Static Black Hole Horizons in Cosmology

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A very brief history of black hole metrics in cosmology

• Black holes and cosmological space times are exciting predictions of Einstein's general theory of relativity and yet there is a lot of debate about a black hole metric embedded in a cosmological space time.

• Schwarzschild-de Sitter metric: Schwarzschild black hole in a cosmology with constant Hubble parameter is well known:

$$ds^2 = -(1 - \frac{l_s}{r} - \frac{r^2}{l^2})dt^2 + \frac{dr^2}{(1 - \frac{l_s}{r} - \frac{r^2}{l^2})} + r^2 d\Omega_2^2$$

This metric has two static horizons which are solutions to $g^{rr} = 1 - \frac{l_s}{r_h} - \frac{r_h^2}{l^2} = 0$.

McVittie's metric

A metric suggested for a point particle embedded in cosmology with varying Hubble parameter.
McVittie postulates the metric to be of the form

$$ds^2 = -e^{\zeta(t,r)}dt^2 + e^{\nu(t,r)}(dr^2 + r^2 d\Omega_2^2).$$

Using assumptions of isotropic pressure $p_r = p_\perp$ and $T_{tr} = 0$, he finds the metric

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + \left(\frac{dr}{\sqrt{1 - \frac{2m}{r}}} - H(t)rdt\right)^2 + r^2 d\Omega_2^2$$

with apparent horizon at $1 - \frac{2m}{r_h} - H(t)^2 r_h^2 = 0$ which is time dependent.

The cosmological coupling of black hole horizon

There are claims that black hole horizons are inevitably cosmologically coupled (see for instance Faraoni, Rinaldi Phys.Rev.D 110 (2024) 6)

This is based on the assumption that the metric takes the form

$$ds^2 = -e^{\zeta(t,r)}dt^2 + e^{\nu(t,r)}(dr^2 + r^2 d\Omega_2^2).$$

In particular note the assumption $r^2 g_{rr} = g_{\theta\theta}$.

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Overview of our results

- We drop this assumption and first show that it is in principle possible to have a black hole metric with a static horizon embedded in cosmology.
- In the second part of this work we study black hole metrics in a cosmology with constant equation of state and show that the black hole's apparent horizon for large cosmological time can tend to to a constant value.

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The feasibility of a static horizon

We begin with pure black hole and cosmological metrics in Painlevé-Gullstrand coordinates:

$$ds_{BH}^2 = -(1 - \frac{l_s}{r})dt^2 + 2\sqrt{\frac{l_s}{r}} dt\, dr + dr^2 + r^2 d\Omega_2^2,$$

$$ds_C^2 = -(1 - \frac{\dot{a}^2}{a^2} r^2)dt^2 - 2\frac{\dot{a}}{a} r\, dtdr + dr^2 + r^2 d\Omega_2^2.$$

The feasibility of a static horizon

Considering the Schwarzschild-de Sitter metric

$$ds^2 = -(1 - \frac{l_s}{r} - \frac{r^2}{l^2})dt^2 - 2\sqrt{\frac{l_s}{r} + \frac{r^2}{l^2}}dt\,dr + dr^2 + r^2 d\Omega_2^2.$$

we can write it in terms of the exact black hole horizon l_1 and de Sitter length l as

$$ds^2 = -(1 - \frac{l_1^2}{l^2})\left(1 - \frac{l_1}{r} - \frac{r^2 - l_1^2}{l^2 - l_1^2}\right)dt^2 - 2\sqrt{(1 - \frac{l_1^2}{l^2})\frac{l_1}{r} + \frac{r^2}{l^2}}dtdr + dr^2 + r^2 d\Omega_2^2.$$

The feasibility of a static horizon

Inspired by this we suggest a metric with a static horizon embedded in cosmology with varying $\frac{\dot{a}}{a}$

$$ds^2 = -\left(1-\frac{l_1^2\dot{a}^2}{a^2}\right)\left(1-\frac{l_1}{r}-\frac{r^2-l_1^2}{\frac{a^2}{\dot{a}^2}-l_1^2}\right)dt^2-2\sqrt{\left(1-\frac{l_1^2\dot{a}^2}{a^2}\right)\frac{l_1}{r}+\frac{r^2\dot{a}^2}{a^2}}dtdr+dr^2+r^2d\Omega_2^2.$$

This can be written in a more suggestive form as

$$ds^2 = -(1-h^2)dt^2 - 2hdtdr + dr^2 + r^2d\Omega_2^2.$$

$$h = \sqrt{\frac{l_1}{r}+\frac{r^3-l_1^3}{r}\frac{\dot{a}^2}{a^2}}.$$

The feasibility of a static horizon

This metric has the property that the black hole and cosmological horizons are detached and it has an static horizon at $r = l_1$.

We can see that the Ricci scalar and curvature invariants like $R^{\mu\nu}R_{\mu\nu}$ and $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ remain finite at the black hole horizon. In particular we have at the horizon $r = l_1$

$$R|_{r=l_1} = -6\, l_1 \frac{\dot{a}^3}{a^3} + 6\, l_1 \frac{\dot{a}\ddot{a}}{a^2} + 12 \frac{\dot{a}^2}{a^2}.$$

The feasibility of a static horizon

We have $T_r^t = 0$ and the large r limit of stress tensor tends to the cosmological values. We have

$$G_t^t = -3\frac{\dot{a}^2}{a^2}, \qquad G_r^t = 0,$$

$$G_t^r = -2\,r\big(1 - \frac{l_1^3}{r^3}\big)\Big(\frac{\dot{a}^3}{a^3} - \frac{\dot{a}\ddot{a}}{a^2}\Big) = g^{tr}(G_r^r - G_t^t),$$

$$G_\theta^\theta = G_\phi^\phi = -3\frac{\dot{a}^2}{a^2} - \frac{l_1(l_1^3 - 7r^3)}{2r^3 g_{tr}^3}\Big(\frac{\ddot{a}\dot{a}}{a^2} - \frac{\dot{a}^3}{a^3}\Big) - \frac{(l_1^3 - r^3)(l_1^3 - 4r^3)}{2r^3 g_{tr}^3}\Big(\frac{\dot{a}^5}{a^5} - \frac{\ddot{a}\dot{a}^3}{a^4}\Big)$$

A systematic derivation

We begin with the metric ansatz

$$ds^2 = -f(t,r)dt^2 - 2h(t,r)dtdr + dr^2 + r^2 d\Omega_2^2$$

and require

a) There is no flux of energy in radial direction, i.e. we assume $T_r^t = 0$.
b) There is a static horizon at $r = l_1$, such that $g^{rr}|_{r=l_1} = 0$.
c) We assume the energy density is the same as the FLRW one, i.e. we assume $G_t^t = -3\frac{\dot{a}^2}{a^2}$.
d) The metric at large r tends to the cosmological FLRW metric.

We note that compared to McVittie's assumptions, we have dropped the $p_\perp = p_r$ assumption and instead we have added assumption b.

A systematic derivation

Requiring

$$G_r^t = -\frac{h(f' + 2hh')}{r(f + h^2)^2},$$

to vanish we can write

$$f(t,r) = \gamma(t) - h(t,r)^2,$$

From assumption b we have $h(t, l_1)^2 = \gamma(t)$.
Considering assumption c, we have

$$G_t^t = -\frac{h(t,r)^2 + rh(t,r)h'(t,r)}{r^2 h(t,l_1)^2} = -3\frac{\dot{a}^2}{a^2}.$$

A systematic derivation

Integrating we find

$$h(t,r)^2 = h(t,l_1)^2 r^2 \frac{\dot{a}^2}{a^2} + \frac{c(t)}{r}.$$

Considering assumption d we have $h(t,l_1) = 1$ and from this we find

$$c(t) = l_1\big(1 - \frac{\dot{a}^2}{a^2} l_1^2\big).$$

Therefore with these assumptions the proposed metric is unique.

• The drawback of this metric is that the Null energy condition (NEC) is violated inside the black hole horizon.

There are still other possibilities

To satisfy NEC we need $\dot{h} < 0$ everywhere.

One possible metric which respects NEC and assumptions a,b,d has

$$h(t,r) = \sqrt{\frac{l_1}{r} + r^2\left(1 - \frac{l_1^3}{r^3}\right)\frac{\dot{a}^2}{a^2} + l_1^2\left(1 - \frac{l_1^3}{r^3}\right)\left(\frac{1}{l_0^2} - \frac{\dot{a}^2}{a^2}\right)}.$$

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The case of constant equation of state

Here we consider the possibility that the black hole metric respects the cosmological stress tensor equation of state. Assuming $T_r^t = 0$, we find

$$f(t,r) = \gamma(t) - h^2(t,r).$$

We assume $p_\perp = p_r = \omega\rho$.
Considering $G_t^r = g^{tr}(G_r^r - G_t^t)$[1], we have from Einstein equations

$$G_t^r = \frac{h}{r\gamma^2}(2\gamma\dot{h} - h\dot{\gamma}) = -\frac{h}{\gamma^2}\kappa_4(1+\omega)\rho,$$

[1] This follows from $G_r^t = 0$ and $G^{tr} = G^{rt}$.

The case of constant equation of state

Therefore

$$\frac{1}{r}(2\gamma\dot{h} - h\dot{\gamma}) = -\kappa_4(1+\omega)\rho.$$

On the other hand from $G_t^t = -\kappa_4\rho$ we find

$$h^2 + 2rhh' + \frac{2r\gamma^2\dot{h} - rh\gamma\dot{\gamma}}{1+\omega} = 0.$$

We also have by considering $G_r^r = \kappa_4\omega\rho$,

$$h^2 + 2rhh' + \frac{2r\dot{h} - rh\frac{\dot{\gamma}}{\gamma}}{1+\omega} = 0.$$

The case of constant equation of state

We find

$$(2r\dot{h} - rh\frac{\dot{\gamma}}{\gamma})(\gamma^2 - 1) = 0.$$

So we either have $\gamma^2 = 1$ or $2\frac{\dot{h}}{h} = \frac{\dot{\gamma}}{\gamma}$.
The second possibility requires $h^2 = c(r)\gamma(t)$. We can show that this leads to $c(r) = \frac{l_s}{r}$ where l_s is a constant length and the metric takes the form

$$ds^2 = -\gamma(t)(1 - \frac{l_s}{r})dt^2 - 2\sqrt{\frac{l_s}{r}\gamma(t)}dtdr + dr^2 + r^2 d\Omega^2,$$

which is the Schwarzschild metric in Painlevé-Gullstrand coordinates. Therefore we will set $\gamma(t) = 1$ in what follows.

The case of $\omega \neq 0$

Assuming $\gamma(t) = 1$, we have from the G_0^0 component of Einstein equations

$$h^2 + 2rhh' + \frac{2}{1+\omega} r\dot{h} = 0.$$

Considering the G_θ^θ component we find

$$\partial_r(r\dot{h} + rhh' + \frac{h^2}{2}) = -\kappa_4 r\omega\rho = \frac{1}{r}(h^2 + 2rhh' + 2r\dot{h}),$$

where we used the G_r^r component as well.

The case of $\omega \neq 0$

In this case we find

$$h^2 + 2rhh' + 2r\dot{h} = \frac{2\omega}{1+\omega} b'(t) r^2.$$

$$h = b(t)r + c(r). \tag{1}$$

$$3b^2(t)r^2 + 4rb(t)c(r) + 2r^2 b(t)c'(r) + \frac{2}{1+\omega} r^2 \dot{b} + c^2(r) + 2rc(r)c'(r) = 0.$$

The only nontrivial solution of this equation which is of cosmological interest is $c(r) = 0$, $b(t) = \frac{2}{3(1+\omega)t}$. Therefore we do not have a cosmological black hole solution that satisfies $p_r = p_\perp = \omega\rho$ for $\omega \neq 0$.

The case of matter with $\omega = 0$

We change the coordinates back to a semi-homogeneous form by setting $r = a(t, R)R$ such that $h = R\dot{a}$. We have

$$ds^2 = -dt^2 + (a + Ra')^2 dR^2 + a^2 R^2 d\Omega_2^2,$$

where $a' = \partial_R a$. In this coordinate $T_t^R = T_R^t = 0$ and the Einstein equations become

$$\frac{3a\dot{a}^2 + R\,a'\dot{a}^2 + 2R\,a\dot{a}\dot{a}'}{a^2(a + R\,a')} = \frac{\partial_R(R^3 a\dot{a}^2)}{R^2 a^2(a + R\,a')} = \kappa_4 \rho,$$

$$(\frac{\dot{a}}{a})^2 + 2\frac{\ddot{a}}{a} = -\kappa_4 p_r,$$

$$\frac{\dot{a}^2 + R\,\dot{a}\dot{a}' + 2a\ddot{a} + R\,a'\ddot{a} + R\,a\ddot{a}'}{a(a + R\,a')} = -\kappa_4 p_\perp.$$

The case of matter with $\omega = 0$

From stress energy conservation we have

$$\frac{\dot{\rho}}{\rho} + 2\frac{\dot{a}}{a} + \frac{\dot{a} + R\,\dot{a}'}{a + R\,a'} = 0,$$

and therefore

$$\rho = \frac{\beta(R)}{a^2(a + R\,a')}.$$

From the G_{tt} we find

$$R^3 a\dot{a}^2 = \kappa_4 \int^R dR'\,R'^2\beta(R').$$

From G_{rr} we find

$$a(t,R) = c(R)\left(\frac{3}{2}t - b(R)\right)^{2/3}, \qquad R^3 c(R)^3 = \kappa_4 \int^R dR'\,R'^2\beta(R').$$

The case of matter with $\omega = 0$

If we only had a point mass contribution to ρ this equation lead to

$$c(R) = \frac{l_s^{1/3}}{R}, \qquad a(t,R) = \frac{l_s^{1/3}}{R}(\frac{3}{2}t - b(R))^{2/3}.$$

In our original central coordinate this means that we had

$$h = R\,\dot{a} = \frac{l_s^{1/3}}{(\frac{3}{2}t - b(R))^{1/3}} = \sqrt{\frac{l_s}{r}}$$

where in the last equality we used $r = R\,a$. This simple case corresponds to the Schwarzschild metric in Painlevé-Gullstrand coordinates.
We also note that in this coordinate the black hole singularity is at the surface $\frac{3}{2}t - b(R) = 0$. There is a freedom in our choice of $b(R)$.

The case of matter with $\omega = 0$

For the case of black hole in matter-dominated cosmology we can add a term related to the cosmological matter density to $\beta(R)$. We note that in the asymptotic case we should have $\kappa_4\beta(R) = \beta_0$ where β_0 is a constant. Considering that this is also the case for large t we set $\kappa_4\beta(R) = \beta_0$ and find

$$c(R) = (\frac{l_s}{R^3} + \frac{\beta_0}{3})^{1/3},$$

and

$$h = R\dot{a} = \frac{(l_s + \frac{\beta_0}{3}R^3)^{1/3}}{(\frac{3}{2}t - b(R))^{1/3}} = \frac{r}{\frac{3}{2}t - b(R)}.$$

The apparent horizon is at $h = 1$. Therefore we have

$$r_H = \frac{3}{2}t - b(R_H).$$

The case of matter with $\omega = 0$

On the other hand using $r = Ra$ we have

$$r_H = (l_s + \frac{\beta_0}{3} R^3)^{1/3} (\frac{3}{2} t - b(R))^{2/3}, \qquad r_H = l_s + \frac{\beta_0}{3} R_H^3.$$

If we choose $b(R) = \frac{\kappa l_s^4}{R^3} - d$, we can find

$$r_H = l_s + \frac{\frac{\beta_0}{3} \kappa l_s^4}{\frac{3}{2} t + d - r_H},$$

with solutions

$$r_H(t) = \frac{1}{4} \left(3t + 2d + 2l_s \pm \sqrt{(3t + 2d - 2l_s)^2 - \frac{16}{3} \beta_0 \kappa l_s^4}\right).$$

For large t one of these tends to $r_H = \frac{3t}{2}$ which is the cosmological horizon and the other tends to $r_H = l_s$ which is the black hole horizon.

The case of matter+cosmological constant

In this case from $-\kappa_4 p_r = \frac{3}{l^2}$ and G_{rr} we find

$$a(t,R) = c(R)\sinh^{2/3}\left(\frac{3t}{2l} - \frac{b(R)}{l}\right).$$

We can also write

$$\partial_R(R^3 a\dot{a}^2) = R^2\beta(R) + \frac{3}{l^2}R^2a^2(a + Ra').$$

and

$$\frac{3}{l^2}c^2(R)(c(R) + Rc'(R)) = \beta(R).$$

$$\frac{1}{l^2}\partial_R(R^3c^3) = R^2\beta(R).$$

The case of matter+cosmological constant

We are again left with two unfixed functions of R, namely $b(R)$ and $\beta(R)$. One can use the degree of freedom in choosing the R coordinate to fix one of them.
If we assume $\beta(R) = \beta_0$ to be a constant we find

$$c(R) = (\frac{l_s l^2}{R^3} + \frac{\beta_0 \, l^2}{3})^{1/3}$$

By studying $h = R\,\dot{a}$ one can see that for $\beta = 0$, which is the case of pure cosmological constant, l_s is the Schwarzschild radius, i.e. for $\beta_0 = 0$ we have

$$h|_{\beta_0=0} = R\,\dot{a} = \sqrt{\frac{l_s}{r} + \frac{r^2}{l^2}}.$$

The case of matter+cosmological constant

From $h = 1$ at the apparent horizon we find

$$h = R\,\dot{a} = \frac{r}{l}\coth\left(\frac{3t}{2l} - \frac{b(R)}{l}\right) = 1.$$

$$\sinh^2\left(\frac{3t}{2l} - \frac{b(R)}{l}\right) = \frac{\frac{r_H^2}{l^2}}{1 - \frac{r_H^2}{l^2}}.$$

Assuming $b(R)$ behaves such that the location of the horizon for large t tends to small R, we can write for large t

$$r_H \approx (l_s l^2)^{1/3}\left(\frac{\frac{r_H^2}{l^2}}{1 - \frac{r_H^2}{l^2}}\right)^{1/3}, \qquad \frac{l_s}{r_H} + \frac{r_H^2}{l^2} = 1.$$

i.e. the asymptotic apparent horizon coincides with Schwarzschild-de Sitter black hole horizon.

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Conclusions

• In the first part of this work, we proposed a black hole metric with a static horizon embedded in FLRW cosmology.

• In the second part of this work we considered a more realistic framework with a constant equation of state (for the cases of matter and radiation) and derived the black hole metric consistent with this assumption.

• We also studied the case of matter+cosmological constant where we assumed the pressure component remains equal to the constant cosmological value. In all of these cases we show that it is possible to have a black hole horizon which asymptotically (for large t) tends to a static horizon.

Thank you.