Quantum Complexity and Thermalization

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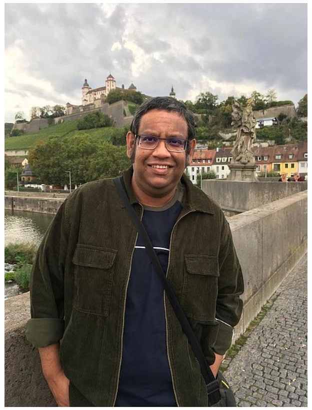
My collaborators in the Past, Present, and Future!



Quantum chaos is an interesting subject though it is difficult to understand. This is due to the fact that the time evolution of quantum mechanic is local and unitary and thus, in general, it is hard to study the emergence of ergodic behavior in quantum system.

Therefore it is of great interest to understand thermal behavior in quantum level in which the eigesnstate thermalization hypothesis, (ETH), plays an important role.

In the classical level the chaotic behavior may be described by the sensitivity of trajectories in the phase space to the initial conditions. Indeed, two initially near by trajectories separate exponentially fast characterized by the Lyapunov exponent.

Nonetheless, to probe the nature of quantum chaos certain quantities have been introduced. These include, for example, out-of-time-order correlators (OTOCs), Complexity ...

Complexity

Suppose our system is in a given state and we would like to map it to another state. Then, intuitively, the complexity tells us that how difficult this task is.

$$|\psi\rangle = U|\psi_0\rangle$$

The operator U may be decomposed as $U = U_1 \cdots U_n$ where $\{U_i\}$ are a set of gates.

Complexity is defined as minimum gates needed to perform the task.

Complexity in Holography

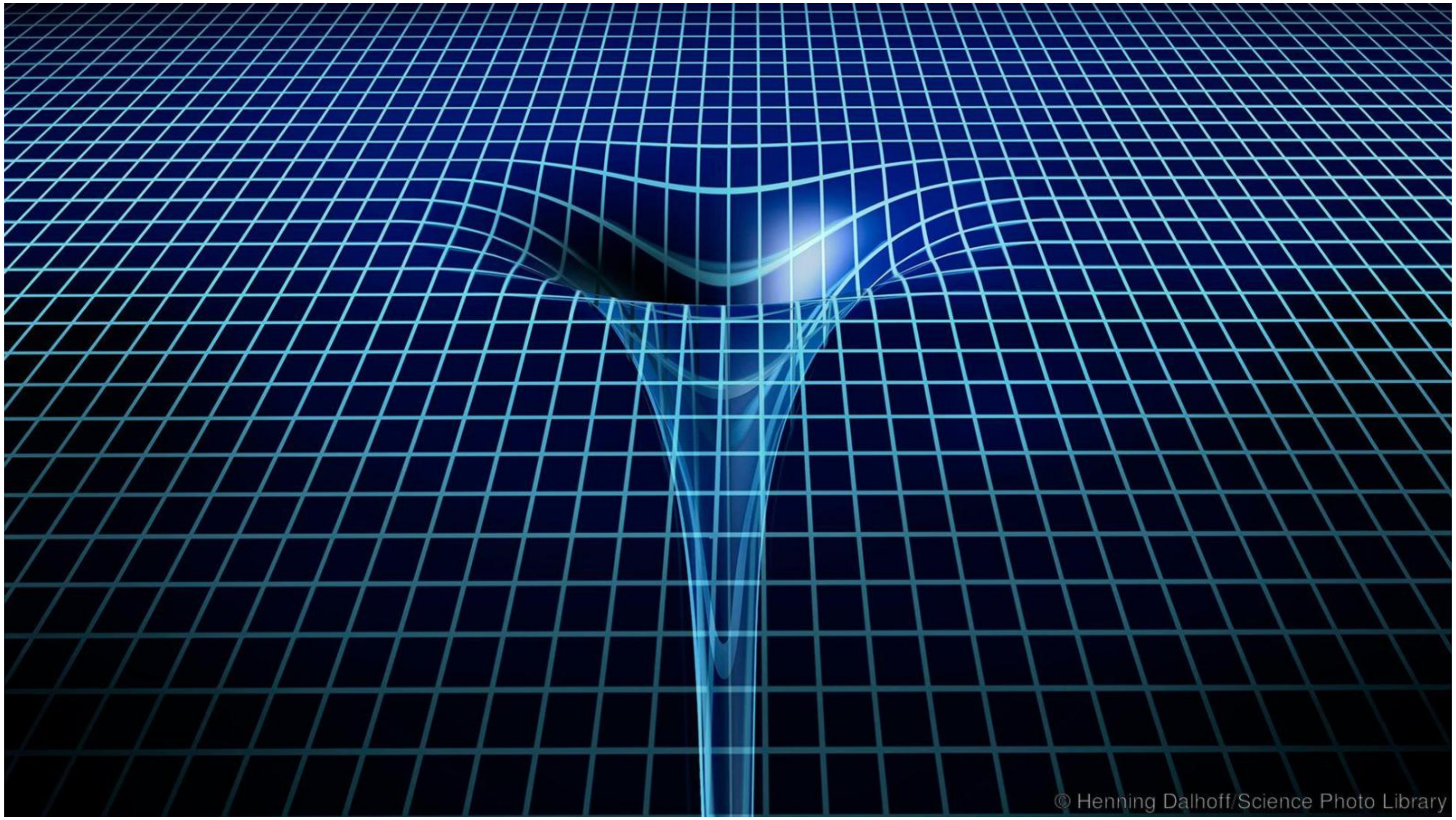
In the context of AdS/CFT correspondence let us consider a black brane solution

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}_d^2, \qquad f(r) = 1 - \frac{r_h^{d+1}}{r^{d+1}}$$

The dual theory is a CFT at finite temperature. This is a static solution. Therefore there is no time dependent phenomena.

According to the AdS/CFT correspondence every thing in the gravity side should be understood from field theory point of view, including the black hole interior.

Here is a paradox

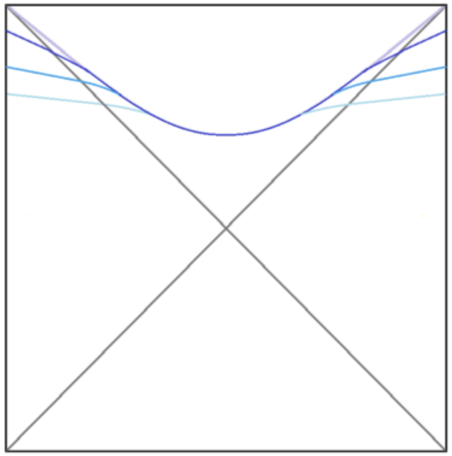


As soon as a black hole was formed, the volume inside the horizon keeps growing linearly with time.

A time dependent phenomena.

Holographic complexity

The boundary field theory reaches thermal equilibrium very quickly, while the ERB continues to grow on much longer timescales. Therefore, there must be some quantity in the field theory that corresponds to this information.



Susskind introduced holographic complexity as the boundary entity whose growth corresponds to the evolution of the ERB.

Susskind, arXiv:1403.5695, arXiv:1402.5674

The complexity of the boundary state is proportional to the volume of a maximal codimension-one bulk surface B that extends to the AdS boundary, and asymptotes to the time slice Σ on which the boundary state is defined

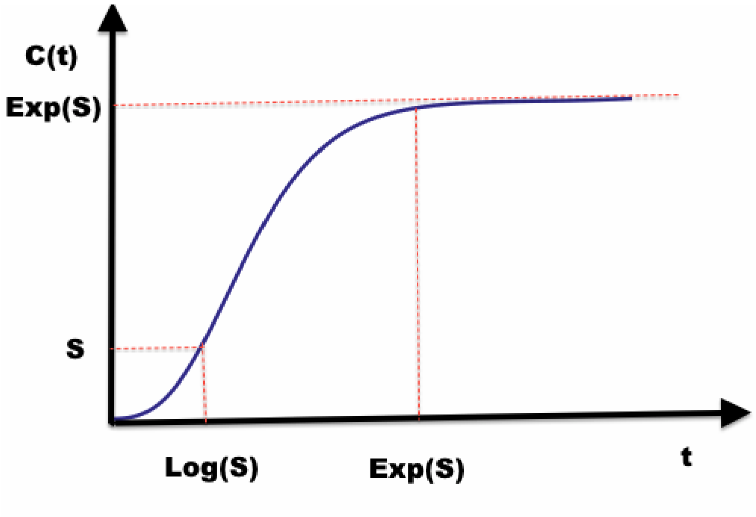
$$\mathcal{C} = \frac{V(B)}{GR}\Big|_{\partial B = \Sigma}$$

This known as CV proposal. There is also another proposal known as CA.

Susskind, arXiv:1403.5695, arXiv:1402.5674

Time dependence of Complexity

We note, however, that after the period of linear growth and at times $t \sim \mathcal{O}(e^S)$ we expect saturation to a plateau of size $C \sim \mathcal{O}(e^S)$.



While semi-classical contributions both in form of the CV and CA conjectures indeed provide the period of growth, the saturation to the plateau, until recently, has been illusive.

L.V. Iliesiu, M. Mezei and G. Sárosi, arXiv:2107.06286, M. A., S. Banerjee and J. Kames-King, arXiv:2205.01150, M. A. and S. Banerjee, arXiv:2209.02441 [hep-th].

Krylov complexity

One of the most useful notation of complexity is the Krylov complexity which we review it in what follows.

Let us consider a quantum system describing by a time independent Hamiltonian H whose eigenstates and eigenvalues are denoted by $|E_a\rangle$ and E_a, respectively. Here $a = 1, 2, \cdots \mathcal{D}$ with $\mathcal{D}$ being the dimension of the associated Hilbert space $\mathcal{H}$.

In this system the time evolution of a state is given by

$$|\psi(t)\rangle = e^{iHt}|\psi(0)\rangle = \sum_n \frac{(it)^n}{n!} H^n|\psi(0)\rangle \, .$$

$|\psi(t)\rangle$ is a wave function in a space expanded by $\{|\psi(0)\rangle, H|\psi(0)\rangle, H^2|\psi(0)\rangle, \cdots\}$

D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi and E. Altman, arXiv:1812.08657, J. L. F. Barbón, E. Rabinovici, R. Shir and R. Sinha, 1907.05393, S. K. Jian, B. Swingle and Z. Y. Xian,, 2008.12274,E. Rabinovici, A. Sánchez-Garrido, R. Shir and J. Sonner, 2009.01862 [hep-th]]. V. Balasubramanian, P. Caputa, J. M. Magan and Q. Wu, arXiv:2202.06957.

Using the Hamiltonian, one can construct an orthonormal and ordered basis associated with any state of the Hilbert space. Denoting the corresponding state by $|\psi(0)\rangle$ the orthonormal, ordered basis $\{|n\rangle, n = 0, 1, 2, \cdots, \mathcal{D}_\psi - 1\}$ may be constructed using the Gram-Schmidt process.

The first element of the basis is indeed the original given state of the Hilbert state $|0\rangle = |\psi(0)\rangle$ (which is assumed to be normalized). Then the other elements are constructed recursively as follows

$$|\widehat{n+1}\rangle = (H - a_n)|n\rangle - b_n|n-1\rangle$$

where $|n\rangle = b_n^{-1}|\widehat{n}\rangle$ and

$$a_n = \langle n|H|n\rangle, \qquad b_n = \sqrt{\langle\widehat{n}|\widehat{n}\rangle}$$

The procedure stops where ever b_n vanishes which occurs for $n = \mathcal{D}_\psi$ which is the dimension of subspace $\mathcal{H}_\psi$ expanded by the basis $\{|n\rangle\}$.

This procedure produces an orthogonal basis together with coefficients a_n and b_n known as Lanczos coefficients.

Then the Krylov complexity is defined as

$$\mathcal{C}(t) = \langle\psi(t)|\mathcal{N}|\psi(t)\rangle = \sum_{n=0}^{\mathcal{D}_\psi - 1} n\,|\psi_n(t)|^2\,,$$

where $\mathcal{N} = \sum_{n=0}^{\mathcal{D}_\psi - 1} n|n\rangle\langle n|$, and

$$|\psi(t)\rangle = \sum_{n=0}^{\mathcal{D}_\psi - 1} \psi_n(t)|n\rangle$$

The probability amplitudes $\psi_n(t)$ may be computed recursively from the following Schrödinger equation

$$\frac{d\psi_n}{dt} = ia_n\psi_n + b_n\psi_{n-1} - b_{n+1}\psi_{n+1},$$

with the boundary conditions $\psi_n(0) = \delta_{n0}$, $\psi_{-1}(t) \equiv 0$.

The behavior of Krylov complexity is encoded in the behavior of Lanczos coefficients. It was conjectured that for a chaotic system, one has large n linear growth; $b_n \approx \alpha n$ for large n.

Actually the large n linear growth is a typical behavior for Lanczos coefficients and has noting to do with the chaotic nature of the system.

Under certain condition it may also exhibit saturation phase on which the Lanczos coefficients saturate to a constant.

The saturation of Lanczos coefficients results in a linear growth for complexity at late times.

A. Dymarsky and M. Smolkin, [arXiv:2104.09514 [hep-th]], B. Bhattacharjee, X. Cao, P. Nandy and T. Pathak,[arXiv:2203.03534 [quant-ph]], A. Avdoshkin, A. Dymarsky and M. Smolkin, [arXiv:2212.14429 [hep-th]], H. A. Camargo, V. Jahnke, K. Y. Kim and M. Nishida, [arXiv:2212.14702 [hep-th]].

Thermalization

Krylov Complexity is a particular example of a more general quantity we are usually dealing with in the context of quantum thermalization in a closed quantum system

$$\langle \psi(t)|\mathcal{O}|\psi(t)\rangle = \langle \psi|\mathcal{O}(t)|\psi\rangle$$

where $|\psi(t)\rangle = e^{iHt}|\psi\rangle$

For a given state $|\psi\rangle = \sum_n c_n|E_n\rangle$ with $\sum_n |c_n|^2 = 1$ one has

$$\langle \mathcal{O}(t)\rangle = \sum_{n,m} c_n^* c_m e^{-i(E_n - E_m)t}\langle E_n|\mathcal{O}|E_m\rangle$$

where $\mathcal{O}$ in an operator in the system. The infinite time average

$$\bar{\mathcal{O}} = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^T dt\langle \mathcal{O}(t)\rangle = \sum_n |c_n|^2\langle E_n|\mathcal{O}|E_n\rangle$$

It is not what we would expect in micro-canonical ensemble!

For chaotic systems the notion of thermalization may be described by the eigenstate thermalization hypothesis (ETH) which gives an understanding of how an observable thermalizes to its thermal equilibrium value.

According to the ETH, thermalization occurs at the level of individual eigenstates of the Hamiltonian. In fact setting

$$\varepsilon = \frac{E_1 + E_2}{2}, \qquad \omega = E_1 - E_2,$$

The ETH states that the matrix elements of observables in the basis of the eigenstates of the Hamiltonian obey the following ansatz

$$\langle E_1|\mathcal{O}|E_2\rangle = \bar{\mathcal{O}}(\varepsilon)\delta_{E_1,E_2} + e^{-S} f(\varepsilon,\omega)\mathcal{R}_{E_1 E_2}$$

where $\bar{\mathcal{O}}(\varepsilon)$ is the micro canonical average of the corresponding operator, S is thermodynamical entropy of the system, $f(\varepsilon,\omega)$ is a smooth function of its arguments and $\mathcal{R}$ is unit variance random function with zero mean.

J. M. Deutsch, Phys Rev A.43 (1991), M. Srednicki, cond-mat/9403051, J. Phys. A32 (1999) .

Let us consider spin$-\frac{1}{2}$ Ising model given by the following Hamiltonian

$$H = -J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z - \sum_{i=1}^{N} (g\, \sigma_i^x + h\, \sigma_i^z)\,.$$

Here $\sigma^{z,x}$ are Pauli matrices and J, g and h are constants which define the model.

By rescaling one may set $J = 1$, and the nature of the model, being chaotic or integrable, is controlled by constants g and h. In particular, for $gh \neq 0$ the model is non-integrable.

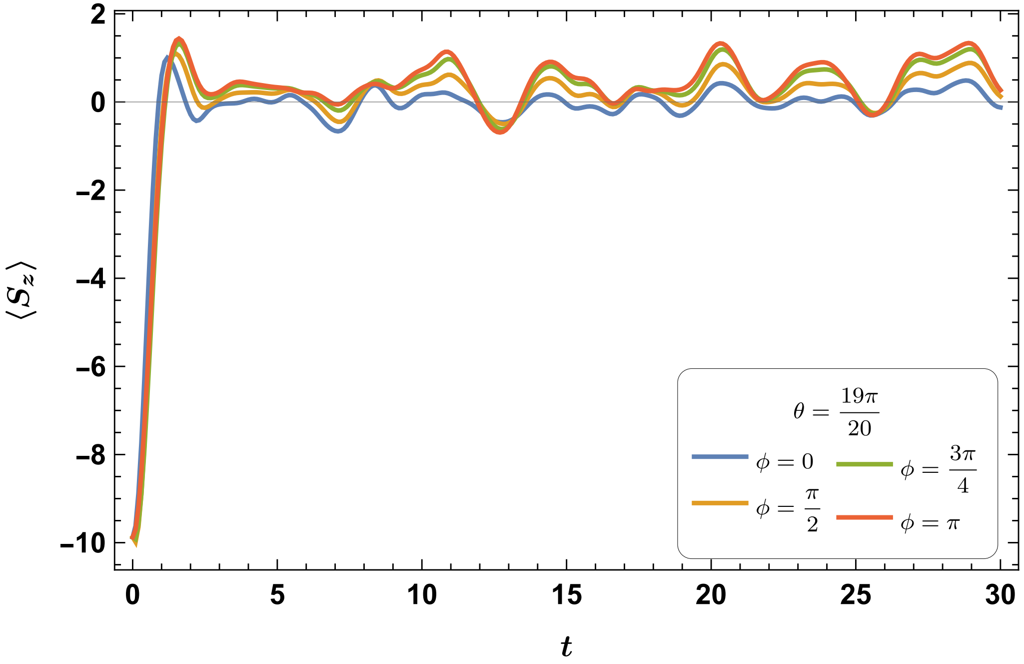
Let us consider an arbitrary initial state in the Bloch sphere which may be parameterized by two angles θ and ϕ as follow

$$|\theta, \phi\rangle = \prod_{i=1}^{N} \left(\cos\frac{\theta}{2}\; |Z+\rangle_i + e^{i\phi} \sin\frac{\theta}{2}\; |Z-\rangle_i \right),$$

where $|Z\pm\rangle$ are eigenvectors of σ^z with eigenvalues $\pm$.

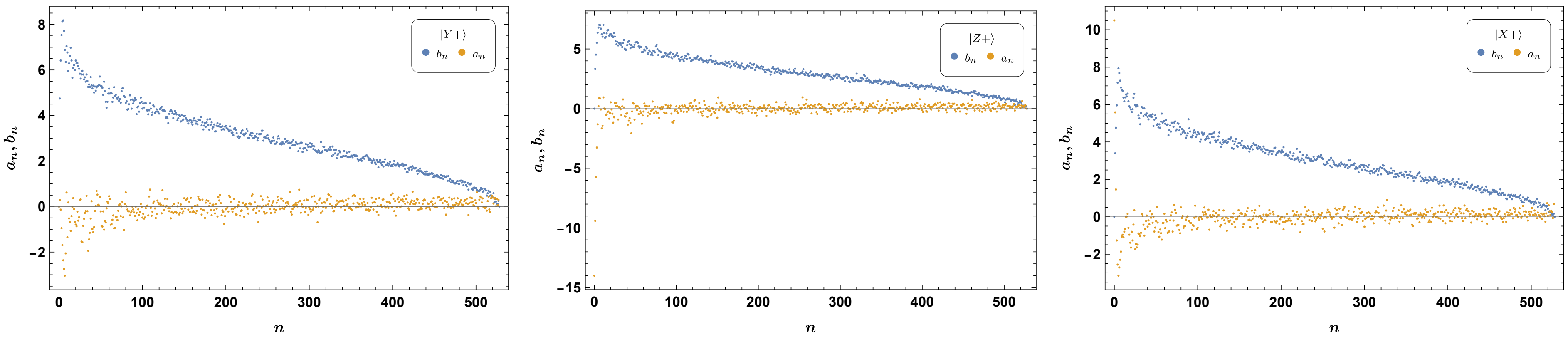
Consider the magnetization in the z direction and compute the following quantity

$$\langle S_z(t)\rangle = \langle \theta, \phi| e^{-iHt} \sum_{i=1}^{N} \sigma_i^z e^{iHt} |\theta, \phi\rangle \,,$$



For $J = -1.05, h = 0.5$ and $N = 10$.

Let us also compute Lanczos coefficients for three different initial states $|Y+\rangle, |Z+\rangle$ and $|X+\rangle$.



Although it is believed that a non-integrable model will generally thermalize, the nature of thermalization might be different in different situations.

Actually, besides Hamiltonian which gives dynamics of the system, the nature of the thermalization may also depend on the initial state, such that, within a fixed model different initial states may exhibit different behaviors.

In general, we would like to study time evolution of expectation value of a local operator (observable) $\mathcal{O}$

$$\langle\psi(t)|\mathcal{O}|\psi(t)\rangle,$$

whose behavior could explore the nature of thermalization whether it is strong or weak.

In the strong thermalization, the expectation value relaxes to the thermal value very fast, while for weak thermalization it strongly oscillates around the thermal value, though its time average attains the thermal value.

M. C. Bañuls, J. I. Cirac and M. B. Hastings, "Strong and Weak Thermalization of Infinite Nonintegrable Quantum Systems," Phys. Rev. Lett. **106** (2011) no.5, 050405]

For the Ising model we considered it has been shown that although the initial state $|Y+\rangle$ exhibits strong thermalization, for initial state $|Z+\rangle$ one observes weak thermalization and for initial state $|X+\rangle$ there is an apparent departure of the thermal expectation value form its thermal value suggesting that there might be no thermalization for this state.

It was proposed that whether we are going to observe strong or weak thermalization is closely related to the effective inverse temperature, β, of the initial state which can be read from the following equation

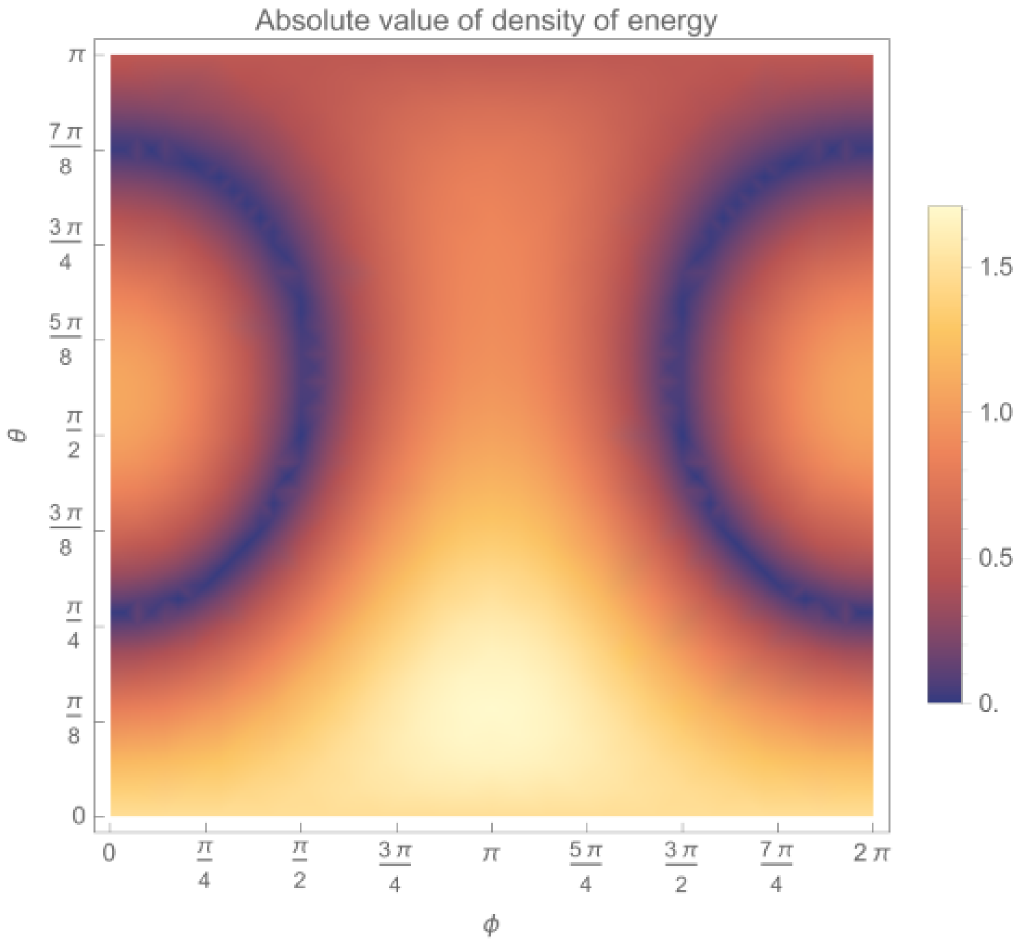
$$\mathrm{Tr}\left(H(\rho_0 - \rho_{th})\right) = 0,$$

where $\rho_{th} = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$ is thermal density state with inverse temperature β.

For a given initial state $|\psi_0\rangle$ this equation may be rewritten as follows

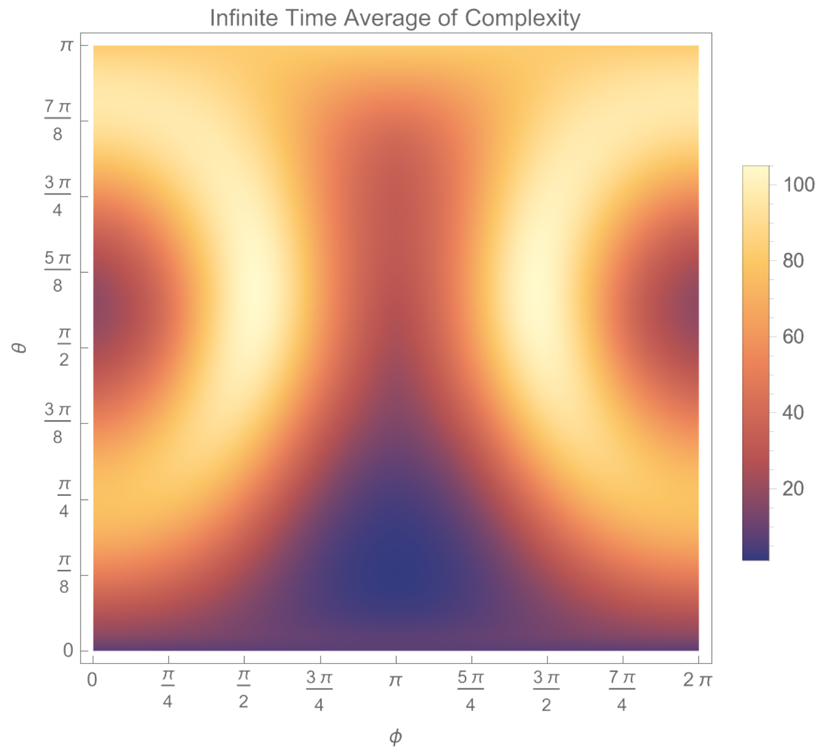
$$\mathrm{Tr}\,(\rho_{th} H)) = \langle\psi_0|H|\psi_0\rangle = E\,,$$

which suggests that the information of the effective inverse temperature could be read from the expectation value of the energy.



This information is also encoded in the infinite time average of Krylov complexity

$$\overline{\mathcal{C}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \mathcal{N}(t) \rangle \, dt$$



M. A. and M. J. Vasli, 2403.06655 [quant-ph]

Summary

Complexity may provide an interesting quantity to understand the physics behind the horizon.

There are different notations of complexity.

Krylov complexity provides a notation of complexuty which looks more tractable.

One may use Krylov complexity to explore the nature of the quantum thermalization.

Thank you