Supersymmetric Solutions of D=3, N=4 Supergravity

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Why 3-dimensions?

- Easier than 4-dimensions to address some difficult problems
- AdS_3/CFT_2
- Relatively less explored

Which Problems?

- New models
- Connection to higher dimensions
- Supersymmetric solutions
- Holographic applications

D=3, N=4, SO(4) Gauged Supergravity

This model comes from a consistent S^3 reduction of the $D = 6$, $N = (1, 0)$ supergravity coupled to a single chiral tensor multiplet whose Lagrangian is:

$$\mathcal{L}_6 = \sqrt{-g}\Big(\mathcal{R} - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{12}e^{-\sqrt{2}\phi}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)$$

The reduction ansatz to compactify this theory on the three-sphere was found in:

A supersymmetric reduction on the three-sphere, NSD, H. Samtleben, O. Sarioglu, and D. Van den Bleeken, Nucl.Phys.B 890 (2014) 350, arXiv:1410.7168.

$$
\begin{aligned}
ds_6^2 &= (\det T^{\frac{1}{4}}) \left(\Delta^{\frac{1}{2}} ds_3^2 + g_0^{-2} \Delta^{-\frac{1}{2}} T_{ij}^{-1} \mathcal{D}\mu^i \mathcal{D}\mu^j \right), \\
\phi &= \frac{1}{\sqrt{2}} \log \left(\Delta^{-1} \det T^{\frac{1}{2}} \right), \\
H &= k_0 (\det T) \, \mathrm{vol}_3 - \frac{1}{6} \epsilon_{ijkl} \left(g_0^{-2} M \Delta^{-2} \mu^i \mathcal{D}\mu^j \wedge \mathcal{D}\mu^k \wedge \mathcal{D}\mu^l \right. \\
&+ \left. 3 g_0^{-2} \Delta^{-2} \mathcal{D}\mu^i \wedge \mathcal{D}\mu^j \wedge \mathcal{D} T_{km} T_{ln} \mu^m \mu^n + 3 g_0^{-1} \Delta^{-1} F^{ij} \wedge \mathcal{D}\mu^k T_{lm} \mu^m \right)
\end{aligned}
$$

where k_0 and g_0 are constants and

$$
\begin{aligned}
\mu^i \mu^i &= 1, \qquad \Delta = T_{ij} \mu^i \mu^j, \qquad M = 2 T_{ik} T_{jk} \mu^i \mu^j - \Delta T_{ii}, \\
\mathcal{D}\mu^i &= d\mu^i + g_0 A^{ij} \mu^j, \qquad \mathcal{D} T_{ij} = dT_{ij} + g_0 A^{ik} T^{kj} + g_0 A^{jk} T^{ki}, \\
F^{ij} &= dA^{ij} + g_0 A^{ik} \wedge A^{kj}, \qquad i, j = 1, \ldots, 4.
\end{aligned}
$$

After the reduction one gets three-dimensional SO(4) gauged, $N = 4$ supergravity with quaternionic sigma model target space SO(4,4)/(SO(4)×SO(4)):

$$\mathcal{L}_3 = \sqrt{-g}\left(R - \frac{1}{4}\, T_{ij}^{-1} T_{kl}^{-1} \mathcal{D}_\mu T_{jk} \mathcal{D}^\mu T_{li} - \frac{1}{8}\, T_{ik}^{-1} T_{jl}^{-1} F_{\mu\nu}^{ij} F^{kl\,\mu\nu} - V\right) + \mathcal{L}_{\rm CS}$$

$$\begin{aligned} V &= \frac{1}{2}\left(k_0^2 \det T + 2 g_0^2 T_{ij} T_{ij} - g_0^2 (T_{ii})^2\right) \\ \mathcal{L}_{\rm CS} &= -\frac{1}{8}\, k_0 \epsilon_{ijkl} \varepsilon^{\mu\nu\rho} A_\mu^{ij} \left(\partial_\nu A_\rho^{kl} + \frac{2}{3}\, g_0 A_\nu^{km} A_\rho^{ml}\right) \end{aligned}$$

General construction of D=3, gauged supergravities is known but explicit construction of any one of them needs effort.

(*Gauged locally supersymmetric $D = 3$ nonlinear sigma models*, B. de Wit, I. Herger, H. Samtleben, Nucl.Phys.B 671 (2003) 175, hep-th/0307006.)

Consistent reduction means that any solution of the 3-dimensional theory is a solution of the 6-dimensional one. This provides an opportunity to construct some complicated solutions in 6-dimensions by uplifting 3-dimensional solutions.

D=3 solutions $\longrightarrow$ D=6 solutions $\longrightarrow$ D=10 solutions

Supersymmetric solutions are more special.

To take advantage of this, we simplify the D=3 model further by truncating it so that only the U(1) $\times$ U(1) $\subset$ SO(4) symmetry is preserved. This corresponds to choosing the T_{ij} matrix as

$$T = \begin{pmatrix} e^{\xi_1} e^{R(\rho,\theta)} \mathbb{I}_2 & 0_2 \\ 0_2 & e^{\xi_2} \mathbb{I}_2 \end{pmatrix} \qquad \text{with} \qquad R(\rho,\theta) = \rho \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix},$$

and the vectors $A_{\mu\, ij}$ as

$$A_\mu = \begin{pmatrix} A^1_\mu & 0 \\ 0 & A^2_\mu \end{pmatrix} \qquad \text{with} \qquad A^{1,2}_\mu = \begin{pmatrix} 0 & \mathcal{A}^{1,2}_\mu \\ -\mathcal{A}^{1,2}_\mu & 0 \end{pmatrix},$$

where $\mathcal{A}^{1,2}_\mu$ are two Abelian vector fields.

The result is

$$\mathscr{L}_3 = \sqrt{-g}\left(R - \frac{1}{2}\left[\partial_\mu\xi_1\,\partial^\mu\xi_1 + \partial_\mu\xi_2\,\partial^\mu\xi_2 + \partial_\mu\rho\,\partial^\mu\rho + \sinh^2\rho\,D_\mu\theta D^\mu\theta\right]\right.$$
$$\left. - \frac{1}{4}\,e^{-2\xi_1}\,\mathcal{F}^1_{\mu\nu}\,\mathcal{F}^{1\,\mu\nu} - \frac{1}{4}\,e^{-2\xi_2}\,\mathcal{F}^2_{\mu\nu}\,\mathcal{F}^{2\,\mu\nu} - V\right) - \frac{k_0}{2}\,\varepsilon^{\mu\nu\rho}\,\mathcal{A}^1_\mu\,\mathcal{F}^2_{\nu\rho}\,,$$

where

$$D_\mu\theta = \partial_\mu\theta + 2\,g_0\,\mathcal{A}^1_\mu\,.$$

The scalar potential does not depend on the field θ and is given as

$$V = -4\,g_0^2\,e^{\xi_1+\xi_2}\,\cosh\rho + 2\,g_0^2\,e^{2\xi_1}\,\sinh^2\rho + \frac{k_0^2}{2}\,e^{2(\xi_1+\xi_2)}\,.$$

This potential has only one supersymmetric vacuum, which is AdS_3 and is located at

$$\rho = 0\,, \qquad e^{\xi_1} = e^{\xi_2} = \frac{2g_0}{k_0}\,.$$

The potential V can be written in terms of the superpotential W

$$V = 2\left[(\partial_{\xi_1} W)^2 + (\partial_{\xi_2} W)^2 + (\partial_\rho W)^2 - W^2\right]\,,$$

where

$$W = \frac{e^{\xi_2}}{2}\,\left(-2\,g_0 + k_0\,e^{\xi_1}\right) - g_0\,e^{\xi_1}\,\cosh\rho\,.$$

Supersymmetric Dyonic Strings in 6-Dimensions from 3-Dimensions, NSD, N. Petri, D. Van den Bleeken, JHEP 04 (2019) 168, arXiv:1902.05325.

The Killing spinor equation is

$$(\partial_\mu + \frac{1}{4}\,\omega_\mu{}^{bc}\,\gamma_{bc})\,\zeta_a + X_\mu\,\epsilon_{ab}\,\zeta^b - \frac{1}{2}\,W\,\gamma_\mu\,\zeta_a = 0\,,$$

where

$$X_\mu = \frac{1}{4}\,(1-\cosh\rho)\,D_\mu\theta - 2\epsilon_\mu{}^{\sigma\rho}\left(\mathcal{F}^1_{\rho\sigma} + \mathcal{F}^2_{\rho\sigma}\right)\,.$$

The BPS conditions from matter fermions are:

$$\begin{aligned}(\gamma^\mu\,\partial_\mu\xi_{1,2})\,\zeta_a - \left(\gamma_\mu\,\epsilon^{\mu\sigma\rho}\,\mathcal{F}^{1,2}_{\rho\sigma}\right)\,\epsilon_{ab}\,\zeta^b + 2\,\frac{\partial W}{\partial\xi_{1,2}}\zeta_a &= 0\,,\\ (\gamma^\mu\,\partial_\mu\rho)\,\zeta_a + \sinh\rho\,\left(\gamma^\mu D_\mu\theta\right)\,\epsilon_{ab}\,\zeta^b + 2\,\frac{\partial W}{\partial\rho}\,\zeta_a &= 0\,.\end{aligned}$$

In these supersymmetry transformations ζ_a's $(a=1,2)$ are defined in terms of Majorana spinors λ_A's $(A=1,2,3,4)$ as $\zeta_1 = \lambda_1 + i\lambda_3$ and $\zeta_2 = \lambda_2 + i\lambda_4$. Here, $\epsilon_{ab} = -\epsilon_{ba}$ such that $\epsilon^{12} = \epsilon_{12} = -1$.

A systematic way of constructing supersymmetric solutions of a supergravity based on Killing spinor bilinears was developed in:

All Metrics Admitting Supercovariantly Constant Spinors, K.P. Tod, Phys. Lett. B 121 (1983) 241.

This is a powerful method and has been applied to many supergravities in different dimensions but not much is done in D=3. We applied this method to the above model in 2 papers:

1) *Rotating AdS_3xS^3 and dyonic strings from 3-dimensions*, NSD, C.A. Deral, A. Saha, O. Sarioglu, JHEP 10 (2024) 185, arXiv:2408.03197.

2) *Timelike supersymmetric solutions of D=3, N=4 supergravity*, NSD, C.A. Deral, Phys.Rev.D 111 (2025) 4, 4, arXiv:2411.04437.

In this method, one starts with assuming the existence of one set of Killing spinors λ_A $(A = 1, 2, 3, 4)$ which we take as commuting and main objects are the spinor bilinears constructed out of them:

$$F^{AB} = \bar{\lambda}^A \lambda^B = -F^{BA} \,,$$
$$V_\mu^{AB} = \bar{\lambda}^A \gamma_\mu \lambda^B = V_\mu^{BA} \,.$$

One then derives algebraic and differential conditions on them from the supersymmetry transformations. A useful tool is the Fierz identity for spinors

$$\lambda\bar{\chi} = \frac{1}{2}(\bar{\chi}\lambda)\mathbb{1} + \frac{1}{2}(\bar{\chi}\gamma_\mu\lambda)\gamma^\mu \,,$$

Using the Fierz identity one finds that, the spinor indices can be split as $A = (a, \tilde{a})$ with $a = \{1, 2\}$ and $\tilde{a} = \{3, 4\}$ and one can choose a basis in which

$$F^{ab} = -f\epsilon^{ab}\,,\; F^{a\tilde{a}} = F^{\tilde{a}\tilde{b}} = 0\,.$$

Consequently, only V_μ^{ab}'s are non-zero, and we define the following vectors using them:

$$V_\mu = V_\mu^{11} + V_\mu^{22}\,, \quad K_\mu = V_\mu^{11} - V_\mu^{22}\,, \quad L_\mu = 2V_\mu^{12}\,,$$

which satisfy the following algebraic conditions

$$\begin{aligned} &V^\mu K_\mu = V^\mu L_\mu = K^\mu L_\mu = 0\,, \quad V_{[\mu}K_{\nu]} = f\,\epsilon_{\mu\nu\sigma}L^\sigma\,, \\ &V^\mu V_\mu = -K^\mu K_\mu = -L^\mu L_\mu = -4f^2\,. \end{aligned}$$

When $f \neq 0$, they constitute an orthogonal basis for the 3-dimensional spacetime. However, when $f = 0$ we can choose a basis in which $V_\mu = K_\mu$ and $L_\mu = 0$.

Supersymmetry breaking conditions are found as

$$V^{\mu}\gamma_{\mu}\lambda_{a} = 2f\epsilon_{ab}\lambda^{b}\,,\; K^{\mu}\gamma_{\mu}\lambda_{a} = 2f\,(-1)^{a}\epsilon_{ab}\lambda^{b}\,,\; L^{\mu}\gamma_{\mu}\lambda_{a} = 2f\,(-1)^{a}\lambda^{a}\,.$$

Only two of these relations are independent.

From the Killing spinor equation we get

$$\partial_{\mu}f = 0\,,$$

and

$$\nabla_{\mu}V_{\nu} = -W\epsilon_{\mu\nu\sigma}V^{\sigma}\,,$$

from which we see that $\nabla_{(\mu}V_{\nu)} = 0$. Since $V^{\mu}V_{\mu} = -4f^{2}$, we conclude that V^{μ} is either a timelike $(f \neq 0)$ or a null $(f = 0)$ Killing vector which can be used to classify solutions.

One also finds

$$\mathcal{L}_V \xi_1 = \mathcal{L}_V \xi_2 = \mathcal{L}_V \rho = 0\,,$$

where $\mathcal{L}_V$ is the Lie derivative in the Killing direction V. Moreover, we have

$$\mathcal{L}_V \mathcal{A}^{1,2}_\mu = 0\,,$$

after choosing the gauge

$$V^\mu \mathcal{A}^{1,2}_\mu = -f \xi_{1,2}\,.$$

When $\rho \neq 0$, with this gauge choice one has

$$\mathcal{L}_V \theta = 2 f g_0 (2 e^{\xi_1} + \xi_1)\,.$$

Null Killing Vector Case

In this case one finds that the most general spacetime metric admitting $V = \partial_v$ as a null Killing vector is

$$ds^2 = dr^2 + 2e^{2U(r)}dudv + e^{2\beta(u,r)}du^2 \,,$$

where $U(r)$ is related to the superpotential W as

$$U'(r) = W.$$

Vector fields take the form

$$\mathcal{A}_u^{1,2} = \chi^{1,2}(u,r) \,, \, \mathcal{A}_v^{1,2} = \mathcal{A}_r^{1,2} = 0.$$

For the scalar fields we have $\theta = 0$ and we assume (ξ_1, ξ_2, ρ) depend only on r.

From these we get the Killing spinors as

$$\lambda^a = (1+i)\, e^{U - \frac{1}{2}\beta}\, \lambda_0^a \,,$$

where λ_0^a is a constant, real spinor that satisfies $(\gamma^1 - \gamma^0)\lambda_0^a = 0$. From matter BPS equations one also obtains

$$\begin{aligned}
\xi_1' &= -k_0 e^{\xi_1+\xi_2} + 2g_0 e^{\xi_1} \cosh\rho \,, \\
\xi_2' &= -k_0 e^{\xi_1+\xi_2} + 2g_0 e^{\xi_2} \,, \\
\rho\,' &= 2g_0 e^{\xi_1} \sinh\rho \,,
\end{aligned}$$

and

$$(1-\cosh\rho) g_0 \chi^1 + 8(\partial_r \chi^1 + \partial_r \chi^2) = 0 \,.$$

The scalar field equations and all the Einstein's field equations except its *uu*-component are satisfied automatically, which implies

$$2\partial_r(e^{2\beta}W) - \partial_r^2(e^{2\beta}) = 4g_0^2\sinh^2\rho(\chi^1)^2 + e^{-2\xi_1}(\partial_r\chi^1)^2 + e^{-2\xi_2}(\partial_r\chi^2)^2\,.$$

Finally, the vector field equations reduce to

$$0 = \partial_r(e^{-2\xi_1}\partial_r\chi^1) - 4g_0^2\sinh^2\rho\chi^1 - k_0\partial_r\chi^2\,,$$
$$0 = \partial_r(e^{-2\xi_2}\partial_r\chi^2) - k_0\partial_r\chi^1.$$

We were able to solve these equations in full generality. Solutions can be classified with respect to the number of active scalars that are distinct which ranges from 0 to 3. They are independent solutions, that is one cannot go from, let's say, the 3-scalars to the 2-scalars solution by setting the extra active scalar to a constant and so on. This is so since the scalars are functionally dependent on each other in these solutions.

Example: Null warped AdS

Assuming all scalar fields are constant we find

$$
\begin{aligned}
ds_3^2 &= -\frac{k_0^2\,Q^2}{4g_0^2}\,R^4\,du^2 + 2R^2\,dudv + \frac{k_0^2}{4g_0^4}\,\frac{dR^2}{R^2}\,,\\
\mathcal{A}_1 &= -\mathcal{A}_2 = Q\,R^2\,du\,,\ e^{\xi_1} = e^{\xi_2} = \frac{2g_0}{k_0}\,,\ \rho = \theta = 0\,.
\end{aligned}
$$

When $Q = 0$ we have the usual AdS_3. This metric with $Q \neq 0$ is called the *null warped* AdS_3 and is also known as the *Schrödinger spacetime* due to its anisotropic scale invariance

$$
R \to \lambda R\,, \quad u \to \frac{u}{\lambda^2}\,, \quad v \to v\,.
$$

After the uplift of the null warped AdS_3 solution we obtain

$$
\begin{aligned}
ds_6^2 &= \omega \left[2R^2 du\, dv + \frac{dR^2}{R^2}\right] - \omega\, Q\, R^2\, \sigma\, du + \omega\, d\Omega_3^2\,, \\
H_3 &= \frac{2}{g_0^2} R\, dR \wedge du \wedge dv - \frac{2}{g_0^2} \mathrm{vol}_{S^3} + \frac{Q}{2g_0^2}\, d\left[R^2\, \sigma \wedge du\right]\,, \\
e^{\sqrt{2}\phi} &= \frac{2g_0}{k_0}\,, \qquad \omega^2 = \frac{k_0}{2g_0^5}\,.
\end{aligned}
$$

Here σ is the $U(1)$ fiber direction of the 3-sphere

$$
d\Omega_3^2 = \tfrac{1}{4}(d\eta^2 + \sin^2\eta\, d\varphi^2 + \sigma^2)\,, \qquad \sigma = d\psi - \cos\eta\, d\varphi\,.
$$

When $Q = 0$, we have the usual, i.e. non-rotating, $AdS_3 \times S^3$ solution. It was obtained before using the TsT solution generating technique:

The Spectrum of Strings on Warped $AdS_3 \times S^3$, T. Azeyanagi, D. M. Hofman, W. Song, and A. Strominger, JHEP 04 (2013) 078, arXiv:1207.5050

	$D = 3$	$D = 6$
Constant scalars	• AdS_3 • Null warped AdS_3,	• $AdS_3 \times S^3$ • Rotating $AdS_3 \times S^3$
Single active scalar	• Uncharged string with waves • Charged string with waves	• Dyonic string with waves • Rotating dyonic string
Two active scalars	• Uncharged string with waves	• A dyonic string distribution
Three active scalars	• Uncharged string with waves	• A dyonic string distribution

Timelike Killing Vector Case

The most general 3-dimensional spacetime metric that admits $V = \partial_t$ as a Killing vector with the constant negative norm $-4f^2 \neq 0$ can be written as:

$$ds^2 = -4f^2\big[dt + M(x,y)dx + N(x,y)dy\big]^2 + e^{2\sigma(x,y)}(dx^2 + dy^2)\,.$$

It turns out that for the gauged model for which constants g_0 and k_0 of the theory are non-vanishing, the only supersymmetric solution is AdS_3. Therefore, we consider the ungauged limit of our model for which we must set $g_0 = k_0 = 0$.

In this case, BPS conditions set scalars $\xi_{1,2}$ and vector fields to zero, and the model effectively reduces to a supergravity coupled to a sigma model. The remaining scalar fields (ρ, θ) describe a sigma model with a hyperbolic target space $\mathbb{H}^2 = SU(1,1)/U(1)$:

$$\mathscr{L}_{\text{sigma model}} = -\frac{1}{2}\sqrt{-g}\,\left[\partial_\mu \rho\, \partial^\mu \rho + \sinh^2 \rho\, \partial_\mu \theta\, \partial^\mu \theta\right].$$

It turns out that all supersymmetric solutions can be expressed in terms of 2 holomorphic functions and their metric has the form:

$$ds^2_{\text{spacetime}} = -dt^2 + e^H e^{-3K(\rho)}(d\rho^2 + \sinh^2 \rho\; d\theta^2)\,.$$

The function $K(\rho)$ in the warp factor is the non-harmonic part of the Kähler potential of the sigma model target space $\mathbb{H}^2$. Whereas, H is a harmonic function determined by the 2 holomorphic functions.

Some Other Work on D=3 Supergravities

* Construction of new D=3 supergravities

Minimal Massive Supergravity, NSD, M. Geiller, J. Rosseel, H. Samtleben, Phys.Rev.Lett. 129 (2022) 17, 17, arXiv:2206.00675, (Also arXiv:2312.12387, arXiv:2410.07964.)

* Supersymmetric solutions and their applications

- Warped AdS black holes

Supersymmetric Warped AdS in Extended Topologically Massive Supergravity, NSD, A. Kaya, H. Samtleben, E. Sezgin, Nucl.Phys.B 884 (2014) 106, arXiv:1311.4583.
(Also arXiv:1602.07263, arXiv:1803.06926.)

- RG flows

Renormalization group flows from $D = 3$, $N=2$ matter coupled gauged supergravities, NSD, JHEP 11 (2002) 025, hep-th/0209188.
(Also arXiv:2402.11586.)

Thank you