Quantum Weak Measurements and Some Applications

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Outline

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Strong measurement of an observable $\hat{A}$

- Total Hamiltonian: $H = H_s + H_d + H_{int}(t)$
- Impulsive limit: $T \to 0$. In this limit, $H_{int}(t)$ dominates the Hamiltonians of the measured system and the measuring device.
- Eigen value equation: $\hat{A}|a_j\rangle = a_j|a_j\rangle$
- Initial state vector: $|t=0\rangle = |\nu\rangle|\Phi_i\rangle = \sum_j c_j|a_j\rangle|\Phi_i\rangle$
- State vector after interaction:

$$|t=T\rangle \;=\; e^{-\frac{i}{\hbar}\int_0^T H_{int}(t')dt'}|t=0\rangle = e^{-\frac{i}{\hbar}g_0\hat{A}\hat{p}}\sum_j c_j|a_j\rangle|\Phi_i\rangle$$

- State vector in $Q-$representation ($Q-$wave function):

$$\langle Q|t=T\rangle \;=\; \sum_i c_j|a_j\rangle\langle Q|e^{-\frac{i}{\hbar}g_0 a_j\hat{p}}|\Phi_i\rangle = \sum_i c_j|a_j\rangle\Phi_i(Q-g_0 a_j)$$

Measurements in practice, errors

In an ideal measurement the initial position of the pointer is *precisely* defined. But in practice, measurements involve uncertainty. To model a source of uncertainty, we can take the initial state of the pointer to be

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- The uncertainty in the initial position of the pointer produces errors of order Δ in the determination of $\hat{A}$; when $\Delta \to 0$ we recover the ideal measurement.
- If the system is initially in an eigenstate of $\hat{A}$ with eigenvalue a_i, then ideal measurements can yield only the result a_i.

Measurements in practice, errors

- But when the pointer itself introduces uncertainty, other results are possible, indeed a scatter of results, with a spread of about Δ, and peaked at the eigenvalue a_i. Any measured value is possible, although large errors are exponentially suppressed. There is no mystery in the appearance of such errors; they are expected, given the uncertainty associated with the measuring device. Measurements of a positive definite operator such as p^2 could even yield negative values.

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- Since these errors originate in the measuring device and not in the system under study, it seems that they cannot depend on any property of the system.
- However, closer analysis of these errors in the context of *sequences* of measurements reveals a pattern which clearly reflects properties of the system under study. The pattern emerges only after selection of a particular final state of the system.

Negative kinetic energy

Consider a particle trapped in an attractive delta function potential:

$$\langle x|\psi_i\rangle \;=\; \sqrt{\alpha}\; e^{-\alpha|x|}, \qquad \alpha^2 = -\frac{2mE}{\hbar^2} > 0$$

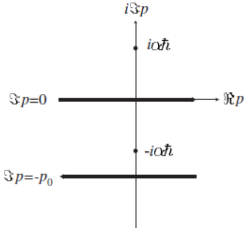
- Interaction Hamiltonian: $\hat{H}_{int} = g(t)\hat{P}\frac{\hat{p}^2}{2m}$.
- Evolved state: $\exp\left[-\frac{i}{\hbar}g_0\hat{P}\frac{\hat{p}^2}{2m}\right]|\psi_i\rangle|\Phi_i\rangle$
- For the final state we choose a Gaussian wave packet with its center far from the potential well

$$\psi_f(x) = (\pi\delta^2)^{-1/4}e^{-x^2/2\delta^2}, \quad \delta > g_0\frac{\alpha\hbar^2}{m\Delta}, \quad \alpha x_0 \gg \left(g_0\frac{|E|}{\Delta}\right)^2$$

Negative kinetic energy

Apart from normalization, *the final state of the measuring device* is

$$\begin{aligned}\Phi_f(Q) &= \langle\psi_f|e^{-\frac{i}{\hbar}g_0\hat{P}\frac{\hat{p}^2}{2m}}|\psi_i\rangle\Phi_i(Q)\\ &= \langle\psi_f|e^{-\frac{i}{\hbar}g_0\hat{P}\frac{\hat{p}^2}{2m}}\left(\int dp|p\rangle\langle p|\right)|\psi_i\rangle\Phi_i(Q)\\ &= \frac{\hbar\alpha}{\pi}e^{\alpha x_0-\alpha^2\delta^2/2}\int dp\frac{e^{-p^2\delta^2/2\hbar^2-ipx_0/\hbar}}{\alpha^2\hbar^2+p^2}\Phi_i(Q-g_0p^2/2m)\end{aligned}$$



Negative kinetic energy

The integral reduces to two terms: a pole term proportional to $\Phi_i(Q + \alpha^2\hbar^2/2m)$ which represents the measuring device with its pointer shifted to the negative value $-\alpha^2\hbar^2/2m$; and a correction, the integral in the above equation with $p - ip_0$ replacing p:

$$\frac{\hbar\alpha}{\pi} e^{\alpha x_0 - \alpha^2\delta^2/2} \int dp \frac{e^{-(p-ip_0)^2\delta^2/2\hbar^2 - i(p-ip_0)x_0/\hbar}}{\alpha^2\hbar^2 + (p - ip_0)^2} \Phi_i(Q - g_0(p - ip_0)^2/2m)$$

The absolute value of this integral depends on x_0 only through the exponential of $(\alpha - p_0/\hbar)x_0$. The rest of this equation is finite. Then, since $\alpha - p_0/\hbar$ is negative, the correction vanishes in the limit $x_0 \to \infty$. The correction is small if x_0 and δ satisfy conditions on them.

An experiment to isolate a particle in a classically forbidden region obtains negative values for its kinetic energy.

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- Imposing weakness (two ways):
  1. limit of small g_0, thus retaining only terms to first order in g_0.
  2. requiring P to remain small (thus, a small uncertainty in P too) which, by the $\Delta P \Delta Q$ uncertainty relation, corresponds to a limit of increasingly broad initial wave functions of the pointer in the Q representation and consequently, a large uncertainty in the measurement.

One-state vector formalism

Taking the pointer's initial wave function as a Gaussian centered around $Q = 0$,

$$\Phi_i(Q) \quad = \quad (\pi\Delta^2)^{-1/4} \exp\left[-\frac{Q^2}{2\Delta^2}\right],$$

one obtains

$$|\Phi_f\rangle \quad = \quad \langle Q|t = T\rangle = \sum_i c_i|a_i\rangle(\pi\Delta^2)^{-1/4} \exp\left[-\frac{(Q - g_0 a_i)^2}{2\Delta^2}\right]$$

for the total state vector in $Q-$representation, assuming that the measurement is impulsive.

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P_f(Q) &= \langle Q|t=T\rangle^\dagger \langle Q|t=T\rangle \\
&= (\pi\Delta^2)^{-1/2} \sum_i |c_i|^2 \exp\left[-\frac{(Q-g_0 a_i)^2}{\Delta^2}\right] \\
&\overset{\Delta \gg g_0 \delta a}{\approx} (\pi\Delta^2)^{-1/2} e^{-Q^2/\Delta^2} \sum_i |c_i|^2 e^{2g_0 a_i Q/\Delta^2} \\
&= (\pi\Delta^2)^{-1/2} e^{-Q^2/\Delta^2} \sum_i |c_i|^2 \left\{1 + \frac{2g_0 a_i Q}{\Delta^2}\right\} \\
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&= (\pi\Delta^2)^{-1/2} e^{-Q^2/\Delta^2} \left\{1 + \frac{2g_0 Q\langle\hat{A}\rangle}{\Delta^2}\right\} \\
&\approx (\pi\Delta^2)^{-1/2} e^{-(Q-g_0\langle\hat{A}\rangle)^2/\Delta^2} \equiv P_i(Q - g_0\langle\hat{A}\rangle)
\end{aligned}
$$

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- Post-selection in practice is achieved by running the process many times with initial state $|\psi_i\rangle$ and after all tasks are completed we perform a further final measurement of the projector Π_{ψ_f} onto $|\psi_f\rangle$ in each run. Then , we retain only those runs for which Π_{ψ_f} yielded 1.

Weak value

- State vector after measurement: After interaction the system and pointer are in joint state

$$e^{-ig_0\hat{A}\hat{P}/\hbar}|\psi_i\rangle|\Phi_i\rangle$$

and we post-select the system on state $|\psi_f\rangle$ resulting in the (sub-normalised) pointer state

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- Imposing weak measurement condition:

$$\begin{aligned} |\Phi_f\rangle &\approx \langle\psi_f|(I - ig_0\hat{A}\hat{P}/\hbar)|\psi_i\rangle|\Phi_i\rangle \\ &= \langle\psi_f|\psi_i\rangle(I - ig_0 A_w\hat{P}/\hbar)|\Phi_i\rangle \\ &\approx \langle\psi_f|\psi_i\rangle e^{-ig_0 A_w\hat{P}/\hbar}|\Phi_i\rangle; \qquad A_w = \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \end{aligned}$$

Weak value

Thus, final wavefunction of the pointer is

$$\Phi_f(Q) \approx \langle\psi_f|\psi_i\rangle\Phi_i(Q - g_0 A_w),$$

which, simply represents a translation of the wave function by the complex value $g_0 A_w$, where A_w is called "weak value" of the observable $\hat{A}$.

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Think of the states $|\phi_n\rangle$ as the possible outcomes of some final measurement on the system, then coefficients $|\langle\phi_n|\psi\rangle|^2$ give us the probabilities $P(n)$ of these outcomes:

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An alternative interpretation of the expectation value: an average of weak values.

Negative kinetic energy as a weak value

$$
\begin{aligned}
(\hat{p}^2/2m)_w &= \frac{\langle\psi_f|\hat{p}^2/2m|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \\
(\hat{p}^2/2m)_w &= (\hat{H})_w - (V(\hat{x}))_w \\
\hat{H}|\psi_i\rangle &= E|\psi_i\rangle \Rightarrow (\hat{H})_w = E \\
V(\hat{x})|\psi_f\rangle &\approx 0 \Rightarrow (V(\hat{x}))_w = 0 \\
(\hat{p}^2/2m)_w &= E = -\alpha^2\hbar^2/2m
\end{aligned}
$$

A measurement of $\hat{A}$ on the PPS ensemble always yields A_w if Δ is sufficiently large. But from the example of negative kinetic energy we see that a measurement may be weak even if Δ is not large. For any given Δ, the kinetic energy measurement is weak if $|\psi_f\rangle$ satisfies $\delta > g_0\alpha\hbar^2/m\Delta$ and $\alpha x_0 \gg (g_0|E|/\Delta)^2$.

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- Initial wavefunction of the pointer: $\Phi_i(Q) = (\Delta^2\pi)^{-1/4} e^{-Q^2/2\Delta^2}$

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- With post-selection:
  1. Strong measurement: The probability distribution of the pointer is localized around the eigenvalues ± 1.
  2. Weak measurement: By increasing Δ, the distribution gradually changes to a single broad peak around $(\sigma_\xi)_w = \sqrt{2}$.

Distribution functions in the absence of post-selection

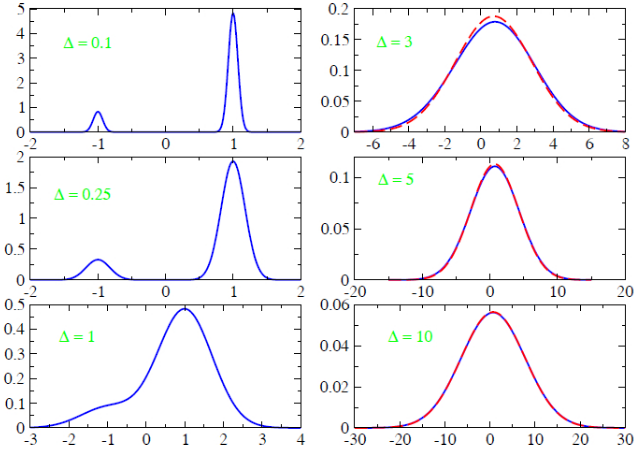
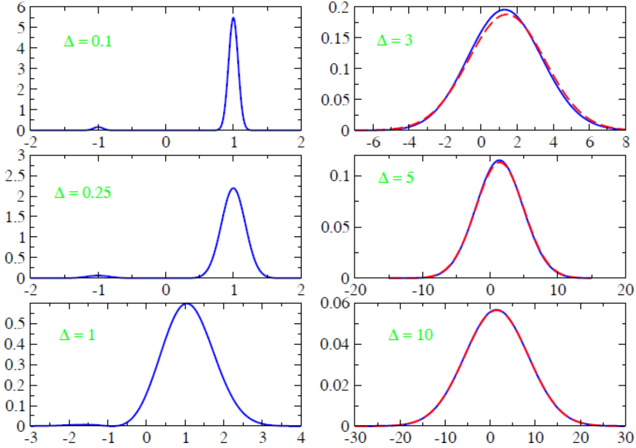


Figure: Dashed red curves show translated initial Gaussian by the amount $\langle \uparrow_x | \hat{\sigma}_\xi | \uparrow_x \rangle = \frac{1}{\sqrt{2}}$.

Distribution functions in the presence of post-selection



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Jozsa's Theorem

After a weak von Neumann measurement interaction on a system with pre- and post-selected states $|\psi_i\rangle$ and $|\psi_f\rangle$, the mean pointer position and momentum satisfy

$$
\begin{aligned}
\langle\hat{Q}\rangle_f &= \frac{\langle\Phi_f|\hat{Q}|\Phi_f\rangle}{\langle\Phi_f|\Phi_f\rangle} = \langle\hat{Q}\rangle_i + g_0 \mathrm{Re}(A_w) + \frac{g_0}{\hbar}\mathrm{Im}(A_w)\left(m\frac{d}{dt}\ \mathrm{Var}_Q\right) \\
\langle\hat{P}\rangle_f &= \frac{\langle\Phi_f|\hat{P}|\Phi_f\rangle}{\langle\Phi_f|\Phi_f\rangle} = \langle\hat{P}\rangle_i + 2\frac{g_0}{\hbar}\mathrm{Im}(A_w)\ \mathrm{Var}_P
\end{aligned}
$$

where, m is the mass of the pointer; and Var_Q and Var_P are respectively variance in Q and P, i.e. $\mathrm{Var}_Q = \langle\Phi_i|\hat{Q}^2|\Phi_i\rangle - \langle\Phi_i|\hat{Q}|\Phi_i\rangle^2$ and $\mathrm{Var}_P = \langle\Phi_i|\hat{P}^2|\Phi_i\rangle - \langle\Phi_i|\hat{P}|\Phi_i\rangle^2$.

Some conclusions of the Josza's theorem

- If A_w is real then $\langle \hat{Q} \rangle_f = \langle \hat{Q} \rangle_i + g_0 A_w$.

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- If A_w is real then $\langle \hat{Q} \rangle_f = \langle \hat{Q} \rangle_i + g_0 A_w$.
- If A_w is complex but the pointer wave function $\Phi_i(Q)$ is real-valued then

$$
\begin{aligned}
\langle \hat{Q} \rangle_f &= \langle \hat{Q} \rangle_i + g_0 \mathrm{Re}(A_w) \\
\langle \hat{P} \rangle_f &= \langle \hat{P} \rangle_i + 2\frac{g_0}{\hbar} \mathrm{Im}(A_w)\ \mathrm{Var}_P
\end{aligned}
$$

Detection probability

- Standard transition probability:
  Consider a standard prepare-and-measure experiment. If a quantum system is prepared in an initial state $|\psi_i\rangle$, the probability of detecting an event corresponding to the final state $|\psi_f\rangle$ is given by the squared modulus of their overlap, $P = |\langle\psi_f|\psi_i\rangle|^2$.

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- Modified transition probability:
  If, however, the initial state is modified by an intermediate unitary interaction $\hat{U}(\epsilon) = e^{-i\epsilon\hat{H}}$, the detection probability also changes to $P_\epsilon = |\langle\psi_f|\hat{U}(\epsilon)|\psi_i\rangle|^2$.

Relative correction to a detection probability

- Weak regime:
  If ϵ is small enough, or in other words if $\hat{U}(\epsilon)$ is "weak", we can consider its Taylor series expansion:

$$P_\epsilon \;=\; |\langle\psi_f|(1-i\epsilon\hat{H})|\psi_i\rangle|^2 = P + 2\epsilon\ \mathrm{Im}\{\langle\psi_i|\psi_f\rangle\langle\psi_f|\hat{H}|\psi_i\rangle\} + O(\epsilon^2)$$

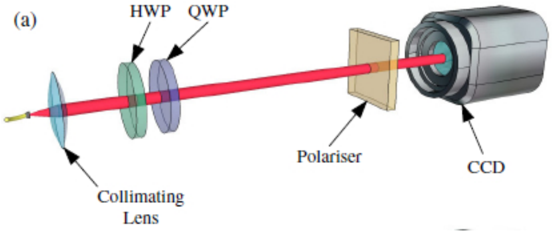
As long as $|\psi_i\rangle$ and $|\psi_f\rangle$ are not orthogonal (i.e., $P \neq 0$), we can divide both sides by P to obtain the relative correction

$$\frac{P_\epsilon}{P} \;=\; 1 + 2\epsilon\ \mathrm{Im}(H_w)$$

Operationally, weak values characterize the relative correction to a detection probability $|\langle\psi_f|\psi_i\rangle|^2$ due to a small intermediate perturbation $\hat{U}(\epsilon)$ that results in a modified detection probability $|\langle\psi_f|\hat{U}(\epsilon)|\psi_i\rangle|^2$.

Measuring weak values: an optical example

Using an optical experimental example, we show how one can measure a complex weak value associated with a polarization observable. Consider the setup shown in Fig. (a).



Measuring weak values: an optical example

- **Preparation(pre-selection):** A collimated laser beam is prepared in an initial state $|i\rangle|\psi_i\rangle$, where $|i\rangle$ is an initial polarization state and $|\psi_i\rangle$ is the state of the transverse beam profile.
  The polarization is prepared through the use of a quarter-wave plate (QWP) and a half-wave plate (HWP).
- **Post-selection:** The beam then passes through a linear polarizer aligned to a final polarization state $|f\rangle$ before impacting the charge coupled device (CCD) image sensor for a camera. Each pixel of the CCD measures a photon of this beam with a detection probability given by

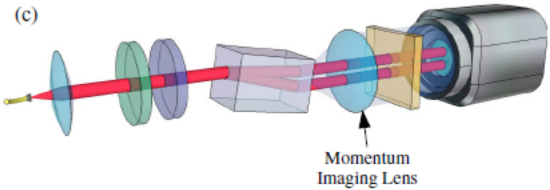
$$P = |\langle f|i\rangle|^2|\langle\psi_f|\psi_i\rangle|^2$$

where, $|\psi_f\rangle$ is the final transverse state post-selected by each pixel.

Measuring weak values: an optical example

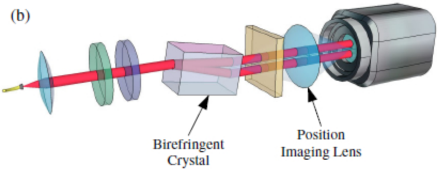
For our purposes, this state corresponds to either a specific transverse position $|\psi_f\rangle = |x\rangle$ or transverse momentum $|\psi_f\rangle = |p\rangle$, depending on whether we image the position or the momentum space onto the CCD [e.g., using a Fourier lens as shown in Fig. (c)]. We refer to this detection probability P as the "unperturbed" probability.

(c)

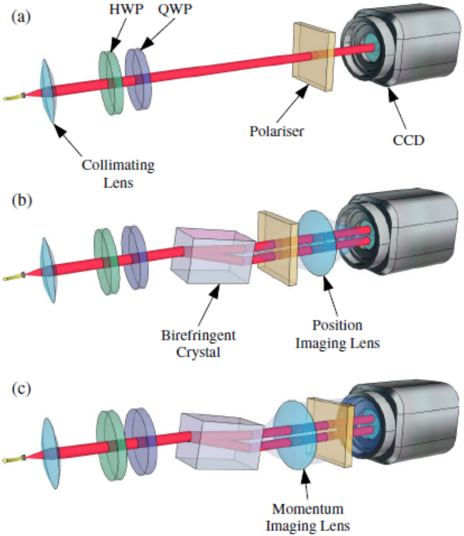


Measuring weak values: an optical example

- **Weak intermediate perturbation:** We now introduce a birefringent crystal between the preparation wave plates and the post-selection polarizer, as shown in Fig. (b). The crystal separates the beam into two beams with horizontal and vertical polarizations. The transverse displacements depend on the birefringence properties of the crystal and on the crystal length.



Measuring weak values: an optical example



Effect of the birefringent crystal

The effect of the birefringent crystal can be expressed by a time evolution operator $\hat{U}(\tau) = e^{-i\tau\hat{H}/\hbar}$ with an effective interaction Hamiltonian

$$\hat{H} \;=\; \epsilon\hat{S}\hat{p}/\tau,$$

Here $\hat{S} = |H\rangle\langle H| - |V\rangle\langle V|$ is the Stokes polarization operator that assigns eigenvalues $+1$ and -1 to the $|H\rangle$ and $|V\rangle$ polarizations, respectively, and $\hat{p}$ is the transverse momentum operator that generates translations in the transverse position x.

Effect of the time evolution operator: $\hat{U}(\tau)$ correlates the polarization components of the beam with their transverse position by translating them in opposite directions. Each pixel of the CCD then collects a photon with a "perturbed" probability given by

$$P_\epsilon \;=\; \langle f|\langle\psi_f|e^{-i\epsilon\hat{S}\hat{p}/\hbar}|i\rangle|\psi_i\rangle.$$

A visual example: A Gaussian beam

As an example, consider a Gaussian beam

$$\langle x|\psi_i\rangle = (2\pi\sigma^2)^{-1/4}\exp\left(-\frac{x^2}{4\sigma^2}\right)$$

with initial polarization preparation:

$$|i\rangle = \frac{|H\rangle - e^{i\phi}|V\rangle}{\sqrt{2}}, \qquad \phi = 0.1$$

and final polarization postselection:

$$|f\rangle = \cos\frac{\theta}{2}|H\rangle + \sin\frac{\theta}{2}|V\rangle, \qquad \theta = \frac{\pi}{2} - 0.2$$

These two polarization states are nearly orthogonal.

A visual example: A Gaussian beam

- Without the crystal present [Fig. 1(a)], the CCD measures the initial Gaussian intensity profile shown as a dashed line in Fig. 3(a) with a total postselection probability given by

$$|\langle f|i\rangle|^2 = \frac{1-\sin\theta\cos\phi}{2} = 0.0124$$

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- When the crystal is present [Fig. 1(b)], the orthogonal polarization components become spatially separated by a displacement ϵ before passing through the postselection polarizer.

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- When the crystal is present [Fig. 1(b)], the orthogonal polarization components become spatially separated by a displacement ϵ before passing through the postselection polarizer.
- The measured profiles for different crystal lengths are shown as the solid line distributions in Fig. 3(a). The dashed line distributions show the unperturbed (but still postselected) profiles for comparison.

A visual example: A Gaussian beam

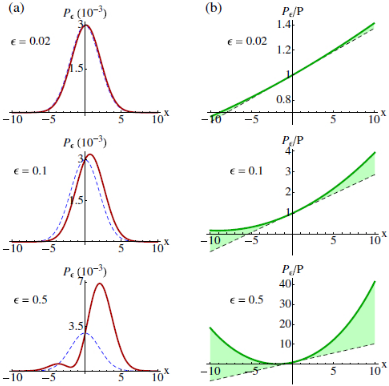

Figure: (a) Perturbed profiles (solid) and a fixed unperturbed profile (dashed). (b) The exact ratio of the two curves (solid) is compared to the first order approximation (dashed).

Weak values of Stokes and transverse momentum operators

In the weak interaction regime, the crystal is short, ϵ is small, and the two orthogonally polarized beams are displaced by a small amount before they interfere at the postselection polarizer. Thus, by expanding the ratio between the perturbed and unperturbed probabilities to first order in ϵ, the linear probability correction term takes the form:

$$\frac{P_\epsilon}{P} - 1 \approx \frac{2\tau}{\hbar}\mathrm{Im}H_w = \frac{2\epsilon}{\hbar}[\mathrm{Re}S_w\mathrm{Im}p_w + \mathrm{Im}S_w\mathrm{Re}p_w]$$

where,

$$S_w = \frac{\langle f|\hat{S}|i\rangle}{\langle f|i\rangle} = \frac{\cos\theta + i\sin\theta\sin\phi}{1-\sin\theta\sin\phi}, \qquad p_w = \frac{\langle\psi_f|\hat{p}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}$$

A clever choice of preselection and postselection states therefore allows an experimenter to separate each of these quantities using different experimental setups.

Isolating real part of S_w

- The procedure for measuring the real part $\mathrm{Re}S_w$ is shown in Fig. 1(b). We image the output face of the crystal onto the CCD so that each pixel corresponds to a postselection of the transverse position $|\psi_f\rangle = |x\rangle$. As a result, the momentum weak value for each pixel becomes

$$p_w = \frac{\langle x|\hat{p}|\psi_i\rangle}{\langle x|\psi_i\rangle} = \frac{-i\hbar\partial_x\psi_i(x)}{\psi_i(x)} = i\hbar\frac{x}{2\sigma^2}$$

Thus,

$$\frac{P_\epsilon}{P} \approx 1 + \epsilon\frac{x}{\sigma^2}\mathrm{Re}S_w$$

effectively isolating the quantity $\mathrm{Re}S_w$ to first order in ϵ.

Isolating imaginary part of S_w

- The procedure for measuring the real part $\mathrm{Im}S_w$ is shown in Fig. 1(c). We image the Fourier plane of the crystal onto the CCD so that each pixel corresponds to a postselection of the transverse momentum$|\psi_f\rangle = |p\rangle$. As a result, the momentum weak value for each pixel becomes

$$p_w = \frac{\langle p|\hat{p}|\psi_i\rangle}{\langle p|\psi_i\rangle} = \frac{p\langle p|\psi_i\rangle}{\langle p|\psi_i\rangle} = p$$

Thus,

$$\frac{P_\epsilon}{P} \approx 1 + \epsilon \frac{2p}{\hbar}\mathrm{Im}S_w$$

effectively isolating the quantity $\mathrm{Im}S_w$ to first order in ϵ.

General point

Note that we could also separate the real and imaginary parts of p_w in a similar manner through a clever choice of polarization postselection states. More generally, one can use this technique to separate weak values of any desired observable by constructing Hamiltonians in a product form and cleverly choosing the preselection and postselection of the auxiliary degree of freedom.

Measurement of polarization state

We aim to determine the complex components of the initial polarization state $|i\rangle$ in the measurement basis $\{|H\rangle, |V\rangle\}$.

- Multiply $|i\rangle$ by constant factor $c = \frac{\langle D|H\rangle}{\langle D|i\rangle} = \frac{\langle D|V\rangle}{\langle D|i\rangle}$,

$$c|i\rangle \;\; = \;\; \frac{\langle D|H\rangle\langle H|i\rangle}{\langle D|i\rangle}|H\rangle + \frac{\langle D|V\rangle\langle V|i\rangle}{\langle D|i\rangle}|V\rangle \equiv H_w|H\rangle + V_w|V\rangle$$

- Since $|H\rangle\langle H| = (\hat{1} + \hat{S})/2$ and $|V\rangle\langle V| = (\hat{1} - \hat{S})/2$ then, $H_w = (1 + S_w)/2$ and $V_w = (1 - S_w)/2$.
- That is, each complex component of the scaled state $c|i\rangle$ can be directly measured as a complex first order weak value. After determining these complex components experimentally, the state can be subsequently re-normalized to eliminate the constant c up to a global phase.

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- That is, each complex component of the scaled state $c|i\rangle$ can be directly measured as a complex first order weak value. After determining these complex components experimentally, the state can be subsequently re-normalized to eliminate the constant c up to a global phase. Thus, we can completely determine the state $|i\rangle$ after the polarization weak value S_w has been measured using the special postselection $|D\rangle = (|H\rangle + |V\rangle)/2$ which is unbiased with respect to both $|H\rangle$ and $|V\rangle$.

Direct measurement of the quantum wave function

- Weakly measure a projector in the position basis, $\pi_x = |x\rangle\langle x|$,
- Post-select the momentum eigenstate, $|\psi_f\rangle = |p\rangle$.
- Then,

$$(\pi_x)_w = \frac{\langle p|x\rangle\langle x|\psi\rangle}{\langle p|\psi\rangle} = \frac{e^{ipx/\hbar}\psi(x)}{\tilde{\psi}(p)}$$

  with $\tilde{\psi}(p)$ the momentum space function.
- Finally, choose $p = 0$

$$\psi(x) \quad = \quad k(\pi_x)_w$$

  with k a constant can be determined later from the normalization of $\psi(x)$.

Directly measuring the quantum state of an arbitrary quantum system

We have the freedom to measure the quantum state in any chosen basis $\{|a\rangle\}$ (associated with observable $\hat{A}$) of the system.

- Weakly measure a projector in this basis, $\pi_a = |a\rangle\langle a|$,
- post-select on a particular value b_0 of the complementary observable $\hat{B}$. In this case, the weak value is

$$(\pi_a)_w = \frac{\langle b_0|a\rangle\langle a|\psi\rangle}{\langle b_0|\psi\rangle} = \langle a|\psi\rangle/\nu,$$

  where, ν is a constant, independent of a. Thus the weak value is proportional to the amplitude of state $|a\rangle$ in the quantum state.
- Step a through all the states in the A basis. This, directly gives the quantum state represented in that basis:

$$|\psi\rangle = \nu \sum_a (\pi_a)_w |a\rangle$$

Bohmian mechanics (BM)

- Complete description: Wavefunction + particle position

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- Quantum equilibrium hypothesis $\Rightarrow$ All statistical predictions of BM agree exactly with those of standard quantum mechanics. In particular,
    1. The uncertainty principle applies, such that it is impossible to precisely observe the trajectory of an individual Bohmian particle.
    2. Superluminal influences experienced by individual Bohmian particles cannot be used for superluminal signaling. Bohmian mechanically, the theory of relativity therefore remains valid, but only in a statistical sense.

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    2. Superluminal influences experienced by individual Bohmian particles cannot be used for superluminal signaling. Bohmian mechanically, the theory of relativity therefore remains valid, but only in a statistical sense.
- Bohmian velocity field in a single-particle system:

$$\mathbf{v}^{\psi}(\mathbf{x}, t) = \frac{1}{m}\Re\frac{\langle x|\hat{p}|\psi(t)\rangle}{\langle x|\psi(t)\rangle}$$

Operational definition of velocity

However, the velocity field for an ensemble of Bohmian particles can be measured weakly in the following way:

- Operational definition for the velocity of a particle at position $\mathbf{x}$:

$$\mathbf{v}(\mathbf{x}, t) = \lim_{\tau \to 0} \tau^{-1} E[\mathbf{x}_{\text{strong}}(t+\tau) - \mathbf{x}_{\text{weak}}(t) | \mathbf{x}_{\text{strong}}(t+\tau) = \mathbf{x}] \,.$$

  Here, $\mathbf{x}_{\text{weak}}$ and $\mathbf{x}_{\text{strong}}$ denote respectively the results of a weak measurement of the position of the particle at time t and a strong measurement of the position a short time τ later. The expectation symbol E refers to the average over a large ensemble of systems, all prepared in the same initial state ψ at time t, and for all of which the result of a strong measurement of position at time $t+\tau$, following the weak measurement at time t, is $\mathbf{x}$.

Equivalence of Bohm and operationally defined velocities

Allowing unitary evolution described by $\hat{U}$ between the measurements, the so-called weak value of an observable $\hat{A}$ is given by,

$$
{}_{\langle\phi|\hat{U}(t+\tau,t)}\langle\hat{A}_w\rangle_{|\psi(t)\rangle} = \Re\frac{\langle\phi|\hat{U}(t+\tau,t)\hat{A}|\psi(t)\rangle}{\langle\phi|\hat{U}(t+\tau,t)|\psi(t)\rangle}
$$

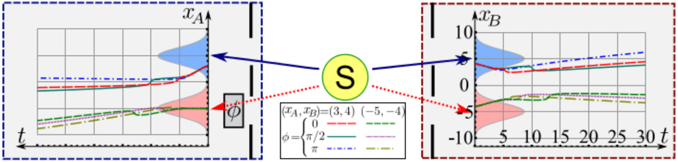
In terms of this, the velocity definition becomes

$$
\begin{aligned}
\mathbf{v}(\mathbf{x},t) &= \lim_{\tau\to 0}\tau^{-1}[\mathbf{x} - {}_{\langle\mathbf{x}|\hat{U}(t+\tau,t)}\langle\hat{\mathbf{x}}_w\rangle_{|\psi(t)\rangle}] \\
&= \lim_{\tau\to 0}\tau^{-1}\left[\mathbf{x} - \Re\frac{\langle\mathbf{x}|\hat{U}(t+\tau,t)\hat{\mathbf{x}}|\psi(t)\rangle}{\langle\mathbf{x}|\hat{U}(t+\tau,t)|\psi(t)\rangle}\right]
\end{aligned}
$$

One easily computes with $\hat{U}(t+\tau,t) = e^{-i\hat{H}\tau/\hbar}$ and $\hat{H} = \hat{\mathbf{p}}^2/2m + V(\hat{\mathbf{x}})$ that $\mathbf{v}(\mathbf{x},t) = \mathbf{v}^{\psi}(\mathbf{x},t)$. This procedure can be easily generalized to multi-particle systems.

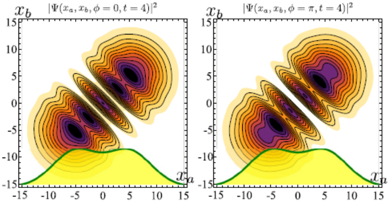
Proposal to Observe the Nonlocality of Bohmian Trajectories

Proposal: The nonlocal character of BM can be seen in an experiment using entangled photon pairs. Path-entangled photons and a double double-slit setup are used, with variable phase shifts between the two slits on one side. It is shown that the Bohmian velocity field (and hence the trajectory) for the particle on the other side depends on the phase shift applied to the first particle.



$$\psi_0(x_A, x_B) \;\;=\;\; \frac{1}{\sqrt{2}}\left[f_u(x_A)f_u(x_B) + e^{i\phi}f_d(x_A)f_d(x_B)\right].$$

Conceptual structure of the proposed experiment



- The system exhibits two-particle interference. The exact location of the fringes depends on the phase ϕ.
- In contrast, there is no single-particle interference. That is, there are no interference fringes and no dependence on ϕ in the marginal single-particle probability distributions $p(x_A) = \int dx_B |\psi(x_A, x_B, t)|^2$ and $p(x_B) = \int dx_A |\psi(x_A, x_B, t)|^2$.

Note that if $p(x_B)$, which is locally measurable, depended on the phase shift ϕ, this would in principle allow superluminal communication between the experimenter on the left and the experimenter on the right.

Bohmian conditional wave function

What is the wave function of a subsystem of a larger system?

- Ordinary quantum mechanics lacks the resources that make it possible to define this notion
- Bohmian mechanics has the resources to construct a single-particle wave function.

Consider for simplicity a system of two spin-0 particles (masses m_1 and m_2, coordinates x and y) each moving in one spatial dimension. According to Bohmian mechanics, the system is then completely described by its wave function $\Psi(x, y, t)$, evolving according to Schrödinger's equation, together with actual particle positions $X(t)$ and $Y(t)$. which evolve according to

$$\frac{dX(t)}{dt} = \frac{\hbar}{i2m_1} \frac{\Psi^* \partial_x \Psi - \Psi \partial_x \Psi^*}{\Psi^* \Psi}\bigg|_{x=X(t),y=Y(t)}$$

$$\frac{dY(t)}{dt} = \frac{\hbar}{i2m_2} \frac{\Psi^* \partial_y \Psi - \Psi \partial_y \Psi^*}{\Psi^* \Psi}\bigg|_{x=X(t),y=Y(t)}$$

Bohmian conditional wave function

The Bohmian conditional wave function (CWF) - for, say, the first particle is simply the ("universal") wave function $\Psi(x, y, t)$ evaluated at $y = Y(t)$:

$$\chi_1(x,t) \quad = \quad \Psi(x,y,t)|_{y=Y(t)}.$$

The evolution law for the position $X(t)$ of particle 1 can be re-written in terms of particle 1's CWF as follows:

$$\frac{dX(t)}{dt} \quad = \quad \frac{\hbar}{i2m_1} \frac{\chi_1^* \partial_x \chi_1 - \chi_1 \partial_x \chi_1^*}{\chi_1^* \chi_1}\bigg|_{x=X(t)}$$

It is thus appropriate to think of $\chi_1(x, t)$ as the guiding or pilot-wave that directly influences the motion of particle 1.

Direct measurement of single-particle wave functions

Consider a two particle system in the entangled state,

$$|\Psi\rangle = \int dx' dy' \Psi(x', y') |x'\rangle |y'\rangle$$

Post-selecting on the final momentum p_x of the particle whose wave function we are trying to measure (here, particle 1) and also post-selecting on the final position Y of particle 2 yields

$$\langle \pi_x \rangle_W^{p_x, y=Y} = \frac{\langle p_x | x \rangle \langle x, Y | \Psi \rangle}{\langle p_x, Y | \Psi \rangle} = \frac{e^{-ip_x x/\hbar} \Psi(x, Y)}{\int dx' \Psi(x', Y) e^{-ip_x x'/\hbar}}$$

Direct measurement of single particle wave functions

Restricting our attention to the cases in which the final measured momentum p_x is zero, we have that

$$\langle \pi_x \rangle_W^{p_x=0, y=Y} \quad \sim \quad \Psi(x, Y) = \chi_1(x)$$

where the right hand side is precisely the Bohmian CWF for particle 1. Note that for Bohmian mechanics, the final position measurement simply reveals the pre-existing location Y of the particle. Thus the two Ys in the analysis the one representing the outcome of the final position measurement of particle 2, and the one, used in the definition of the Bohmian CWF, representing the actual position of particle 2 are, for Bohmian mechanics, the same.

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- An ensemble of *single photons* are sent through a two-slit interferometer,

Average trajectories of single photons

- An ensemble of *single photons* are sent through a two-slit interferometer,
- a weak measurement is performed on each photon to gain a small amount of information about its momentum, followed by a strong measurement that postselects the subensemble of photons arriving at a particular position.

Average trajectories of single photons

- An ensemble of *single photons* are sent through a two-slit interferometer,
- a weak measurement is performed on each photon to gain a small amount of information about its momentum, followed by a strong measurement that postselects the subensemble of photons arriving at a particular position.
- The polarization degree of freedom of the photons is used as a pointer that weakly couples to and measures the momentum of the photons. This weak momentum measurement does not appreciably disturb the system, and interference is still observed.

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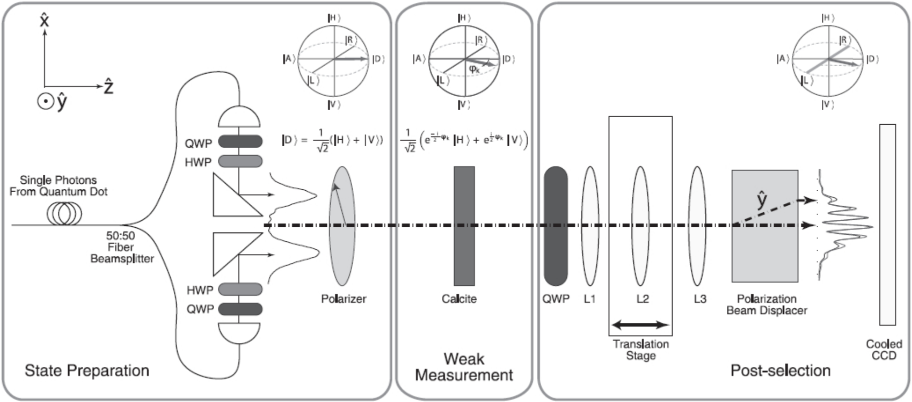
- The polarization degree of freedom of the photons is used as a pointer that weakly couples to and measures the momentum of the photons. This weak momentum measurement does not appreciably disturb the system, and interference is still observed.
- The two measurements must be repeated on a *large ensemble* of particles in order to extract a useful amount of information about the system. From this set of measurements, the average momentum of the photons reaching any particular position in the image plane is determined, and, by repeating this procedure in a series of planes, trajectories over that range is are reconstructed.

Experimental setup for measuring the average photon trajectories



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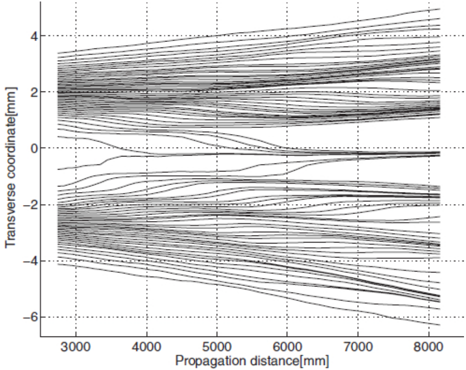

Photons are not constrained to follow these precise trajectories. Rather, these trajectories represent the average behavior of the ensemble of photons when the weakly measured momentum in each plane is recorded imposing the final position at which a photon is observed.

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