Revisiting Quantization of Gauge Field Theories: Sandwich Quantization Scheme

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- To construct a quantum theory we should apply a *quantization scheme* on a given classical theory.
- There are different, but physically equivalent, quantization procedures.
- A quantization scheme can be applied to particle theory, yielding standard Heisenberg formulation of quantum mechanics, as well as to quantum field theories (QFT).
- Particle theory is a $0+1$ dimensional field theory residing on the particle worldline.
- Our discussions here hold for a particle and field theory alike.

Canonical quantization is one of them:

- work out solution phase space of the classical theory,
- promote the Poisson brackets to commutators, and
- observables to operators and a Hilbert space over which the operators act.
- Hilbert space is constructed based on a *vacuum state*.
- The Hilbert space of a (local) QFT are constructed by the action of local operators on the vacuum state.
- By construction, we have operator-state correspondence. (Vacuum state then corresponds to the identity operator.)
- Physical observables are VEV of products of physical operators.

- Path integral is the other quantization scheme.
- In the path integral quantization does not need Hilbert space.
- Path integral computes all observables, i.e. generic n-point functions, vacuum expectation value (VEV) of products n generic local operators.
- Gauge theories and gauge symmetries have been the cornerstone of physical formulations in the last century.
- Gauge symmetries,
  - are not symmetries (subject to Noether's first theorem), are rather *redundancies of description*.
  - are guiding principle for fixing interactions among fields and
  - help making other symmetries of the theory manifest.

- Some of the degrees of freedom (d.o.f) in the Lagrangian of gauge theories are gauge d.o.f.
- Gauge d.o.f have vanishing conjugate momentum; thus, are not propagating and dynamical.
- Gauge theories are hence constrained systems:

  Equations of motion for the gauge d.o.f. are not second order in time derivative; they are constraints

- So, for quantizing gauge theories, Dirac's method or other suitable procedures should be invoked.

- Treatment of gauge field theories starts with gauge-fixing.
- For a system with N gauge d.o.f, gauge-fixing amounts to imposing N arbitrary relations among gauge fields and possibly their derivative.
- Gauge-fixing relations,
  - by definition, are not gauge invariant;
  - may be viewed as constraints which together with EoM for the gauge d.o.f form a well-posed (first-class) constraint system.
- EoM for the gauge d.o.f, by the very definition of gauge theory, are gauge invariant.
- One can construct solution space of the classical gauge-fixed theory and perform the standard canonical quantization procedure.

- Alternatively, one may choose the path integral method, in which one does not use EoM.
- Gauge-fixing condition can be inserted into the path integral through a delta-function.
- This delta-function can be exponentiated using ghosts and the BRST symmetry guarantees consistency of quantization.
- We revisit quantization of gauge field theories, especially the canonical quantization.
- We observe that EoM of the gauge d.o.f. need not be imposed as in the standard textbooks.
- One can impose them as *sandwich conditions*:

physical Hilbert space of the gauge field theory is obtained by requiring vanishing of the EoM of gauge d.o.f, when sandwiched between any two physical states.

Outline of the talk

- Quick review of classic treatment of gauge theories.
- Derivation of Sandwich Quantization Scheme
- Solving Sandwich Constraints
- Summary and outlook

Gauge Theory, Classic Treatment

- Consider a gauge field theory described by the Lagrangian $\mathcal{L}(\Phi)$, Φ being a generic set of fields that includes gauge d.o.f. φ_i $(i = 1, 2, \cdots, N)$ and other fields ψ_A.

- This theory has two important features:

(1) Gauge d.o.f φ_i, by definition, are the fields with identically vanishing conjugate momentum. If t denotes the time direction, that is,

$$\Pi^i := \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi_i)} \equiv 0. \qquad (1)$$

(2) The action is invariant under gauge transformations $\Phi \to \Phi + \delta_{\lambda_i}\Phi$,

$$\delta_{\lambda_i} S = \frac{\delta S}{\delta \lambda_i} = 0 \qquad \text{off-shell}, \qquad S := \int_M \mathcal{L}(\Phi). \qquad (2)$$

Gauge Theory, Classic Treatment

- Both gauge d.o.f φ_i and other fields ψ_A transform under gauge transformations and
- number of gauge d.o.f is equal to the number of independent gauge parameters λ_i.
- Consequently, EoM for gauge d.o.f φ_i, $\mathcal{C}^i$, are not dynamical equations; they are constraints:

$$\frac{\delta S}{\delta \varphi_i} = 0 \quad \Longrightarrow \quad \mathcal{C}^i := -\partial_t \Pi^i + \frac{\partial \mathcal{L}}{\partial \varphi_i} = \frac{\partial \mathcal{L}}{\partial \varphi_i} = 0. \qquad (3)$$

- By the definition of gauge invariance, field equations are gauge covariant (their set is gauge invariant).
- So, the constraints $\mathcal{C}^i = 0$ are gauge covariant and their set closes onto itself under gauge transformations.

Gauge Theory, Classic Treatment

- As a prime example, consider the Maxwell theory where Φ are the gauge field components A_μ with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4}$$

- The conjugate momenta to A_μ are

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t A_\mu)} = F^{t\mu}. \tag{5}$$

- Π_t identically vanishes and hence A_t is the gauge d.o.f.
- So, φ_i is A_t and ψ_A are the spatial components of the gauge field A_a and under generic gauge transformations,

$$A_\mu \to A_\mu + \partial_\mu \lambda.$$

- Both A_t and A_a transform under gauge transformation.

Gauge Theory, Classic Treatment

- The constraint (EoM for A_t) is the Gauss law:
  $\vec{\nabla}\cdot\vec{E}=0$, $E^a = F^{ta}$ is the electric field strength.
  E^a is the momentum conjugate to A_a.
- The constraint, the Gauss law, is gauge invariant.
- Gauge-fixing: the freedom in defining gauge fields can be used to fix λ_i (gauge parameters) can be fixed through imposing,

$$\mathcal{G}_i(\varphi_i, \psi_A) = 0, \tag{6}$$

  where $\mathcal{G}_i$ specify the desired gauge-fixing of our choice.
- Consistency of the gauge-fixing requires that (6) is compatible with the evolution of the system; it should hold at all times.
- In particular

$$\mathcal{G}_i = 0, \qquad \mathcal{C}^i = 0$$

  should hold simultaneously.

Gauge Theory, Classic Treatment

- This consistency requirement may be analyzed and carried out in the Hamiltonian formulation, yielding infinite chain of secondary, but first-class constraints [Henneaux-Teitelboim (1992)] or in the Lagrangian (action) formulation [Weinberg (1995), Peskin (1995)] with physically identical results.
- A handy gauge-fixing, that we adopt here, is to fix the functional form of gauge d.of. as

$$\boxed{\varphi_i - \varphi_i^0 = 0,} \tag{7}$$

where φ_i^0 is a given function with no dependence on other fields; temporal gauge-fixing is an example of this choice.
- Gauge inequivalent classical field configurations are given by

ψ_A subject to their EoM with $\varphi_i - \varphi_i^0 = 0$ and $\mathcal{C}^i = 0$ (at all times).

Gauge Theory, Classic Treatment

Right-Action Quantization Scheme.

- Having the classical physical solution space, one may proceed with the canonical quantization:
  - Promote the physical field configurations and their momenta to operators that satisfy the canonical commutation relations.
  - This guarantees the expectation that one-particle Hilbert space of the quantum theory and classical field configurations are in one-to-one relation. (As presented in QFT textbooks).
- **Notations.**
  - $\mathfrak{O}$ operator associated with a generic function(al) of fields $\mathcal{O}[\Phi]$;
  - $\mathcal{H}_{\mathsf{t}}$ Hilbert space of all field configurations;
  - $\mathcal{H}_{\mathsf{p}}$ Hilbert space of all physical field configurations.
  - Eq.(7) and the EoM of φ_i respectively denoted by

$$\varphi_i - \varphi_i^0 = 0 \qquad \mathfrak{C}^i = 0$$

Right-Action quantization scheme

- Whenever we discuss vanishing of an operator we should specify over which Hilbert space it vanishes.
- $\varphi_i - \varphi_i^0$ and $\mathfrak{C}^i = 0$ do not vanish over $\mathcal{H}$ but over $\mathcal{H}_\mathsf{p}$.
- In other words, $\mathcal{H}_\mathsf{p}$ may defined as a subset of $\mathcal{H}_\mathsf{t}$, such that

$$(\varphi_i - \varphi_i^0)|\psi\rangle = 0, \qquad \mathfrak{C}^i|\psi\rangle = 0, \quad \forall\ |\psi\rangle \in \mathcal{H}_\mathsf{p}. \tag{8}$$

- The above is what we find in the QFT textbooks and is shown to yield a consistent quantization scheme for gauge theories.
- We call this the right-action quantization scheme, as the right-action of the constraint operators on the physical Hilbert space is required to vanish.

Is right-action scheme necessary?

Is the right-action quantization scheme is really required for passage from classical physical (gauge inequivalent) field configurations to physical quantum Hilbert space?

We argue in next that indeed a weaker condition can fulfill the requirement.

Operators on a Hilbert space $\mathcal{H}$ may customarily be viewed as matrices over the Hilbert space, i.e.

$$\mathfrak{O} \text{ over } \mathcal{H} \quad \Longleftrightarrow \quad \langle\psi|\mathfrak{O}|\phi\rangle \text{ for any two states } |\phi\rangle, |\psi\rangle \in \mathcal{H}.$$

Sandwich Quantization Scheme

- So, a natural choice/proposal to replace vanishing of an operator is vanishing of its matrix elements.
- Explicitly, one may explore replacing right-action with the *sandwich constraints*:

$$\langle\phi|(\varphi_i-\varphi_i^0)|\psi\rangle=0,\quad \langle\phi|\mathfrak{C}^i|\psi\rangle=0,\quad \forall\ |\psi\rangle,|\phi\rangle\in\mathcal{H}_{\mathrm{p}}. \qquad (9)$$

- Physical Hilbert space $\mathcal{H}_{\mathrm{p}}$ can be defined upon (9).
- All solutions to (8) are also solutions to (9) but the reverse is not necessarily true.
- We consistently require the gauge-fixing condition as a right-action while $\mathfrak{C}^i$ are imposed as a sandwich constraint:

$$(\varphi_i-\varphi_i^0)|\psi\rangle=0,\quad \langle\psi'|\mathfrak{C}^i|\psi\rangle=0,\quad \forall|\psi\rangle,|\psi'\rangle\in\mathcal{H}_{\mathrm{p}} \qquad (10)$$

Path Integral Derivation of Sandwich Constraints

- Let $\mathcal{O}_i(x)$ denote *gauge invariant* local operators in the gauge field theory we study:

$$\delta_{\lambda_j}\mathcal{O}_i = \frac{\delta\mathcal{O}_i[\Phi]}{\delta\Phi}\delta_{\lambda_j}\Phi = 0. \tag{11}$$

- Physical observables are then generic n-point functions of these operators, VEV of time-ordered product of these operators:

$$G_n(x_1, x_2, \cdots, x_n) = \langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle. \tag{12}$$

- We are assuming that vacuum state $|0\rangle$ is a state in $\mathcal{H}_{\rm p}$. One can show that this assumption is a consistent one within the sandwich quantization scheme.

Path Integral Derivation of Sandwich Constraints

- The above n-point function which is gauge invariant by definition, may be computed using the path integral,

$$G_n(x_1, x_2, \cdots, x_n) = \int \mathcal{D}\Phi\ \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\ e^{\frac{i}{\hbar}S[\Phi]} \quad (13)$$

- To perform the path integral,
    - either define the measure $\mathcal{D}\Phi$ by modding it out by the volume of the gauge orbits or
    - equivalently insert the gauge-fixing condition, which in our case is $\varphi_i - \varphi_i^0 = 0$. Let us denote the modded out measure by $\overline{\mathcal{D}\Phi}$.
- To study the constraints at quantum level in path integral formulation, we can/should consider a generic n-point function with the insertion of $\mathcal{C}^i$.

Path Integral Derivation of Sandwich Constraints

- This is a very common analysis in the study of anomalies and gauge-fixings in gauge field theories [Weinberg QFT book] and/or string theory [Polchinski String theory book].

$$\langle \mathcal{O}_1(x_1)\cdots \mathcal{C}^i(x)\cdots \mathcal{O}_n(x_n)\rangle = \int \overline{\mathcal{D}\Phi}\ \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\ \mathcal{C}^i(x)\ e^{\frac{i}{\hbar}S[\Phi]}$$
$$= -i\hbar \int \overline{\mathcal{D}\Phi}\ \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\ \frac{\delta}{\delta\varphi_i(x)} e^{\frac{i}{\hbar}S[\Phi]}$$
$$= i\hbar \int \overline{\mathcal{D}\Phi}\ \frac{\delta}{\delta\varphi_i(x)} \left(\mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\right)\ e^{\frac{i}{\hbar}S[\Phi]}$$

in the second line we used the fact that $\mathcal{C}^i$ are EoM for the gauge d.o.f φ^i and in the third line, integration by-part.

Path Integral Derivation of Sandwich Constraints

- To proceed further we recall that one may replace φ_i with the gauge parameters, i.e. $\varphi^i = \varphi^i[\lambda^i]$ and as such,

$$\langle \mathcal{O}_1(x_1)\cdots\ \mathcal{C}^i(x)\cdots \mathcal{O}_n(x_n)\rangle =$$
$$= i\hbar \int \overline{\mathcal{D}\Phi}\ \frac{\delta\lambda_j(y)}{\delta\varphi_i(x)}\delta_{\lambda_j(y)}\Big(\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\Big)\ e^{\frac{i}{\hbar}S[\Phi]} \tag{14}$$

- Recalling that $\mathcal{O}_i$ are gauge invariant operators, we learn that

$$\langle \mathcal{O}_1(x_1)\cdots\ \mathcal{C}^i(x)\cdots \mathcal{O}_n(x_n)\rangle = 0, \quad \forall \text{ local gauge invariant operators } \mathcal{O}_i \tag{15}$$

Sandwich Quantization Scheme

- Recall operator-state correspondence: to any local gauge invariant operators $\mathfrak{O}_i$ one may associate the state $|\mathcal{O}_i\rangle$.
- Vacuum state $|0\rangle$ corresponds to identity operator and a generic (multi-particle) state may be associated with products of such operators.
- Thus, (15) is equivalent to sandwich constraints (10).
- We have presented a derivation of the sandwich quantization scheme (10) from the path integral formulation.
- This derivation implies that if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are two physical operators ($|\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle \in \mathcal{H}_{\mathsf{p}}$), then $\mathfrak{O}_1\mathfrak{O}_2|0\rangle$ should also be in $\mathcal{H}_{\mathsf{p}}$; see [arXiv:2412.12436] for related arguments.
- The RHS of (14) is proportional to $\hbar$. Sandwich quantization scheme is an option appearing at quantum level, while at classical level we deal with $\mathfrak{C}^i = 0$.

Solving Sandwich Conditions

- We utilize the ideas developed in [arXiv:2409.16152, 2412.12436].
- Consider a Hermitian operator $\mathfrak{C}$ that acts on a total Hilbert space $\mathcal{H}_{\mathsf{t}}$ such that

$$\mathcal{H}_{\mathsf{p}} \subset \mathcal{H}_{\mathsf{t}}, \qquad \langle \psi' | \mathfrak{C} | \psi \rangle = 0.$$

- To solve the sandwich condition, decompose $\mathcal{H}_{\mathsf{t}}$ into physical and complement parts:

$$\mathcal{H}_{\mathsf{t}} = \mathcal{H}_{\mathsf{p}} \ \cup \ \mathcal{H}_{\mathsf{c}} \quad s.t. \quad \langle \psi^c | \psi^p \rangle = 0, \ \forall |\psi^c\rangle \in \mathcal{H}_{\mathsf{c}}, |\psi^p\rangle \in \mathcal{H}_{\mathsf{p}}$$

- One then has TWO options:

I. $\mathfrak{C}|\psi\rangle \in \mathcal{H}_{\mathsf{p}}, \ \forall |\psi\rangle \in \mathcal{H}_{\mathsf{p}}$

II. $\mathfrak{C}|\psi\rangle \in \mathcal{H}_{\mathsf{c}}, \ \forall |\psi\rangle \in \mathcal{H}_{\mathsf{p}}$.

The two options

- For **Class I** case, $\mathfrak{C}$ is an operator defined over $\mathcal{H}_{\rm p}$ and hence $\mathcal{H}_{\rm p}$ consists of all zero eigenstates of $\mathfrak{C}$.
- For **Class II**, however, $\mathfrak{C}$ is defined over the total Hilbert space $\mathcal{H}_{\rm t}$ and not $\mathcal{H}_{\rm p}$.

– Explicit construction of Class II states:

- Let $|C\rangle$ denote eigenstates of $\mathfrak{C}$,

$$\mathfrak{C}|C\rangle = C|C\rangle. \tag{16}$$

- Zero-eigenstate with $C = 0$ are **Class I** physical states.
- For **Class II** states $C \neq 0$.

Class II states

- Assume that $\mathfrak{C}$ has a $\mathbb{Z}_2$ symmetric spectrum such that for every state with eigenvalue $|C|$ there is another state with eigenvalue $-|C|$.
- One can construct states of the form

$$|\mathtt{C}\rangle_{\pm} := \frac{1}{\sqrt{2}}(|\mathtt{C}\rangle \pm |-\mathtt{C}\rangle), \qquad \mathtt{C} := |C| \tag{17}$$

- For these states we find:

$$\mathfrak{C}|\mathtt{C}\rangle_{\pm} = \mathtt{C}\ |\mathtt{C}\rangle_{\mp}, \qquad {}_{\mp}\langle\mathtt{C}'|\mathtt{C}\rangle_{\pm} = 0. \tag{18}$$

- Thus, evidently

$${}_{\pm}\langle\mathtt{C}'|\mathfrak{C}|\mathtt{C}\rangle_{\pm} = 0, \quad \forall \mathtt{C}, \mathtt{C}'$$

Solving Sandwich Conditions

- So, collection of all states $|\mathtt{C}\rangle_+$ (or $|\mathtt{C}\rangle_-$) form a physical Hilbert space.
- To include $C = 0$ solutions in the same convention, we choose $\mathcal{H}_{\mathsf{p}}$ to be spanned by $|\mathtt{C}\rangle_+$ states.
- $\mathtt{C} = 0$ sector of $\mathcal{H}_{\mathsf{p}}$ respects the $\mathbb{Z}_2$ and generic $\mathtt{C} \neq 0$ does not exhibit this symmetry.
- $\mathfrak{C}$ maps a **Class II** physical state onto a state in $\mathcal{H}_{\mathsf{c}}$, while $\mathfrak{C}^2$ takes a physical state to a physical state.
- **Class I & II** Hilbert spaces are two super-selection sectors in the physical Hilbert space defined through the sandwich quantization scheme.

Summary and Outlook

- Revisiting the old problem of quantization of gauge field theories, we noted a small, but important, point that has slipped the attention of physicists.
- We introduced the sandwich quantization scheme.
- We showed how **Class I** and **Class II** physical physical Hilbert spaces arise. The latter is not discussed in QFT textbooks.
- While **Class I** states are a direct generalization of classical gauge-fixing procedure, **Class II** case arises in the quantized theory, with no classical counterpart.
- For **Class I**, EoM for gauge d.o.f $\mathcal{C}^i = 0$, is an operator defined on $\mathcal{H}_{\text{p}}$, whereas in **Class II**, it is defined over $\mathcal{H}_{\text{t}}$ (while $(\mathcal{C}^i)^2$ is defined over $\mathcal{H}_{\text{p}}$).

Summary and Outlook

- There are some steps remaining to complete the sandwich quantization proposal and show its consistency and sufficiency.
- **Consistency and sufficiency** of the proposal:

  Work through the BRST invariance of the proposal and show that physical states satisfying the sandwich conditions are defined up to BRST equivalence classes, i.e. our ***Class II*** *Hilbert space is BRST invariant.*

- **Completion** of the proposal amounts to uncovering the physical meaning of the **Class I** physical Hilbert space:
  - **Class II** states by definition are orthogonal to **Class I** states and the gauge invariant dynamics of the theory will never mix the two classes; one can consistently formulate the theory using only **Class I** states and recover QFT textbook results.
  - There is another option: physics may be formulated based on **Class II** Hilbert space.

Outlook

- Based on the example explored in [arXiv:2409.16152] we propose the *sandwich equivalence principle*:

  Physical observables of gauge field theories should equivalently be described by **Class I or II** *physical Hilbert spaces.*

- The above is similar to what we have in Einstein's Equivalence Principle and the choice of different observers.
- Sandwich quantization scheme, seemingly, was first proposed with a different motivation and argument: *quantization of null strings*, i.e. worldsheet theory of tensionless strings whose worldsheet is a 2d null surface [Bagchi et al (2020)].
- The same quantization scheme may be invoked for generic string worldsheet theory [arXiv:2409.16152] and also null p-brane theory [arXiv:2412.12436]

Outlook

- New insights on quantization our new scheme? Can it be useful in addressing interesting physics questions?
- Two examples could be of particular interest:
  - Quantization of worldline theory of a particle, as a $0+1$ dimensional theory with the worldline reprarametrization invariance as gauge symmetry.
  - Quantization of General relativity with spacetime diffeomorphism as gauge symmetry. This can provide a new venue beyond the Wheeler-DeWitt framework and may lead to a generally covariant arrow of time as a quantum feature [To appear].

Thank you for your attention.