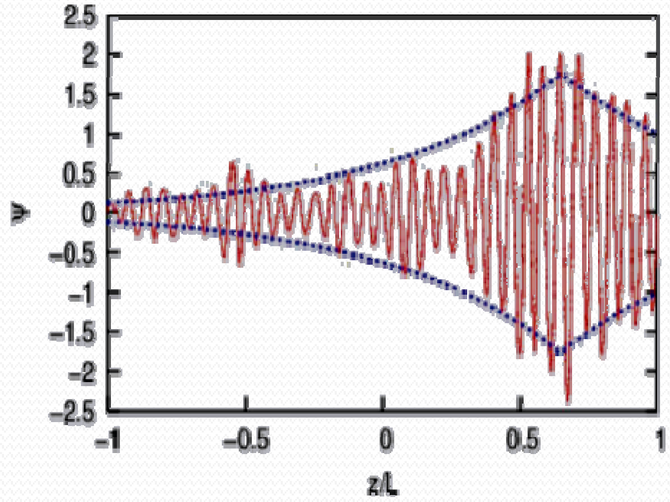
Localization Length and Lyapunov Exponent(LE)

1D-Anderson model (Localization length)

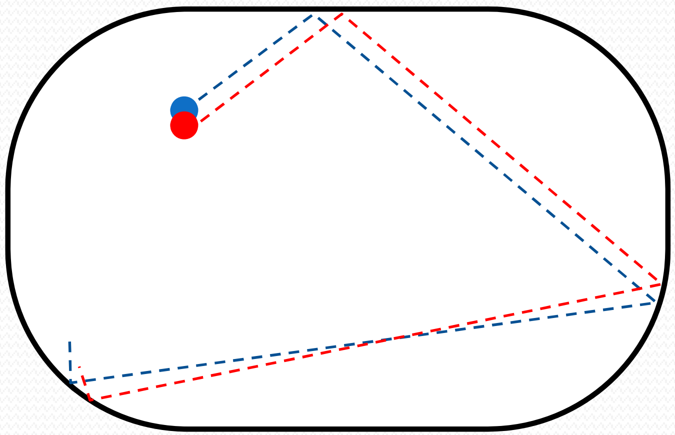
$$\Psi_{n+1} + (\varepsilon_n - E)\Psi_n + \Psi_{n-1} = 0$$

$$|\Psi_i| \sim \Psi_0 e^{-|x_i - x_0|/\xi}$$



Chaotic Stadium (LE)

$$z(t) = \begin{pmatrix} \delta q(t) \\ \delta p(t) \end{pmatrix} \Rightarrow |z(t \to \infty)| \sim e^{\gamma t}$$



Products of Random Matrices

1D-Anderson model:

$$\begin{pmatrix} \Psi_{t+1} \\ \Psi_t \end{pmatrix} = \begin{pmatrix} E - \varepsilon_t & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \varepsilon_{t-1} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - \varepsilon_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_0 \end{pmatrix}$$

Chaotic Map:

$$z(t+1) = \mathbf{B}(t)\mathbf{B}(t-1)\cdots\mathbf{B}(1)z(0)$$

Simple Examples:

1) Numbers

$$P_N = x_1 x_2 \cdots x_N \sim e^{\gamma N}$$
$$\gamma = \overline{\ln|x|}$$

2) Comutative & diagonalizable matrices

$$P_N = X_1 X_2 \cdots X_N$$
$$\gamma = \max_i [\overline{\ln|x^d_{ii}|}]$$

3) Guirvarch'h 1983

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma = \begin{cases} \ln|a|, & p = 1 \\ 0, & p < 1 \end{cases}$$

Furstenberg Theorem: maximum LE

Assume $\overline{\ln^{+} \| \mathbf{X} \|} < \infty$ which average taken over the measure $d\mu(\mathbf{X}) = \mathcal{P}(\mathbf{X})d\mathbf{X}$

then

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \ln \| \mathbf{P}_N \|$$

exists with probability 1, moreover it is nonrandom: $\gamma = \bar{\gamma}$

In case of D x D matrices with determinant equal to 1 and nondegenerate spectrum it is positive.

This theorem neglects finer structure of lower growth rates

Application to 1D models: K. Ishii ≈1970

Osceledec Theorem (Multiplicative Ergodic Theorem)

Consider a stationary sequence of DxD matrices, with $\overline{\ln \| \mathbf{X} \|} < \infty$ then

$$\lim_{N\to\infty} [\mathbf{P}_N^\dagger(\omega)\mathbf{P}_N(\omega)]^{1/2N} = \mathbf{V}(\omega)$$

exists.

$\mathbf{V}(\omega)$ has D real positive eigenvalues $\exp(\gamma_i(\omega))$ where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_D$

If the sequence of matrices is ergodic then Lyapunov spectrum does not depend on particular spectrum ω

Note: eigenvalues of $\mathbf{P}_N$ itself are complex in general and their real part are not equal to γ except for special cases (Orszag et al. 1987)

Virster Theorem

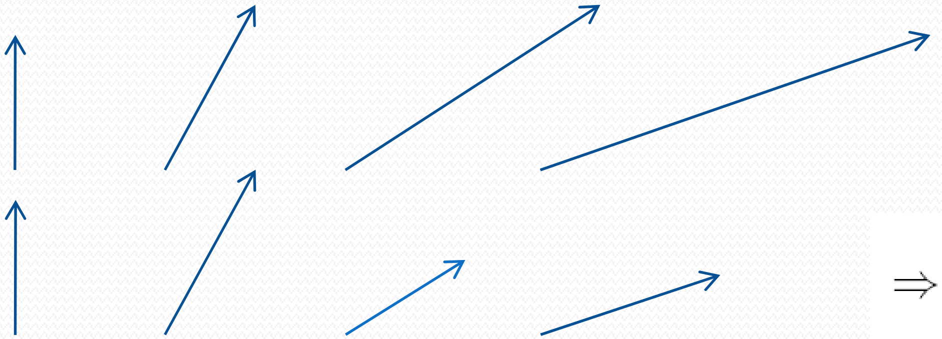
For a sequence of independent distributed 2Dx2D symplectic* random matrices, Lyapunov exponents are nondegenerate:

$$\gamma_1 > \gamma_2 > \cdots > \gamma_D > 0$$

and $\gamma_{2D-i+1} = -\gamma_i$

* Symplectic matrix: $S^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

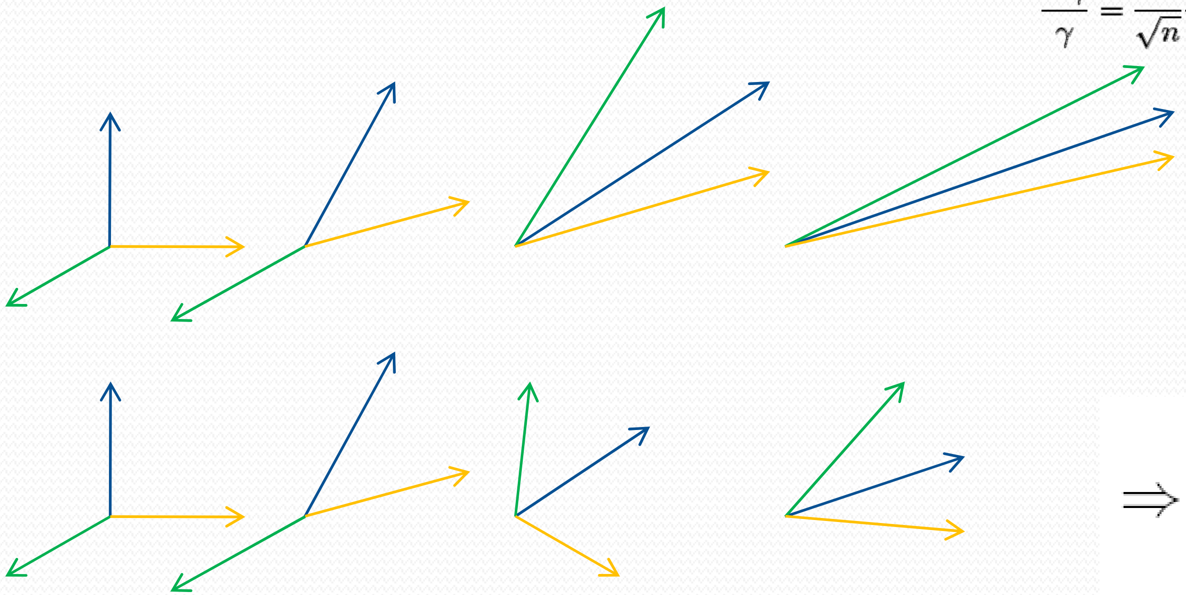
Numerical computation of LE spectrum (Benettin et al. 1980)



... overflow

$$\Rightarrow \gamma_{max} = \frac{1}{n\tau}\sum_{\alpha=1}^{n} \ln(d_\alpha)$$

$$\frac{\Delta\gamma}{\gamma} = \frac{1}{\sqrt{n}}\frac{\sqrt{\langle(\ln d_\alpha)^2\rangle - \langle \ln d_\alpha\rangle^2}}{\langle \ln d_\alpha\rangle}$$



$$\Rightarrow \gamma_i = \frac{1}{n\tau}\sum_{\alpha=1}^{n} \ln(d_\alpha^{(i)})$$

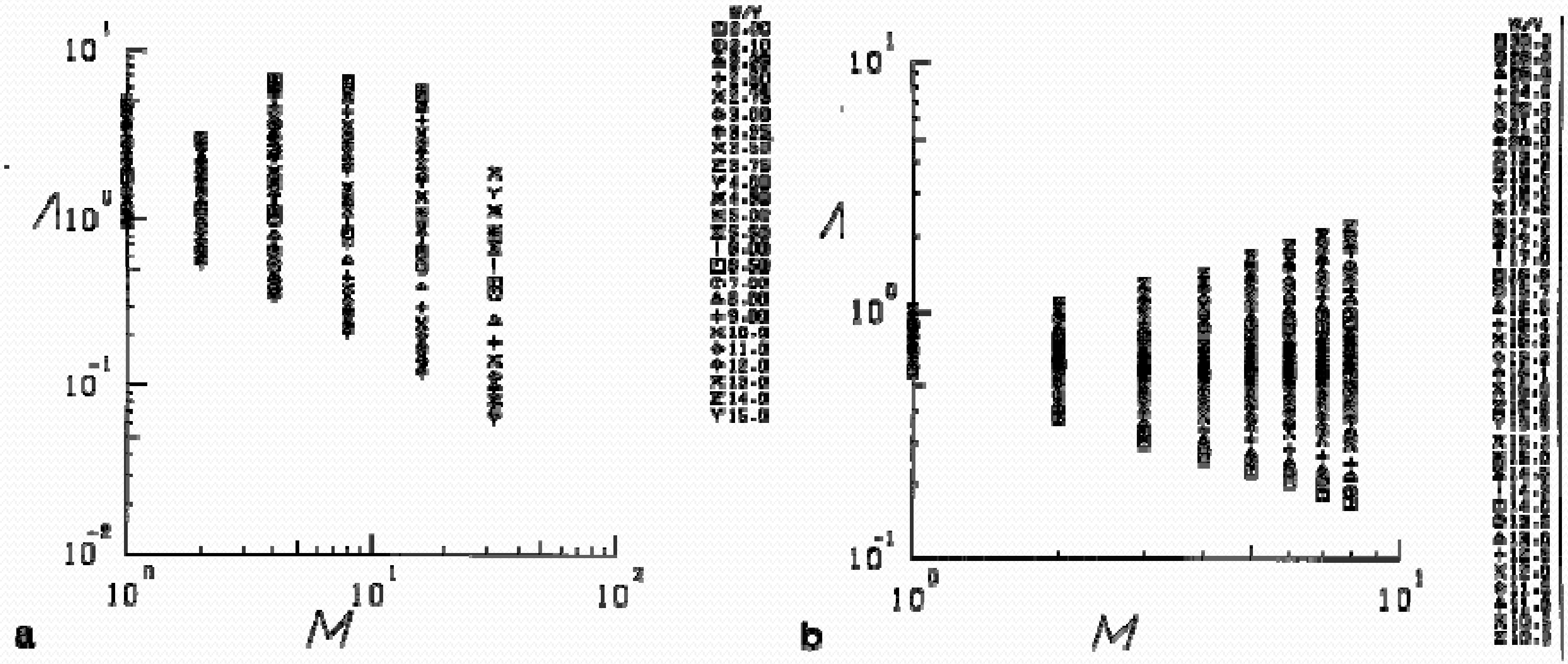
Gram-Schmidt

2D and 3D Anderson model (Pichard et al. 1981, MacKinnon et al. 1981)

$$H = \sum_i \varepsilon_i \hat{a}_i^\dagger \hat{a}_i + t \sum_{\langle ij \rangle} \hat{a}_i^\dagger \hat{a}_j \qquad \varepsilon_i \in [-W/2, W/2] \qquad \Lambda(W, M) = \lambda_M(W)/M$$

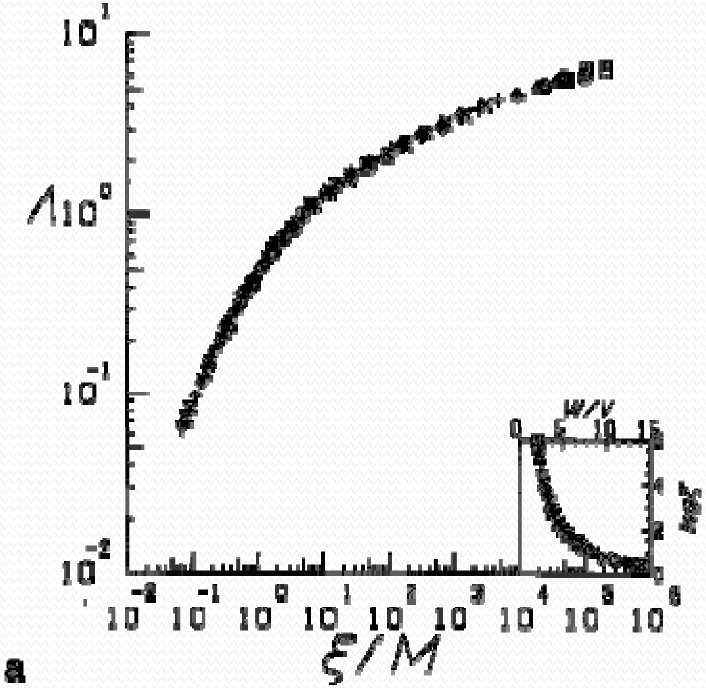
2D

3D

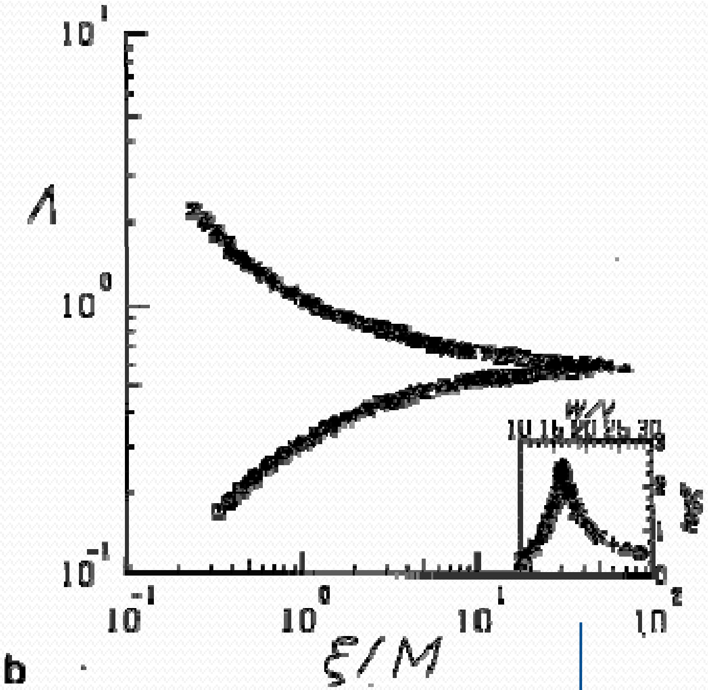


After MacKinnon and Kramer

Single parameter scaling: $$\Lambda(W, M) = \lambda_M(W)/M = f\left(\frac{\xi(W)}{M}\right)$$



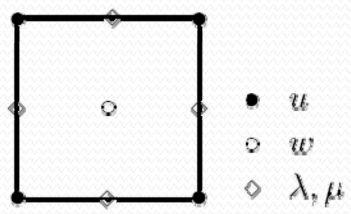
a

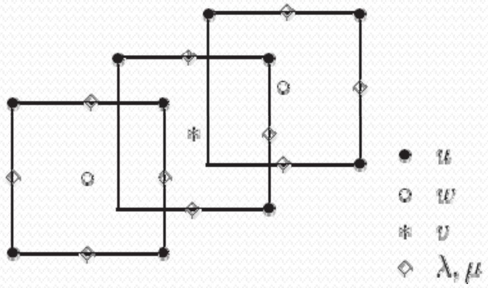


b

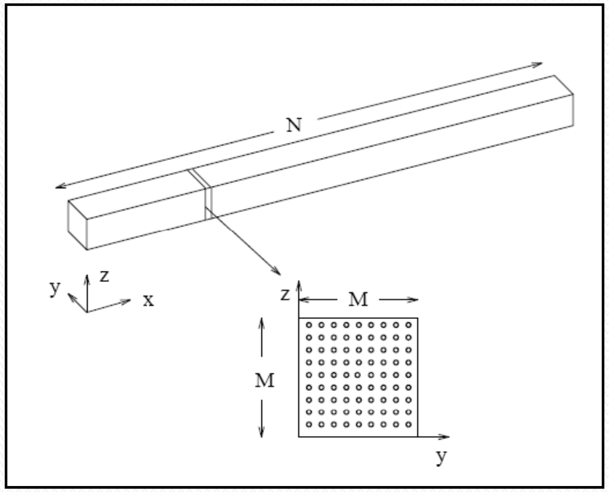
$$\xi(W) = |W - W_c|^{-\nu}$$

Elastic wave: discrete wave equation



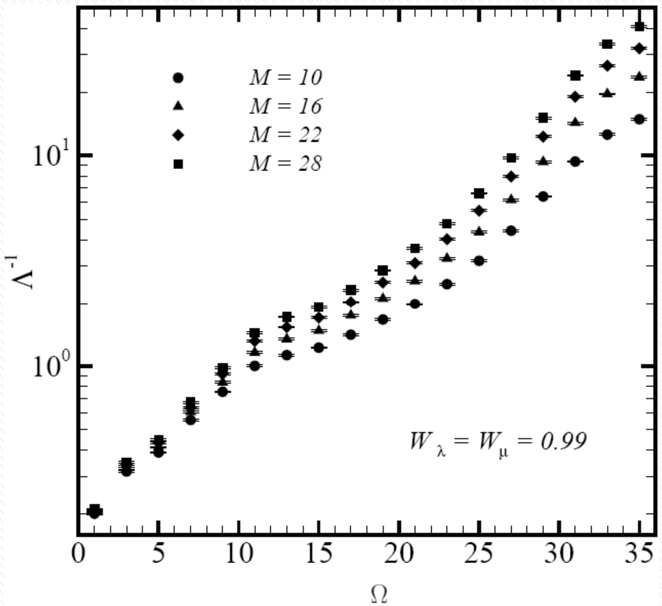


n	$M\gamma_m$	error	n	$M\gamma_m$	error
1	15.421216	0.018	13	-0.146944	0.007
2	13.044478	0.015	14	-0.384629	0.007
3	10.675876	0.012	15	-0.769397	0.008
4	8.019660	0.010	16	-1.404398	0.009
5	6.411892	0.009	17	-2.559604	0.010
6	5.138542	0.008	18	-3.831230	0.010
7	3.831297	0.008	19	-5.138530	0.009
8	2.559524	0.008	20	-6.411741	0.010
9	1.404456	0.007	21	-8.019935	0.012
10	0.769496	0.006	22	-10.675919	0.014
11	0.384616	0.006	23	-13.044517	0.016
12	0.146903	0.006	24	-15.421134	0.018

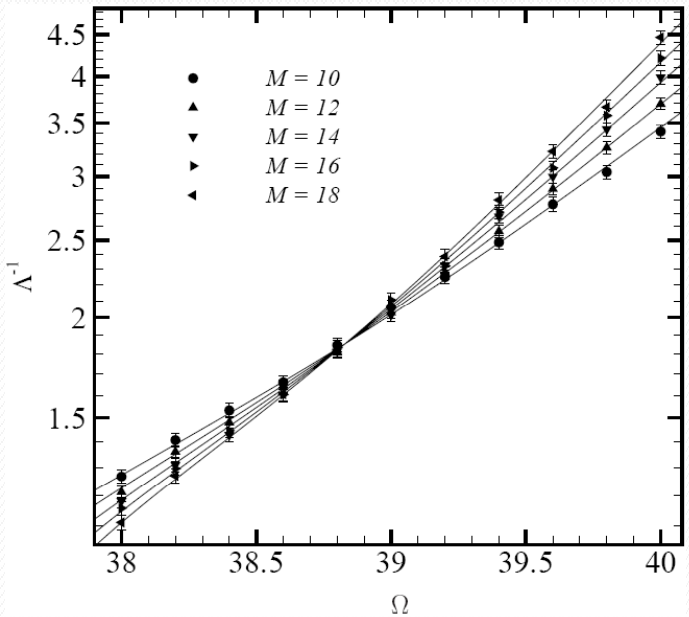


Transfer Matrix

D=2



D=3



Localization in 2D is stronger than in 3D

Finite size scaling

$$\xi \sim |\Omega - \Omega_c|^{-\nu}$$

$$\Lambda^{-1} = F(\chi M^{1/\nu}) = \sum_{i=0}^{n} a_i \chi^i M^{i/\nu}$$

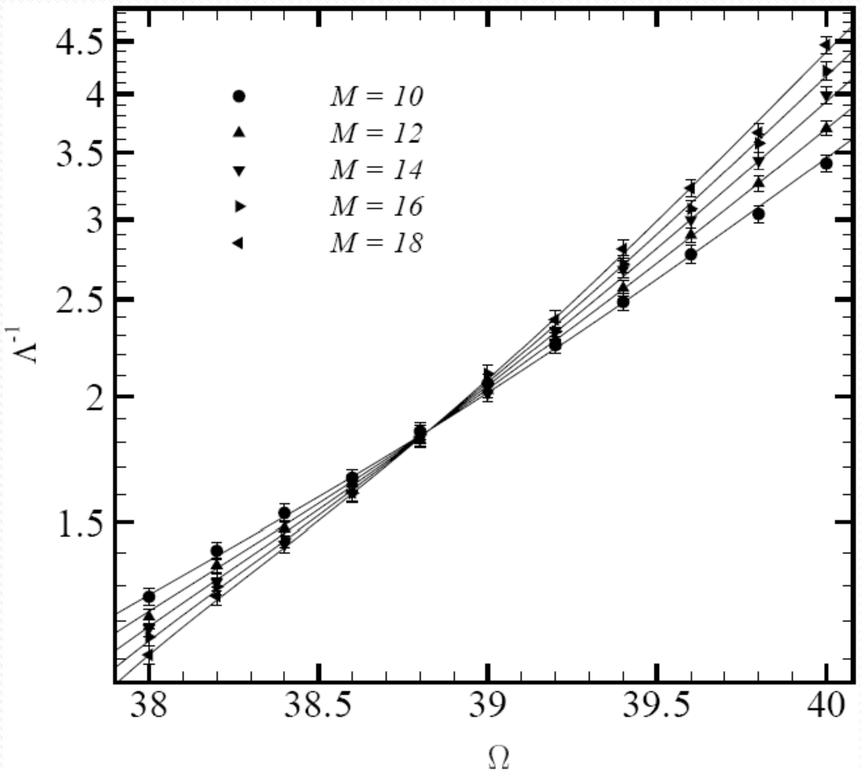
$$\chi = \sum_{i=1}^{m} b_i \Omega_r^i$$

$$\Omega_r = (\Omega - \Omega_c)/\Omega_c$$

$n = 3$ and $m = 2$

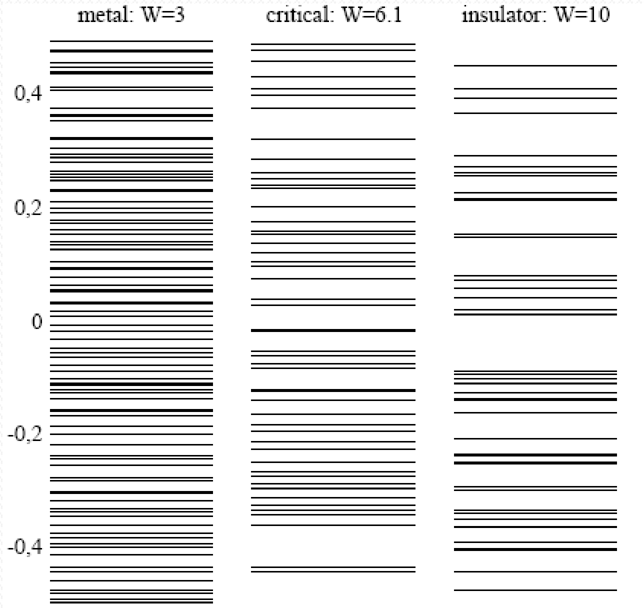
And Levenberg-Marqurt multifunction fitting

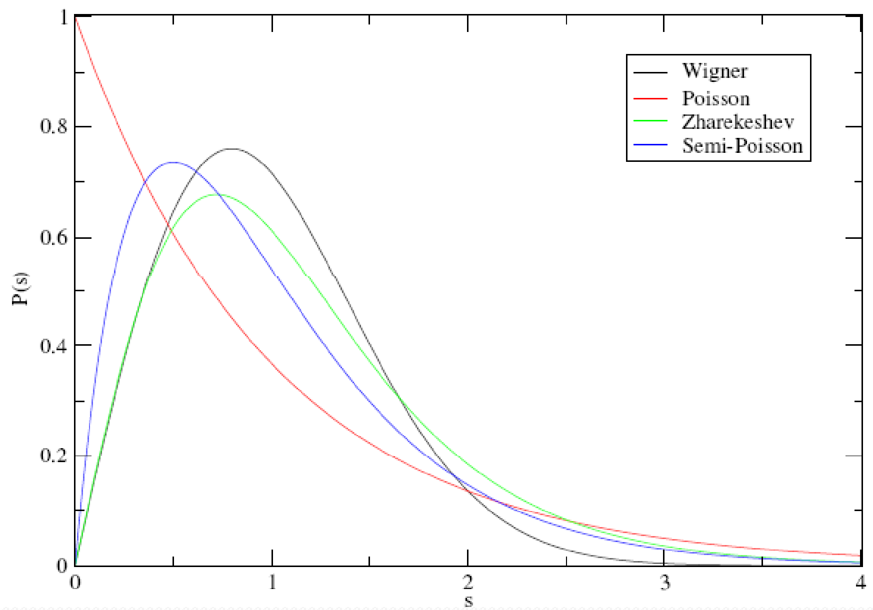
⇓



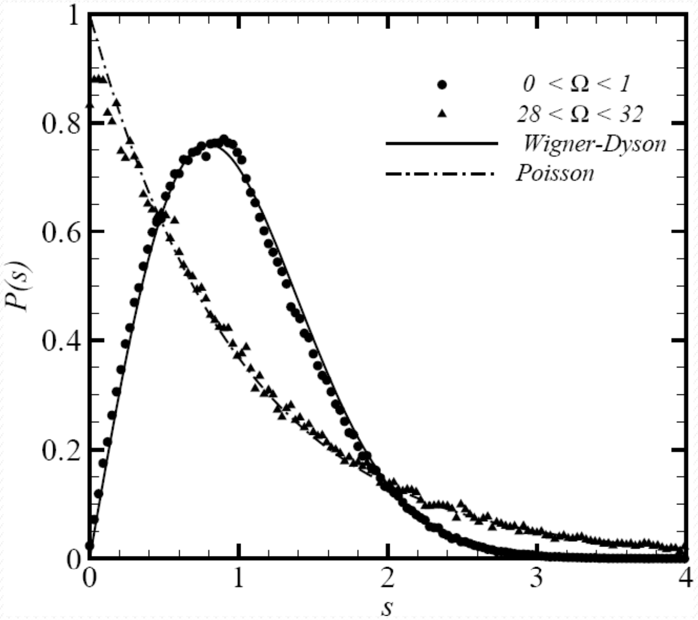
$$\nu \simeq 1.89 \pm 0.17\ , \quad \Lambda_c^{-1} \simeq 1.84 \pm 0.06\ , \quad \Omega_c \simeq 38.82 \pm 0.06$$

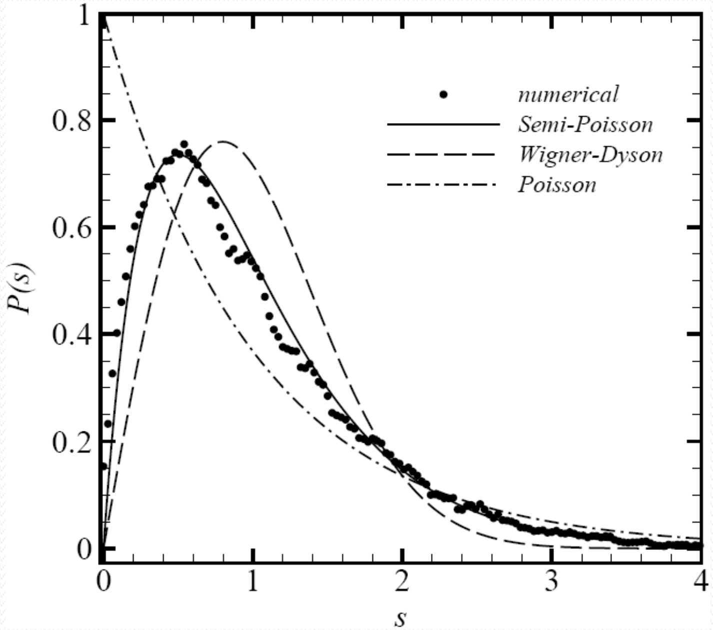
Statstics of energy levels:





Elastic wave





D=2

D=3, Critical region

Orthogonal Universality class

Thanks