Quantum effects in some problems with time-dependent boundary conditions

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Outline:

- Diffraction in time (DIT)
- Supperarrivals (SA)

DIT

- Diffraction in time was initially introduced by Moshinsky.
- A beam of particles impinging from the left on a totally absorbing shutter located at the origin which is suddenly turned off in an instant.
- The transient current has a close mathematical resemblance with the intensity of light in the Fresnel diffraction by a straight edge.

- An interesting feature of the solutions for cut-off initial waves, occurring both in the free case and in the presence of a potential interaction is that, if initially there is a zero probability for the particle to be at $x > 0$, as soon as $t = 0+$, there is instantaneously a finite, though very small, probability of finding the particle at any point $x > 0$.

- This non-local behavior of the Schrödinger solution is due to its non-relativistic nature and not as a result of the quantum shutter setup. The application of the Klein–Gordon equation to the shutter problem shows that the probability density is restricted to the accessible region $x < ct$ (c is the speed of light).

Optics and Quantum mechanics

$$\psi(\vec{r},t) \quad = \quad \left[\frac{m}{۲\pi i\hbar(t-t')}\right]^{۳/۲} \int d\vec{r}' \, e^{\frac{im}{۲\hbar}\frac{(\vec{r}-\vec{r}')^{۲}}{t-t'}} \psi(\vec{r}',t')$$

$$E(x,y,z) \quad = \quad \frac{e^{-ik(z-z')}}{\lambda(z-z')} \int\int dx'dy' e^{\frac{-ik}{۲}\frac{(x-x')^{۲}+(y-y')^{۲}}{z-z'}} E(x',y',z')$$

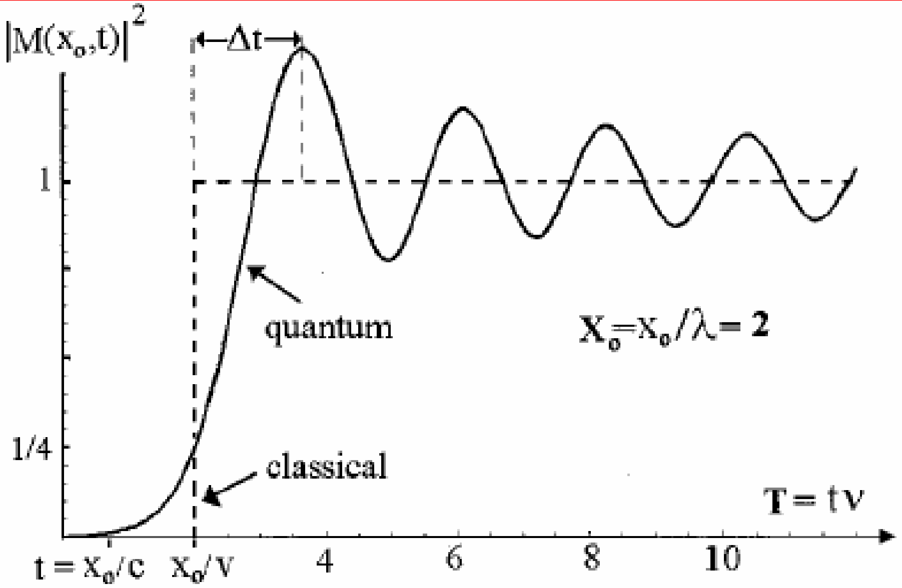
Wave function for shutter problem

$$\begin{aligned} \psi(x,\text{۰}) &= e^{ipx/\hbar} & x \leq \text{۰} \\ &= \text{۰} & x > \text{۰} \end{aligned}$$

$$M(x,t) = \frac{e^{-i\pi/\text{۴}}}{\sqrt{\text{۲}}} e^{i(px-Et)/\hbar} \left\{ \left[C\left(\xi(x,t)\right) + \frac{\text{۱}}{\text{۲}} \right] + i \left[S\left(\xi(x,t)\right) + \frac{\text{۱}}{\text{۲}} \right] \right\}$$

$$\xi(x,t) = \sqrt{\frac{m}{\pi\hbar t}} \left(\frac{p}{m} t - x \right)$$

Moshinky's quantum DIT

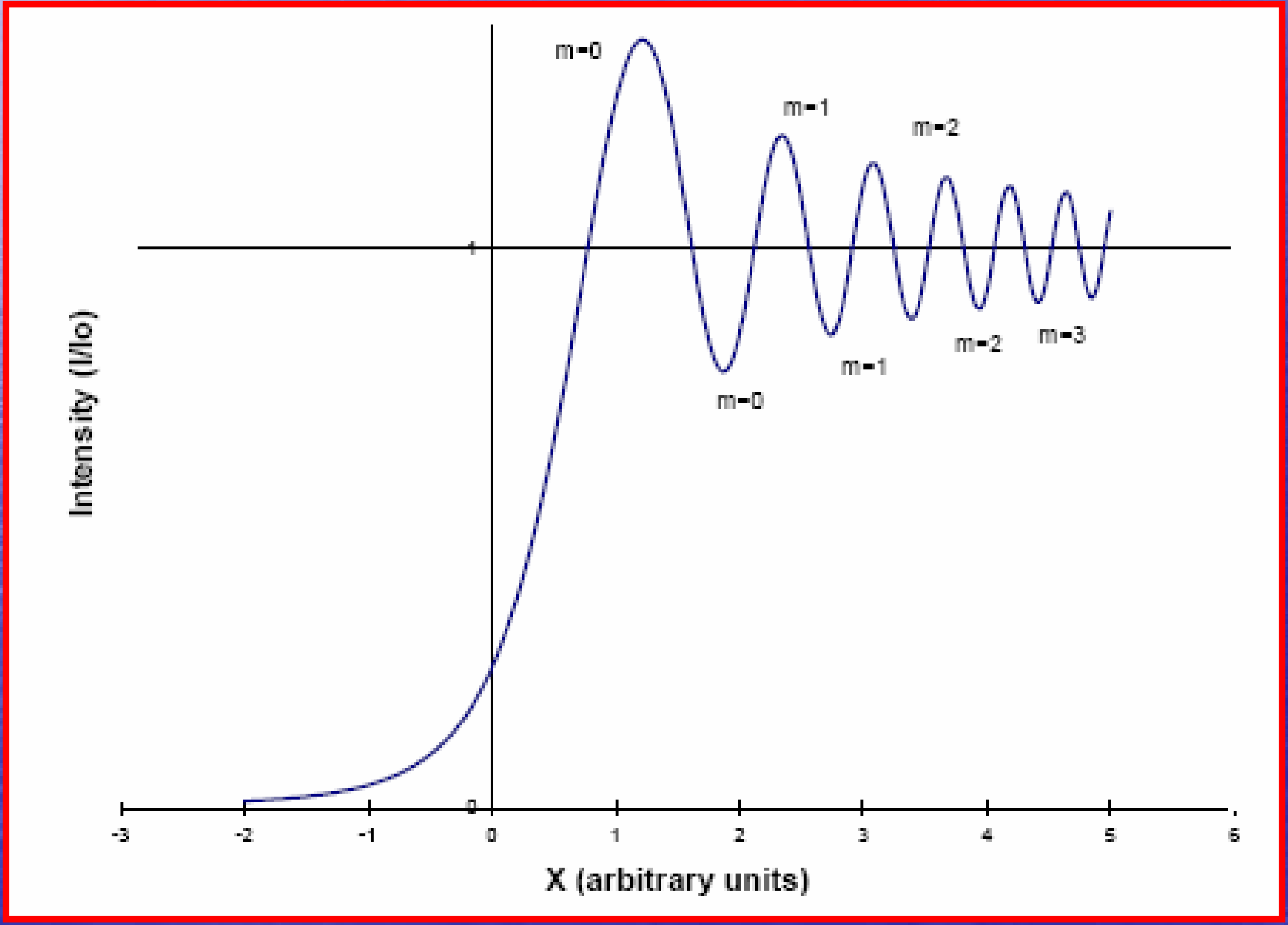


Moshinsky's quantum diffraction in time. $X \equiv x/\lambda$ and $T \equiv t\nu$ are dimensionless quantities. Observation point $X_0 = 2$.

$$X \equiv x/\lambda = xp/2\pi\hbar$$

$$T \equiv t\nu = t\varepsilon/2\pi\hbar$$

Diffraction by a straight edge



DIT of a particle in box with reflecting wall

$$\psi_n(x,t) = \sqrt{\frac{m}{2\pi \mathrm{i}\hbar t}}\sqrt{\frac{2}{L}}\int_0^L \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar t}(x-x')^2}\sin(k_n x')\,\mathrm{d}x',$$

$$= \sqrt{\frac{m}{4\pi \mathrm{i}^3\hbar t L}}\int_0^L \mathrm{e}^{\frac{\mathrm{i}m}{2\hbar t}(x-x')^2}(\mathrm{e}^{\mathrm{i}k_n x'} - \mathrm{e}^{-\mathrm{i}k_n x'})\,\mathrm{d}x',$$

$$\equiv \psi_{n,+}(x,t) + \psi_{n,-}(x,t),$$

$$\psi_{n,+}(x,t) = \frac{1}{\sqrt{4\mathrm{i}^3 L}}\,\mathrm{e}^{\mathrm{i}k_n x - \mathrm{i}E_n t/\hbar}[F_n(x-L,t) - F_n(x,t)],$$

$$\psi_{n,-}(x,t) = \frac{1}{\sqrt{4\mathrm{i}^3 L}}\,\mathrm{e}^{-\mathrm{i}k_n x - \mathrm{i}E_n t/\hbar}[F_n(L-x,t) - F_n(-x,t)],$$

$$F_n(x,t) = \int_0^{\xi_n(x,t)} \mathrm{d}u\,\mathrm{e}^{\mathrm{i}\pi u^2/2},$$

$$\xi_n(x,t) = \sqrt{\frac{m}{\pi\hbar t}}\,(v_n t - x)$$

Probability density

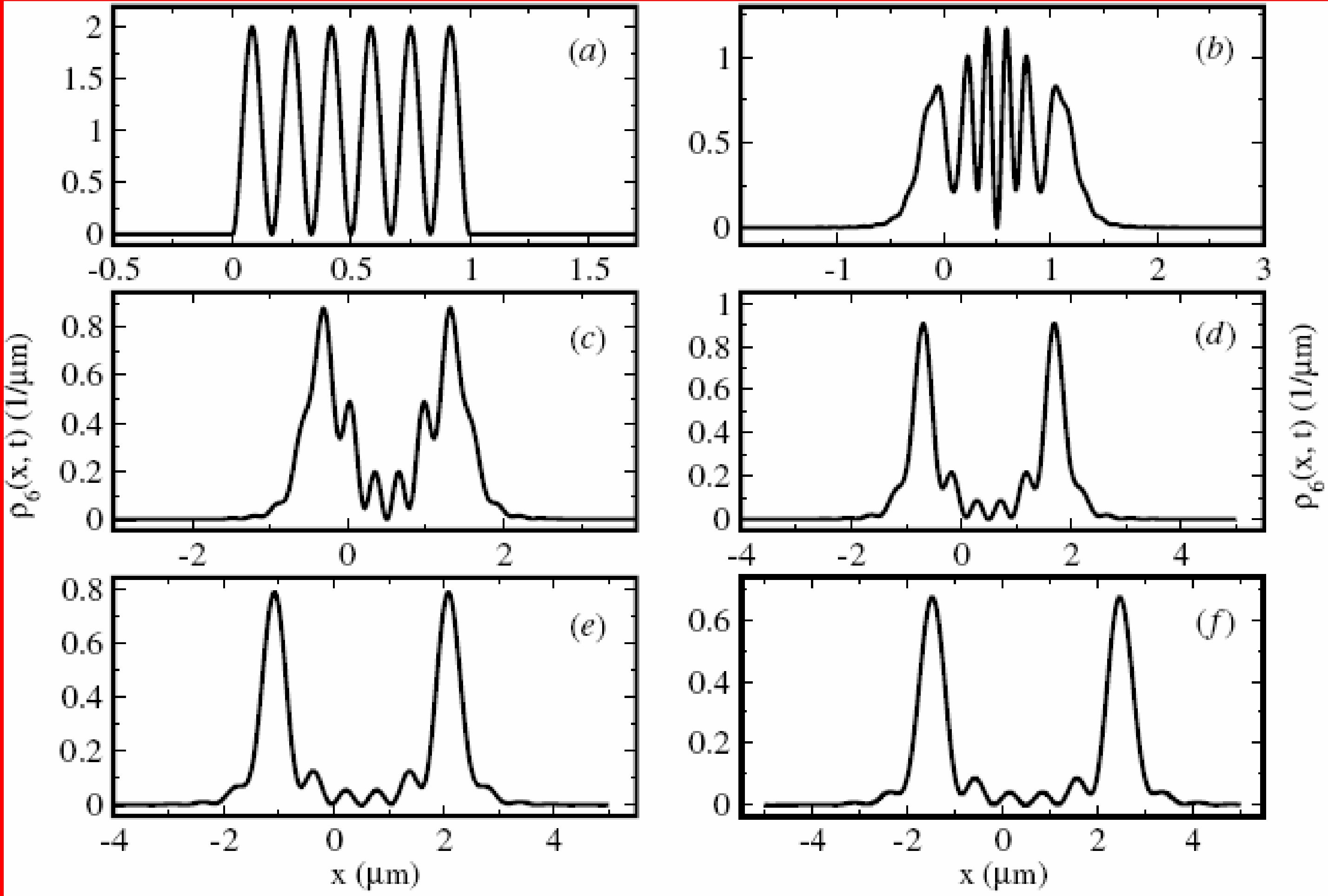


Figure 1. Probability density versus distance $x(\mu\text{m})$ for state $n = 6$ at times (*a*) $t = 0$, (*b*) $t = 0.03$ ms, (*c*) $t = 0.06$ ms, (*d*) $t = 0.09$ ms, (*e*) $t = 0.12$ ms and (*f*) $t = 0.15$ ms.

Probability current density

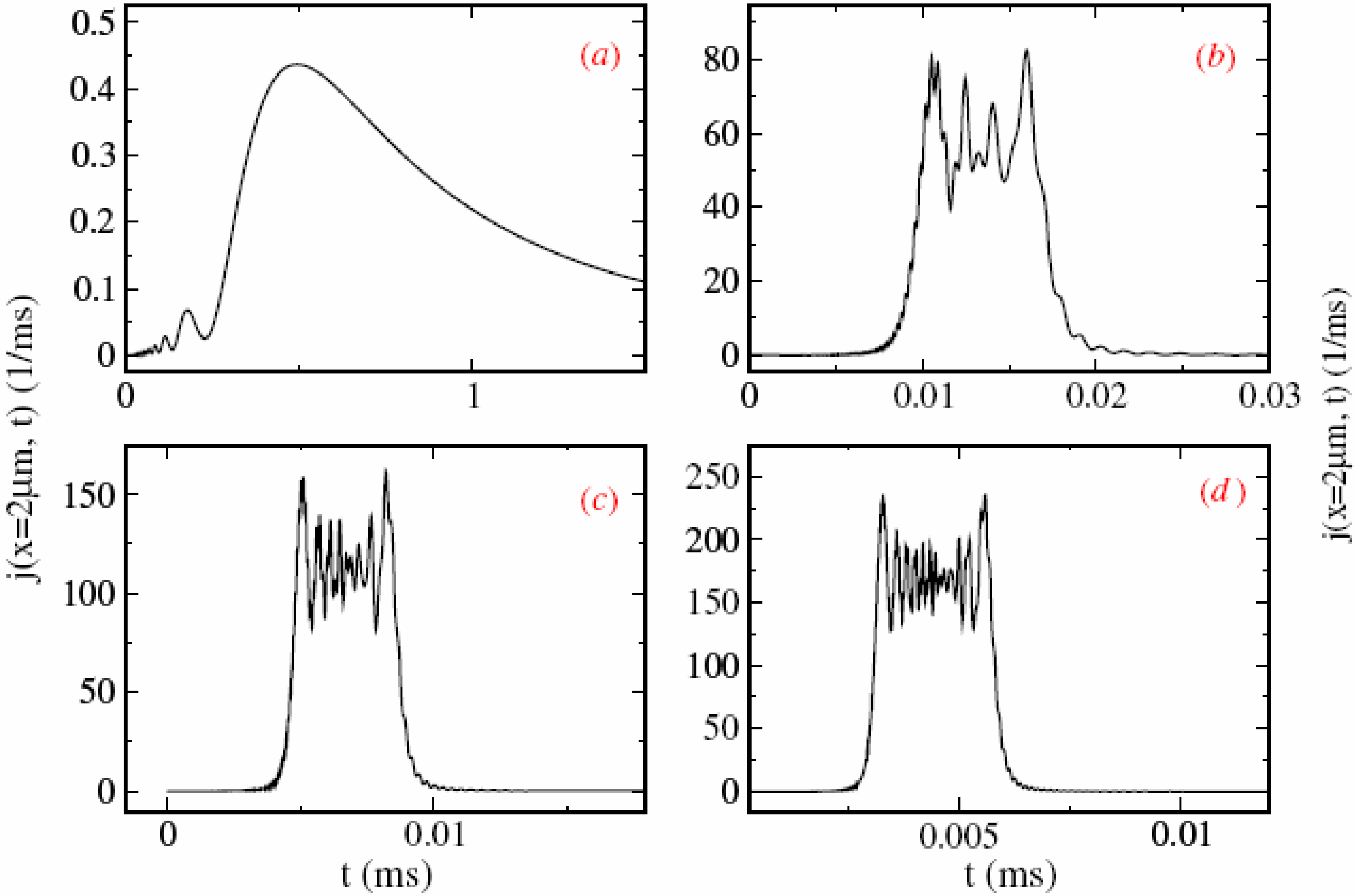


Figure 2. Probability current density (1 ms^{-1}) as a function of time t (ms) at observation point $x = 2\ \mu$m for states (*a*) $n = 1$, (*b*) $n = 50$, (*c*) $n = 100$ and (*d*) $n = 150$.

Bohmian trajectories

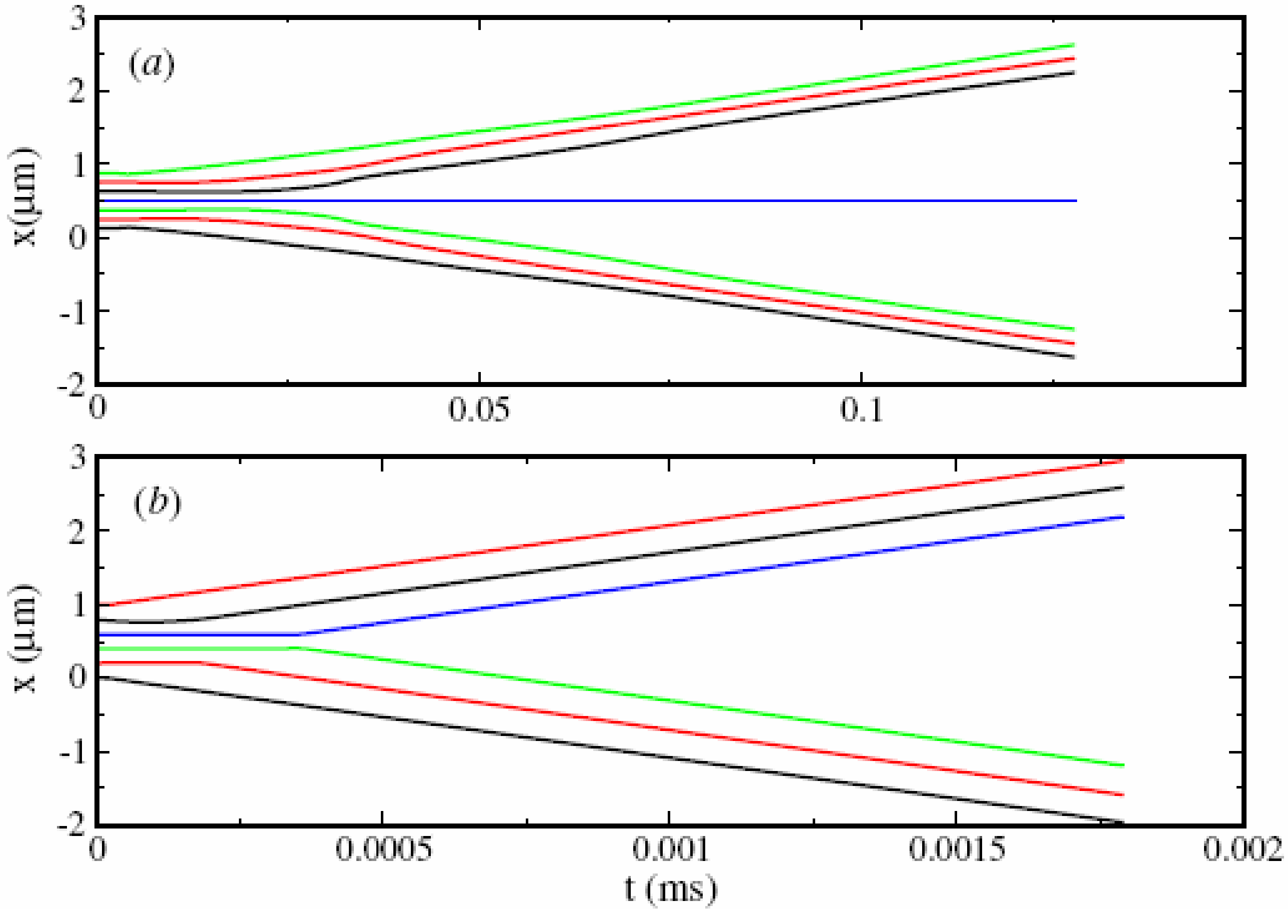


Figure 3. A selection of Bohmian paths for states (*a*) $n = 7$ and (*b*) $n = 500$.

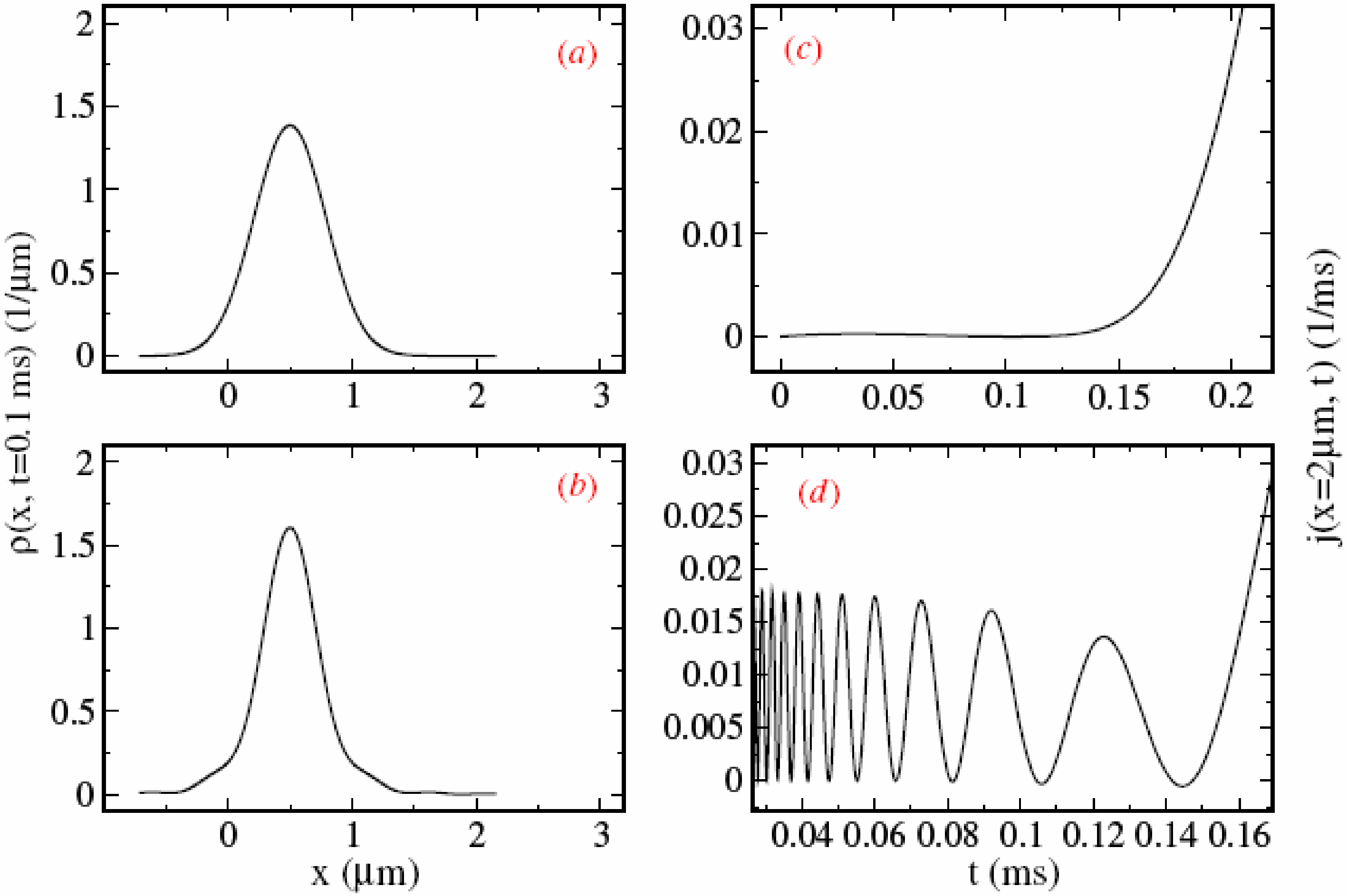
DIT of a particle in a box with absorbing wall

- Now, we consider the case in which initial wave function is a motionless localized Gaussian wave-packet inside the box with negligible overlap with the walls of the well.

$$\psi_0(x) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} e^{-\frac{(x-x_0)^2}{4\sigma_0^2}} \chi_{[0,L]}(x)$$

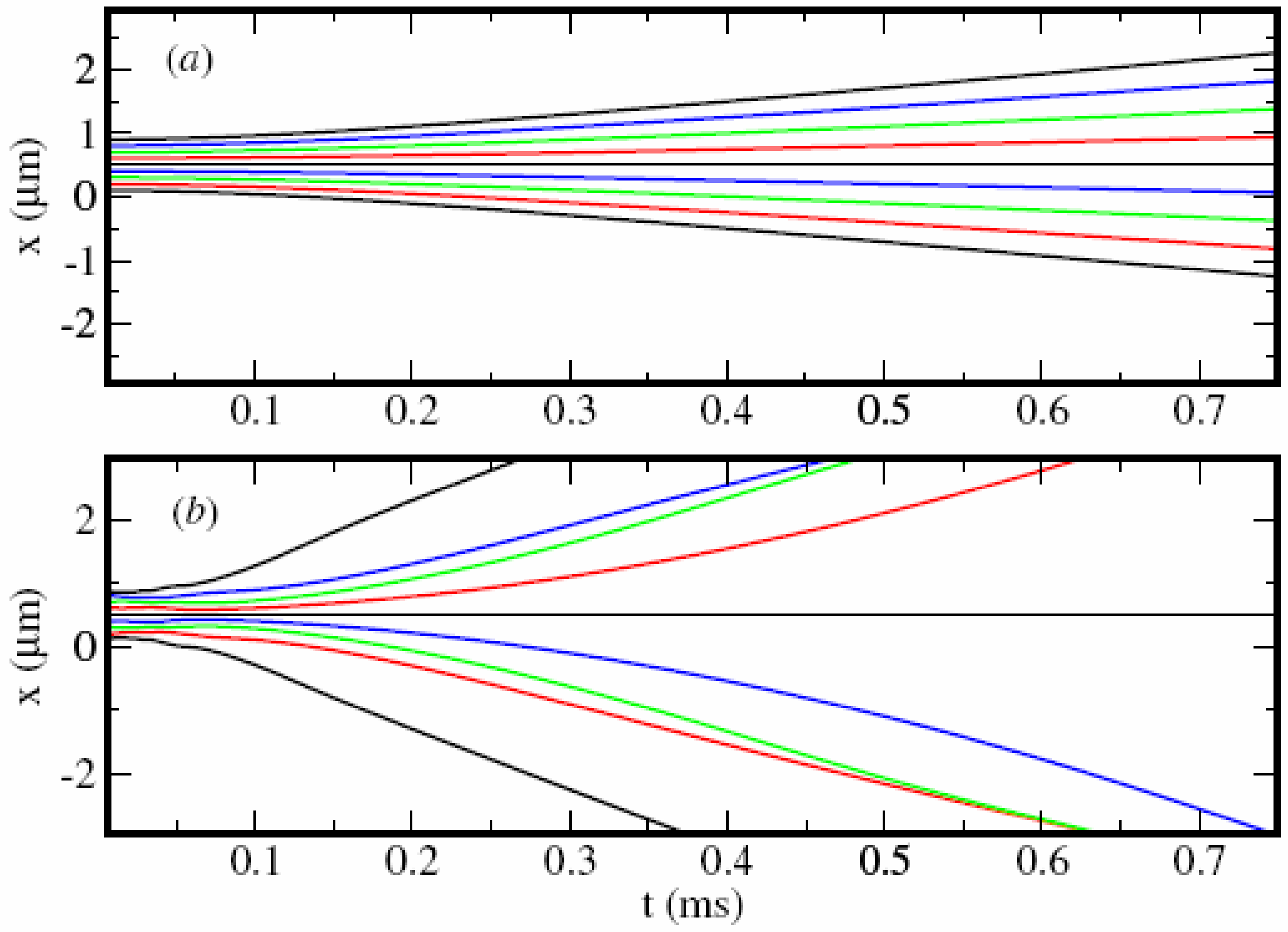
$$\psi(x,t) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \sqrt{\frac{m}{2\pi i\hbar t}} \int_0^L \mathrm{d}x' \, e^{-\frac{(x'-x_0)^2}{4\sigma_0^2} + \frac{im}{2\hbar t}(x-x')^2}$$

Probability and probability current density



Probability density ($1\ \mu m^{-1}$) versus distance $x\,(\mu m)$ at time $t = 0.1$ ms for (a) a free motionless Gaussian wave-packet and (c) a motionless truncated Gaussian wave-packet initially confined in a box. Probability current density ($1\ ms^{-1}$) versus time t (ms) at observation point $x = 2\ \mu m$ for (b) a free motionless Gaussian wave-packet and (d) a motionless truncated Gaussian wave-packet initially confined in a box.

Bohmian paths



A selection of Bohmian paths for (*a*) a free motionless Gaussian wave-packet and (*b*) a motionless truncated Gaussian wave-packet initially confined in a box.

Superarrivals

- This effect was discovered by S. Bandyopadhyay, A. S. Majumdar, D. Home in year 2002 by calculating the *time evolving* probability of reflection of a Gaussian wave packet from a rectangular potential barrier *while* it is perturbed by *reducing* its height.
- The time evolving reflection probability is given by

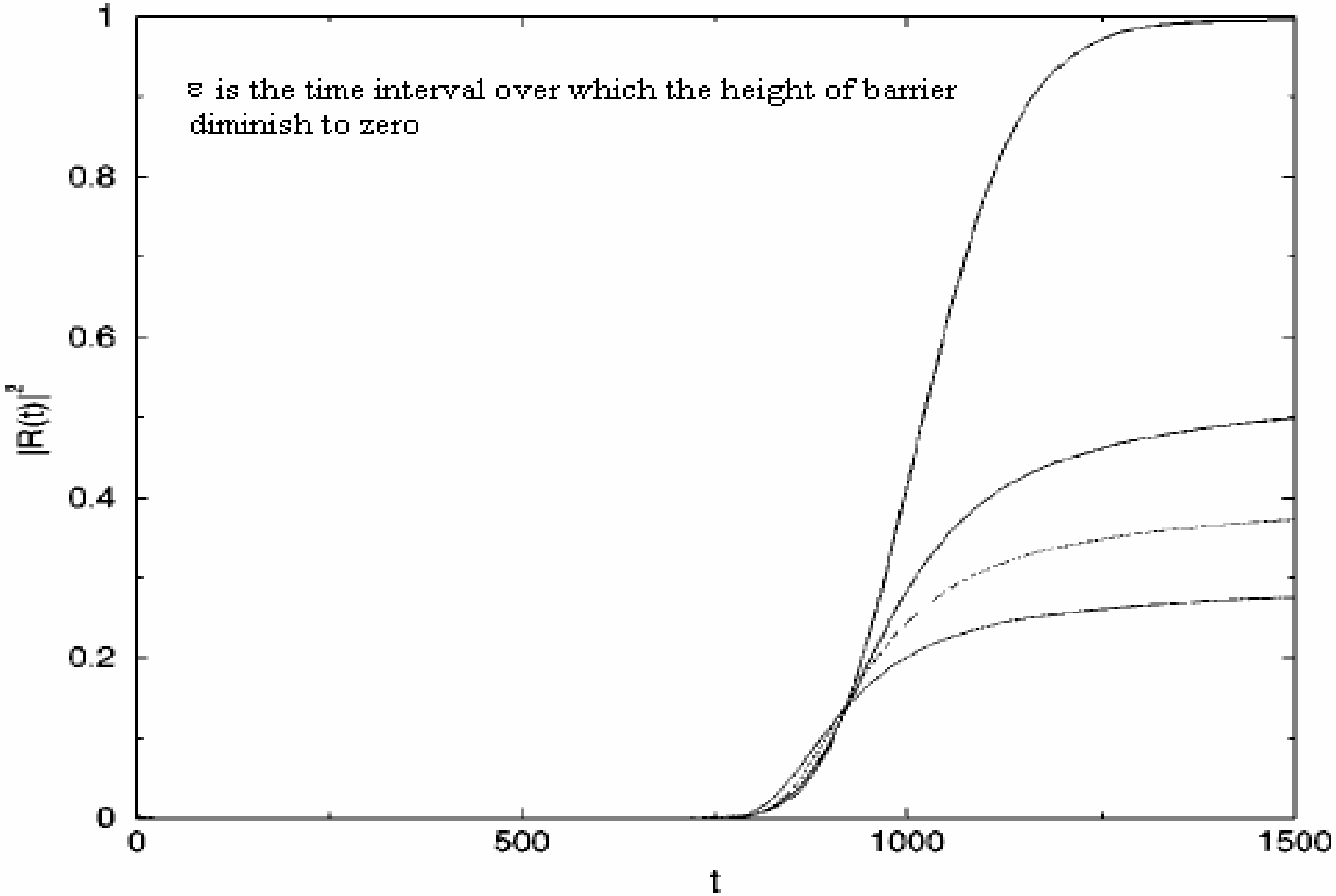
$$|R(t)|^2 = \int_{-\infty}^{x'} |\psi(x,t)|^2 dx$$

- A finite time interval is found during which $|R(t)|^2$ shows an *enhancement (*superarrivals)! in the perturbed case even though the barrier height is reduced. This time interval and the amount of enhancement depend on the *rate* at which the barrier height is made zero.
- The phenomenon of superarrivals is inherently

quantum mechanical.

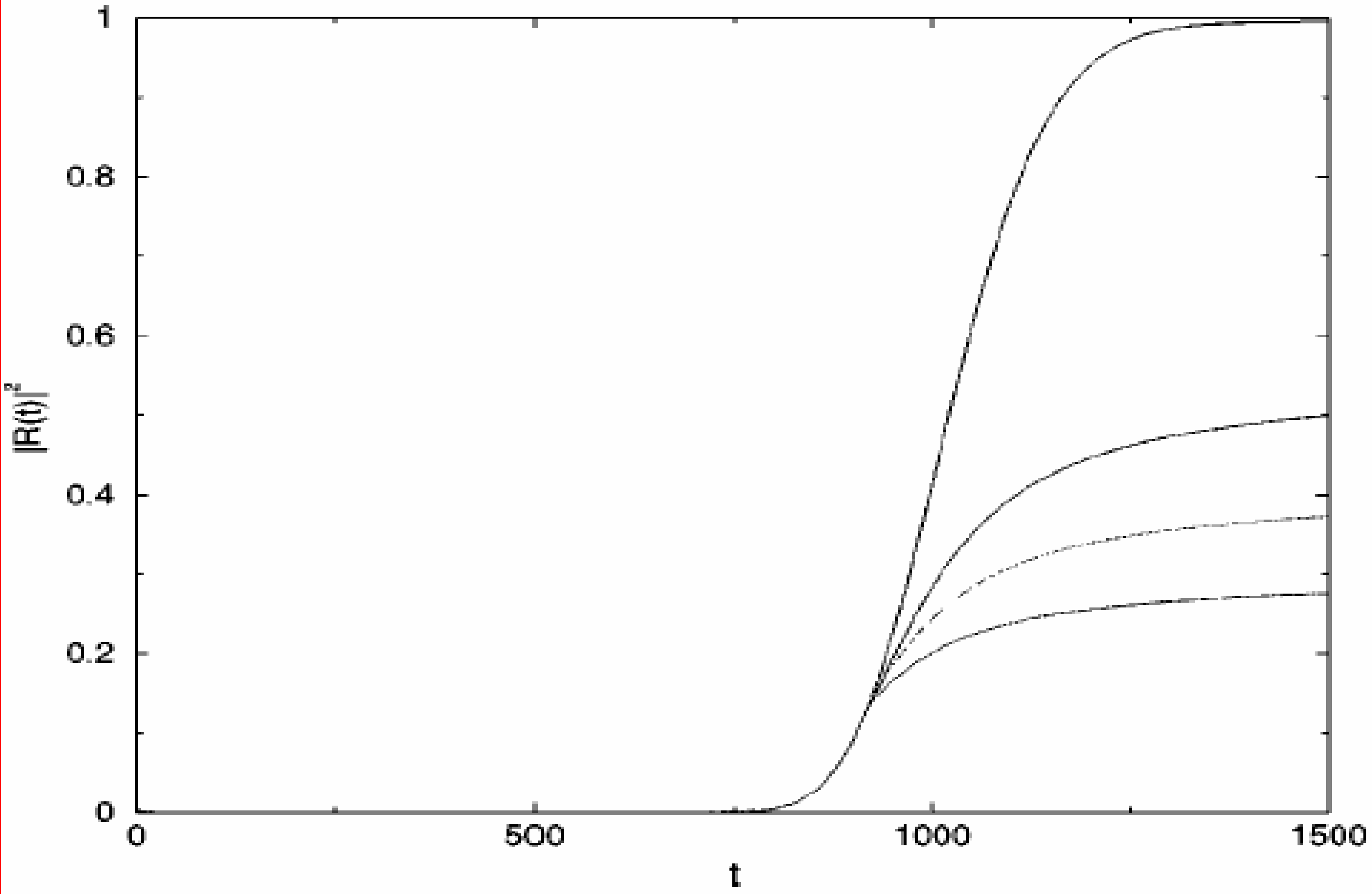
- The *origin* of superarrivals may be understood by considering the wave function to act as a ‘‘field’’ through which a disturbance from the ‘‘kick’’ provided by perturbing the barrier travels with a definite speed.

Quantum treatment of reflection probability



The top curve corresponds to the static case and reaches value 1 asymptotically. $|R(t)|^2$ for other curves correspond to various values of ϵ. The curve with the lowest asymptotic value corresponds to the smallest value of ϵ chosen for this set. As one increases ϵ, superarrivals are slowly wiped off.

Classical treatment of reflection probability



The time-varying reflection probability for the classical evolution is plotted for the same values of ϵ as previous Fig. The absence of superarrivals in this case demonstrates the nonclassical nature of this phenomenon.

Three intervals

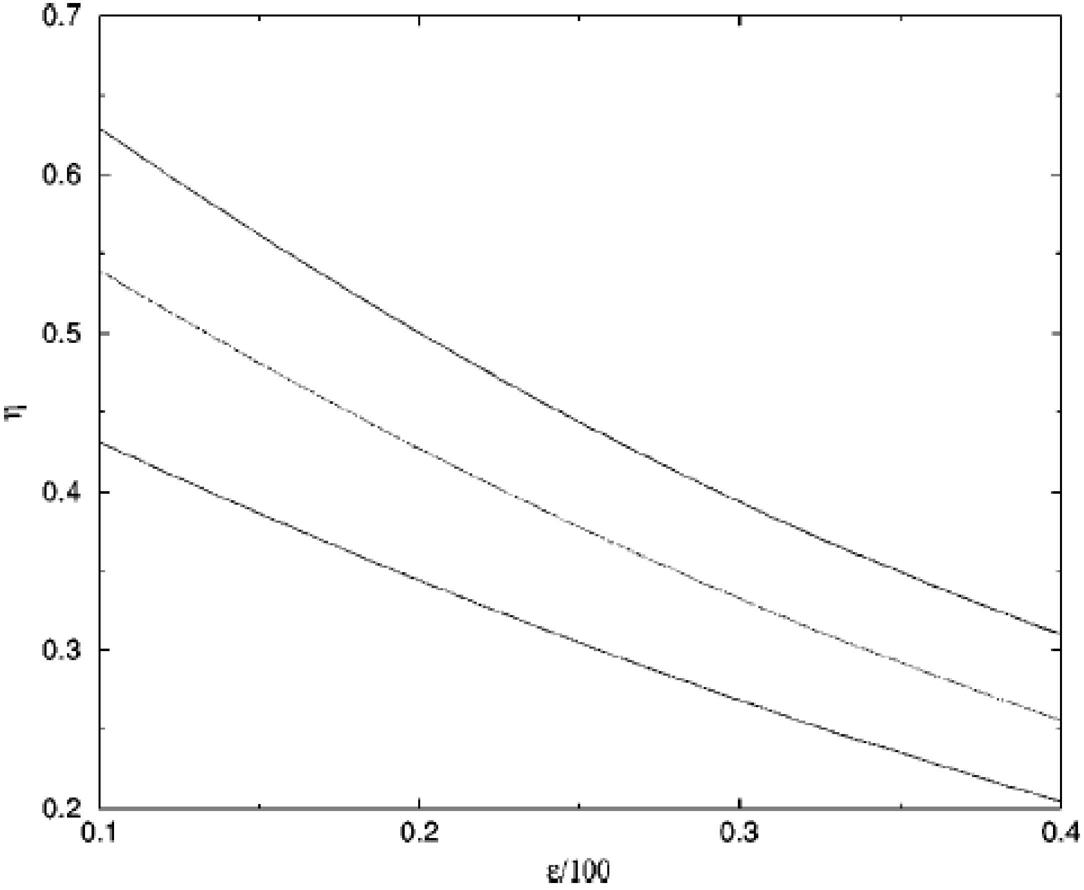
$$|R_p(t)|^2 = |R_s(t)|^2, \qquad t \leq t_d,$$

$$|R_p(t)|^2 > |R_s(t)|^2, \qquad t_d < t \leq t_c,$$

$$|R_p(t)|^2 < |R_s(t)|^2, \qquad t > t_c,$$

- *tc* is the instant when the two curves cross each other, and
- *td* is the time from which the curve corresponding to the perturbed case starts deviating from that in the unperturbed case

Quantitavive measure of SA



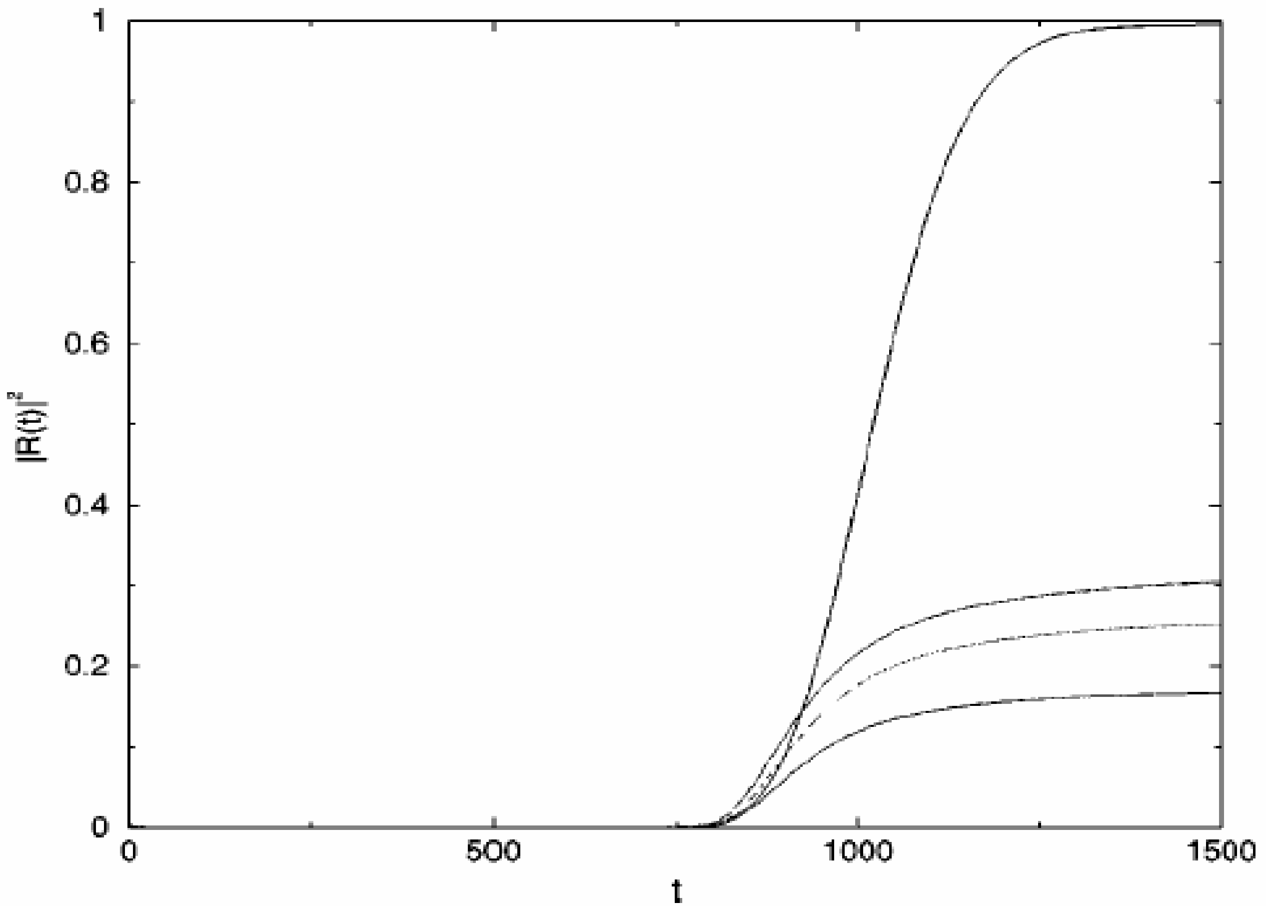
$$\eta = \frac{I_p - I_s}{I_s}$$

$$I_p = \int_{\Delta t} |R_p(t)|^2 dt$$

$$I_s = \int_{\Delta t} |R_s(t)|^2 dt$$

The magnitude of superarrivals η diminishes with an increase in ϵ, the time taken for barrier height reduction. This behavior is seen for three different detector positions $x' = -0.4$, -0.5, and -0.6.

Dependence of SA to width of the barrier



Here we show that superarrivals diminish by decreasing the width of the barrier. They completely disappear for a small enough barrier width. The three curves for the perturbed cases correspond to the widths of 0.016, 0.008, and 0.004.

Results:

(a) There exists a finite time interval Δt during which an *increase* in the reflection probability (superarrivals) occurs for the perturbed cases compared to the unperturbed situation.

(b) Superarrivals are inherently *nonclassical*.

(c) The magnitude of superarrivals η is appreciable only in cases where the wave packet has some *significant overlap* with the barrier during its switching off. Both η and Δt (duration of superarrivals) *fall off* with *increasing* ϵ.

(d) Superarrivals given by η gradually reduce to *zero* upon *decreasing* the barrier width, while keeping the initial barrier height B fixed.

Signal velocity:

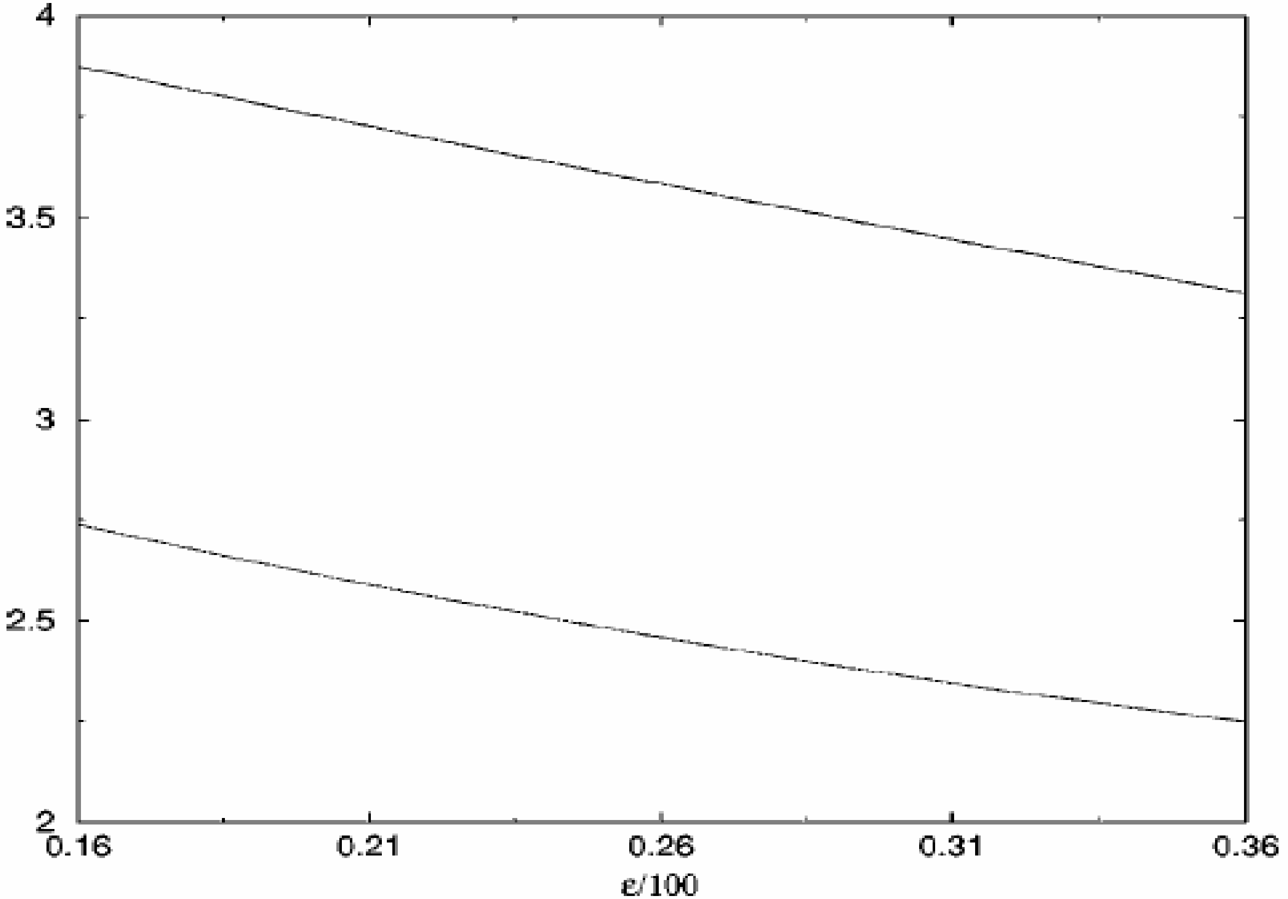
- *How fast* the influence of barrier perturbation travels across the wave packet?

$$v_e = \frac{D}{t_d - t_p}$$

D is the distance of detector from the barrier

t_p is the time at which perturbation is started

Dependence of duration of SA and signal velocity to the rate of perturbation:



The upper curve represents a plot of Δt (duration of superarrivals) versus ϵ. The lower curve is a plot of v_e/v_g versus ϵ. Here the detector position $x' = -0.4$.

0rigin of SuperArrivals

From such behaviors of Δt, η, and υe we infer the following *explanation* for the origin of superarrivals. The barrier perturbation imparts a ‘‘kick’’ on the impinging wave packet that spits, and a part of it is reflected with a distortion. A finite disturbance proportional to this ‘‘kick’’ or the rate of perturbation propagates from the reducing barrier to the reflected packet, which results in a proportional magnitude of superarrivals. Note that information about the barrier perturbation reaches the detector at the instant *td* with a velocity υe, which decreases with the decreasing magnitude of impulse imparted to a wave packet. These results therefore suggest that information about the barrier perturbation propagates with a *definite speed* across the wave function that plays the role of a ‘‘field.’’

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