Amplification of tachyonic perturbations at super-Hubble scales

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Outline of the talk

- Essentials of cosmological perturbation theory
- Behaviour of curvature perturbation in standard slow-roll inflation
- Driving inflation with tachyon field
- Amplification of tachyonic perturbations at super-Hubble scales
  - A specific example
  - Entropy perturbations as the source of amplification
- Criterion for amplification–long wavelength approximation
- Discussions and Conclusions

NOTE: The overdot denotes the derivative with respect to the cosmic time t while the prime denotes the derivative with respect to the conformal time η.

Essentials of cosmological perturbation theory

Taking into account the scalar perturbations, the FRW line element can be written as[a]

$$\mathrm{d}s^2 = a^2(\eta)\ \left[(1+2A)\,\mathrm{d}\eta^2 - 2\,(\partial_i B)\,\mathrm{d}x^i\,\mathrm{d}\eta - \left((1-2\psi)\,\delta_{ij} + 2\,\partial_i\partial_j E\right)\ \mathrm{d}x^i\,\mathrm{d}x^j\right].$$

where A, B, ψ and E are the scalar functions that describe the perturbations. The two gauge-invariant Bardeen variables that characterize the two degrees of freedom describing the scalar perturbations are given by

$$\Phi \equiv A + \left(\frac{1}{a}\right)\,\left[(B-E')\,a\right]' \qquad \text{and} \qquad \Psi \equiv \psi - \mathcal{H}\,(B-E')\,.$$

In the absence of anisotropic stress, $\Phi = \Psi$. The first order Einstein equations can be written as

$$\begin{aligned}
\nabla^2\Phi - 3\,\mathcal{H}\,(\Phi' + \mathcal{H}\,\Phi) &= \left(4\pi G\,a^2\right)\,\left[\delta\rho + \rho'\,(B-E')\right],\\
\partial_i\,(\Phi' + \mathcal{H}\,\Phi) &= \left(4\pi G\,a^2\right)\,\left[\delta q_i + (\rho+p)\,\partial_i\,(B-E')\right],\\
\Phi'' + 3\,\mathcal{H}\,\Phi' + \left(2\,\mathcal{H}' + \mathcal{H}^2\right)\,\Phi &= \left(4\pi G\,a^2\right)\,\left[\delta p + p'\,(B-E')\right].
\end{aligned}$$

[a] For a comprehensive review, see, V. Mukhanov, H. Feldman and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

Essentials of cosmological perturbation theory *contd..*

The Einstein equations can then be combined to lead to the following differential equation for the Bardeen potential Φ

$$\Phi'' + 3\,\mathcal{H}\,\left(1 + c_{_{\rm A}}^2\right)\Phi' - c_{_{\rm A}}^2\,\nabla^2\Phi + \left[2\,\mathcal{H}' + \left(1 + 3\,c_{_{\rm A}}^2\right)\mathcal{H}^2\right]\Phi = 4\pi G a^2\,\delta p_{\rm nad}.$$

where we have

$$\delta p = c_{_{\rm A}}^2\,\delta\rho + \delta p_{\rm nad},$$

with $c_{_{\rm A}}^2 \equiv (p'/\rho')$ denoting the adiabatic speed of sound and $\delta p_{\rm nad}$ representing the non-adiabatic pressure component which is related to the entropy perturbation S as[a]

$$\delta p_{\rm nad} = \left(\frac{p'}{\mathcal{H}}\right) S.$$

The curvature perturbation $\mathcal{R}$ is defined in terms of the Bardeen potential Φ as

$$\mathcal{R} = \Phi + \frac{2\,\rho}{3\,\mathcal{H}}\left(\frac{\Phi' + \mathcal{H}\,\Phi}{\rho + p}\right).$$

[a]See, for instance, C. Gordon, D. Wands, B. A. Bassett and R. Maartens, Phys. Rev. D **63**, 023506 (2001).

Essentials of cosmological perturbation theory *contd..*

The differential equation for the Bardeen potential Φ leads to the following equation

$$\mathcal{R}' = -\frac{\mathcal{H}}{(\mathcal{H}' - \mathcal{H}^2)}\left[4\pi G a^2 \delta p_{\rm nad} + c_{_{\rm A}}^2\, \nabla^2 \Phi\right].$$

Note that for adiabatic perturbations $\delta p_{\rm nad} = 0$, therefore $\mathcal{R}' = 0$ *i.e.* $\mathcal{R}$ is conserved for adiabatic perturbations at super-Hubble scales. The total entropy perturbation is then given by

$$\mathcal{S} = \mathcal{H}\left(\frac{\delta p}{p'} - \frac{\delta \rho}{\rho'}\right).$$

In the standard slow-roll inflationary scenario driven by a canonical single scalar field, the curvature perturbation $\mathcal{R}$ remains conserved at super-Hubble scales.

Driving inflation with tachyon field

The action of the tachyon field minimally coupled to gravity is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} - V(T)\sqrt{1 - g_{\mu\nu}\partial^\mu T\partial^\nu T}\right).$$

The background energy density and pressure for the tachyon field are given by

$$\rho = \frac{V(T)}{\sqrt{1-\dot{T}^2}}, \qquad p = -V(T)\sqrt{1-\dot{T}^2}$$

The condition for inflation is

$$(\rho + 3p) < 0 \;\Rightarrow\; \dot{T}^2 < 2/3.$$

In a spatially flat FRW background, the equation of motion for T is given by

$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V_T}{V} = 0.$$

where $V_T \equiv (dV/dT)$.

Cosmological perturbation theory for the tachyonic case

The Bardeen potential Φ induced by the tachyonic perturbations satisfies the following differential equation:

$$\Phi'' + 3\,\mathcal{H}\,\left(1 + c_{_{\rm A}}^2\right)\,\Phi' - c_{_{\rm A}}^2\,\nabla^2\Phi + \left[2\,\mathcal{H}' + \left(1 + 3\,c_{_{\rm A}}^2\right)\,\mathcal{H}^2\right]\,\Phi = \left(c_{_{\rm S}}^2 - c_{_{\rm A}}^2\right)\,\nabla^2\Phi,$$

where $c_{_{\rm S}}^2 = (1 - \dot{T}^2)$ is referred to as the effective speed of sound[a]. The curvature perturbation on a comoving hypersurface is defined as

$$\mathcal{R} = \frac{|v|}{z} = \left[\Phi + \left(\frac{a}{z}\right)\delta\mathcal{T}\right].$$

where $\delta\mathcal{T}$ is the gauge invariant tachyonic field perturbation and the quantity z is given by

$$z \equiv a\left[\frac{\rho + p}{c_{_{\rm S}}^2\,H^2}\right]^{1/2} = \frac{\sqrt{3}\,M_{_{\rm P}}\,a\,\dot{T}}{\sqrt{1 - \dot{T}^2}}$$

and $M_{_{\rm P}} = (8\pi G)^{-1/2}$ denotes the Planck mass.

[a] See, for instance, D. A. Steer and F. Vernizzi, Phys. Rev. D **70**, 043527 (2004).

Cosmological perturbation theory for the tachyonic case...

The corresponding entropy perturbation for the tachyonic case is given by

$$\mathcal{S} = \left(\frac{\mathcal{H}}{4\pi G\, a^2\, p'}\right)\, \left(c_{_{\rm S}}^2 - c_{_{\rm A}}^2\right)\, \nabla^2\Phi.$$

Therefore the equation describing the evolution of curvature perturbation simplifies to

$$\mathcal{R}' = -\left(\frac{4\pi G\, a^2\, p'}{\mathcal{H}' - \mathcal{H}^2}\right)\, \left(\frac{c_{_{\rm S}}^2}{c_{_{\rm S}}^2 - c_{_{\rm A}}^2}\right) \mathcal{S} = -\left(\frac{\mathcal{H}\, c_{_{\rm S}}^2}{\mathcal{H}' - \mathcal{H}^2}\right)\, \nabla^2\Phi.$$

The Fourier modes of the curvature perturbation $\mathcal{R}_k$ are then described by the following differential equation

$$\mathcal{R}_k'' + 2\left(\frac{z'}{z}\right)\mathcal{R}_k' + k^2\, c_{_{\rm S}}^2\, \mathcal{R}_k = 0.$$

The power spectrum of curvature perturbation $\mathcal{R}_k$ is given by

$$\mathcal{P}_{_{\mathcal{R}}}(k) = \frac{k^3}{2\pi^2}\left|\mathcal{R}_k\right|^2.$$

Amplification at super-Hubble scales–A specific example

In the case of the tachyonic scalar field, the slow-roll parameters are defined as

$$\epsilon_1 = \frac{3}{2}\dot{T}^2, \quad \epsilon_2 = \frac{2\,\ddot{T}}{H\dot{T}}, \quad \epsilon_{i+1} = \frac{\dot{\epsilon}_i}{H\epsilon_i}.$$

Note that $\epsilon_1 = 1$ indicates the end of inflation. The quantity (z'/z) that appears in the equation describing the evolution of the curvature perturbation can be expressed in terms of the slow-roll parameters as

$$\frac{z'}{z} = aH\left[1 + \frac{\epsilon_2}{2}\left(\frac{1}{1-2\epsilon_1/3}\right)\right].$$

The quantity (z'/z) becomes negative if

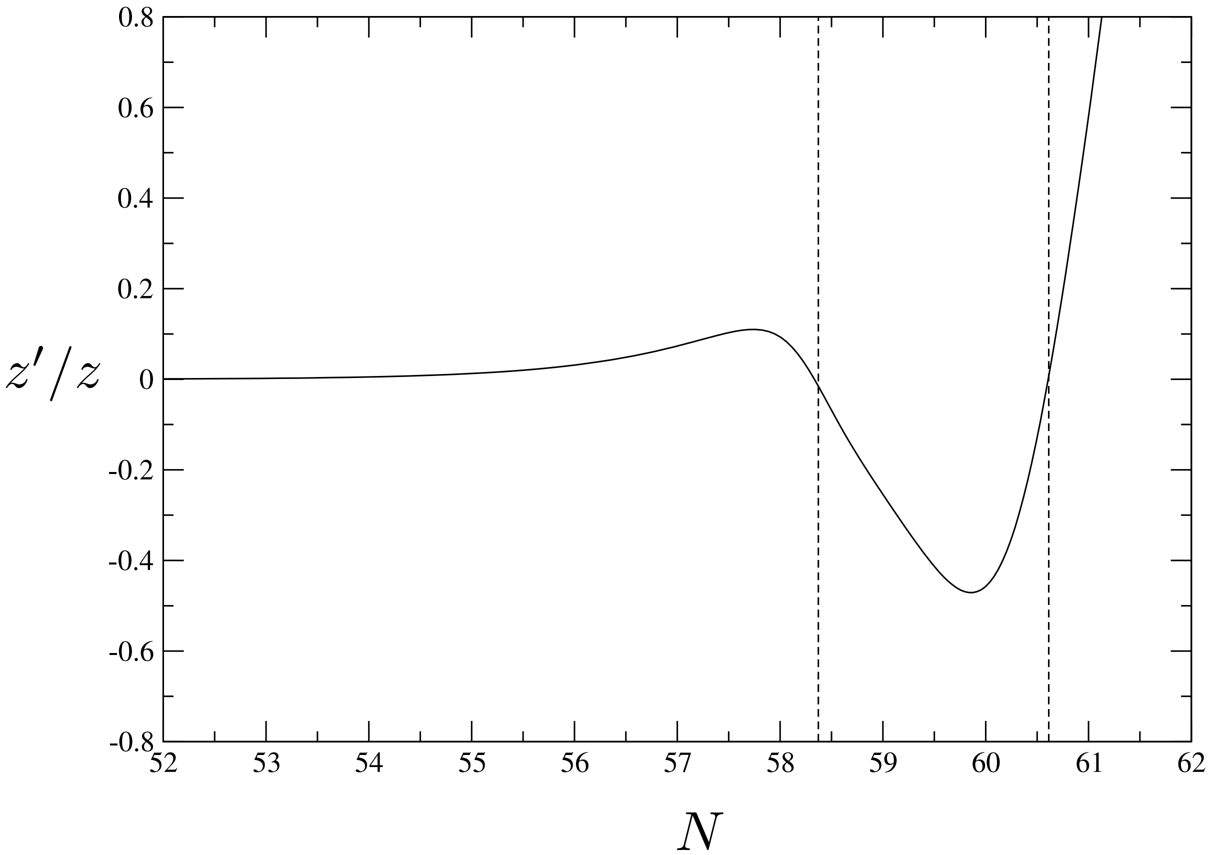
$$\epsilon_2 < -2\left(1 - \frac{2}{3}\epsilon_1\right).$$

We take the potential as

$$V(T) = V_0\left(1 + V_1\,T^4\right).$$

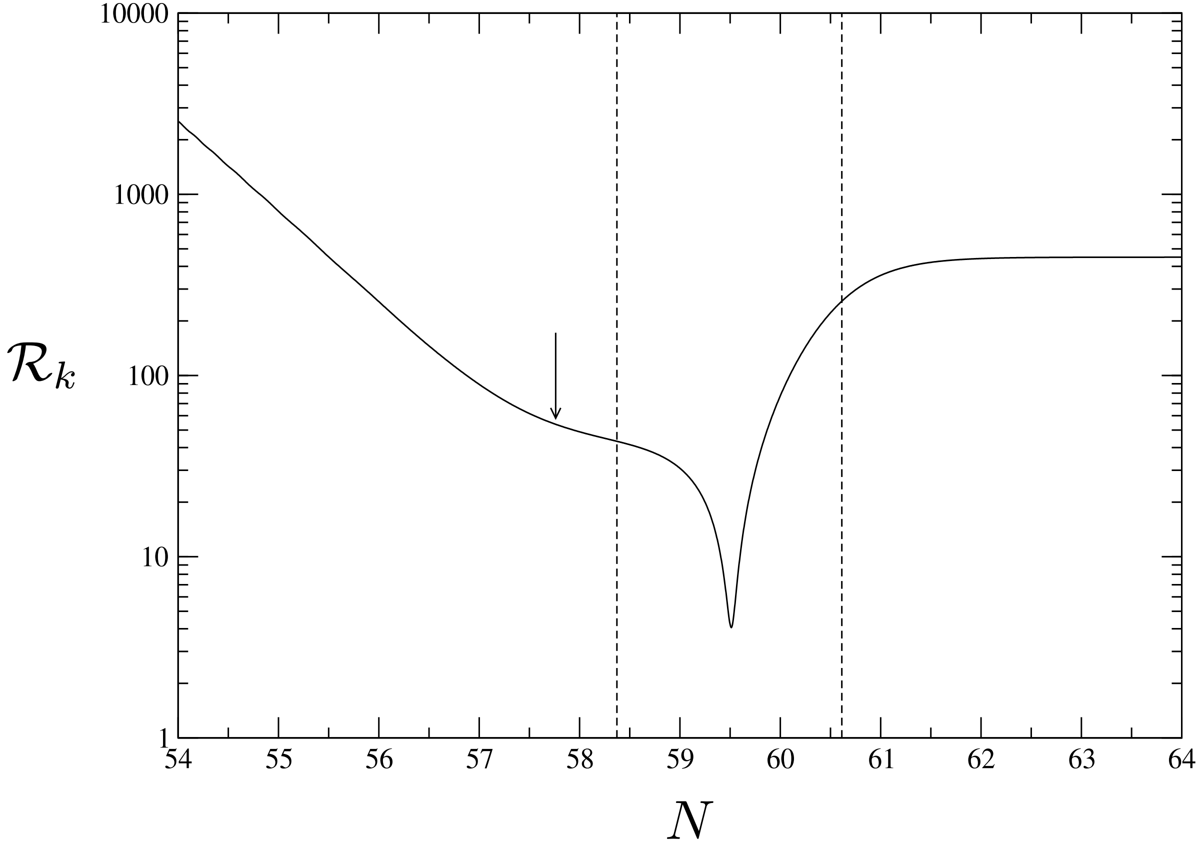
where V_0 and V_1 are constants.

Amplification at super-Hubble scales–A specific example



The evolution of the quantity (z'/z) plotted as a function of the number of e-folds N for $V_0 = 0.5$ and $V_1 = 0.18$. The vertical lines indicate the regime where (z'/z) is negative.

Evolution of the curvature perturbation



The evolution of the curvature perturbation $\mathcal{R}_k$ plotted as a function of the number of e-folds N. The wave number of the mode is $k = 0.1$ and the arrow indicates the time at which the mode leaves the Hubble radius, i.e. when $(k\, c_{\mathrm{S}}) = (a\, H)$. The curvature perturbation is amplified at super-Hubble scales from its value at Hubble exit.

Entropy perturbations as the source of amplification

The differential equation describing the evolution of $\mathcal{R}$ can be written in Fourier space as the following two first order equations [a]

$$\frac{\mathcal{R}_k'}{aH} = \mathcal{A}\,\mathcal{S}_k,$$
$$\frac{\mathcal{S}_k'}{aH} = \mathcal{B}\,\mathcal{S}_k - \mathcal{C}\left(\frac{k^2}{a^2 H^2}\right)\mathcal{R}_k.$$

where

$$\mathcal{A} = (3-2\epsilon_1)\left(\frac{3-6\epsilon_1+\epsilon_2}{6-12\epsilon_1+\epsilon_2}\right),$$
$$\mathcal{B} = \frac{1}{(3-2\epsilon_1)(6+\epsilon_2-12\epsilon_1)(3+\epsilon_2-6\epsilon_1)}\Big[\epsilon_2\big(36\epsilon_1-9\epsilon_3+24\epsilon_1\epsilon_2+2\epsilon_1\epsilon_2^2-144\epsilon_1^2$$
$$-12\epsilon_1^2\epsilon_3-144\epsilon_1^3-24\epsilon_1^2\epsilon_2\big)-(6+\epsilon_2-12\epsilon_1)(3+\epsilon_2-6\epsilon_1)(9-9\epsilon_1+2\epsilon_1^2+3\epsilon_2)\Big],$$
$$\mathcal{C} = \left(\frac{1}{3}\right)\left(\frac{6-12\,\epsilon_1+\epsilon_2}{3-6\,\epsilon_1+\epsilon_2}\right).$$

[a] See, for instance, S. M. Leach and A. R. Liddle, Phys. Rev. D **63**, 043508 (2001).

Evolution of the entropy perturbations

In the slow roll approximation and ignoring the $\mathcal{R}$ term, the behavior of the entropy perturbation $\mathcal{S}$ can be approximated as

$$\frac{\mathcal{S}'}{aH} \simeq -3\mathcal{S}.$$

which can be solved as

$$\mathcal{S} \sim \mathrm{const} \times e^{-3N}.$$

The late time asymptotic behaviour of $\mathcal{S}$ is given by

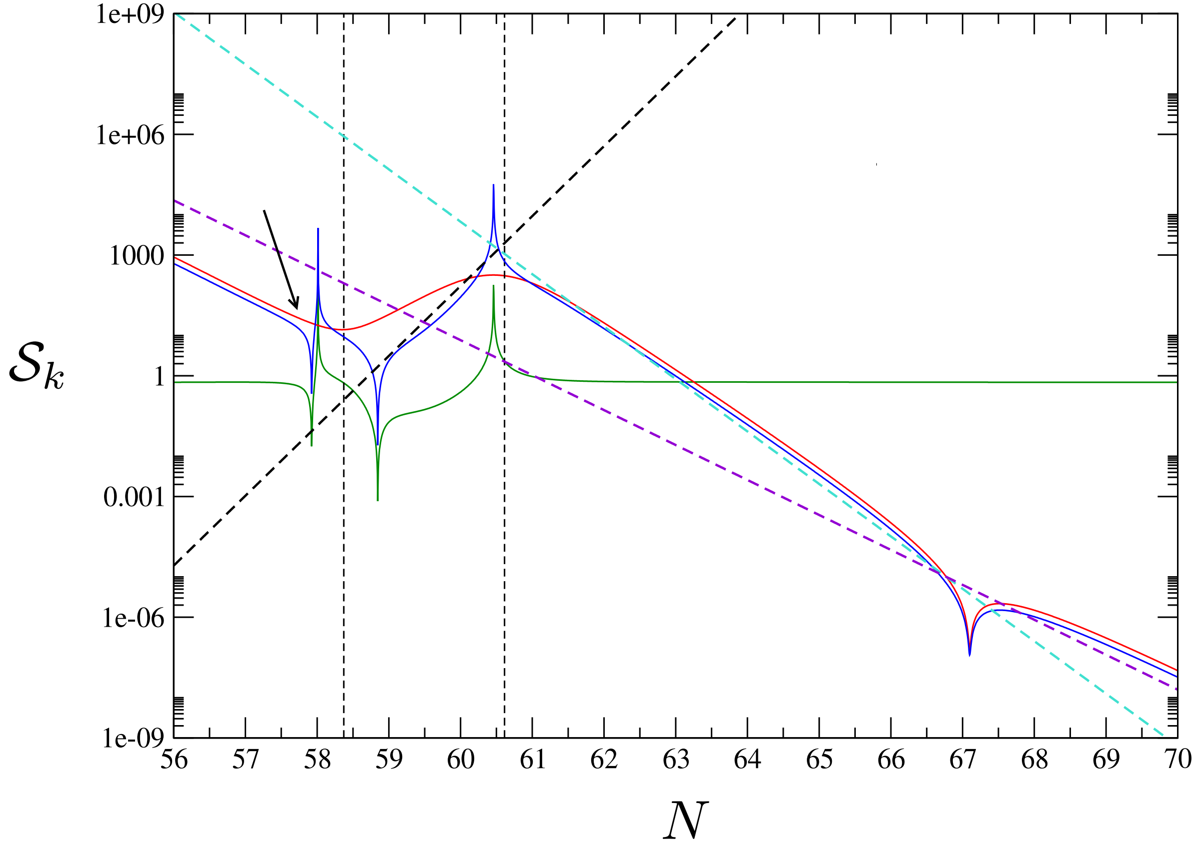
$$\mathcal{S} \sim \mathrm{const} \times e^{-2N}.$$

During the fast roll regime, ϵ_2 becomes large negative so the behavior of $\mathcal{S}$ becomes

$$\mathcal{S} \sim \mathrm{const} \times e^{4N}.$$

which is large enough to have a significant impact on $\mathcal{R}$.

Evolution of the entropy perturbations



The evolution of the entropy perturbation $\mathcal{S}_k$ plotted as a function of the number of e-folds N. The conclusions as we have discussed – e^{4N} growth of the entropy perturbation during fast roll, the intermediate e^{-3N} slow roll behavior and the late time e^{-2N} slow roll decay are evident from the figure.

Criterion for amplification of perturbations[a]

Let $u(\eta)$ and $v(\eta)$ be two solutions of the equation for $\mathcal{R}_k$ for any given k then the decaying mode $v(\eta)$ which vanishes as $\eta \to \eta_*$, can be expressed in terms of the growing mode $u(\eta)$ as

$$v(\eta) \propto u(\eta) \int_{\eta_*}^{\eta} \frac{d\eta'}{z^2(\eta')u^2(\eta')}$$

We assume that $v = u$ at some initial epoch shortly after horizon crossing, $\eta = \eta_k (< \eta_*)$ then $v(\eta)$ is expressed as

$$v(\eta) = u(\eta) \frac{D(\eta)}{D(\eta_k)},$$

where

$$D(\eta) = 3\mathcal{H}_k \int_{\eta_*}^{\eta} d\eta' \frac{z^2(\eta_k)u^2(\eta_k)}{z^2(\eta')u^2(\eta')}.$$

The general solution of $\mathcal{R}_k$ can then be expressed as

$$\mathcal{R}_k(\eta) = \alpha u(\eta) + \beta v(\eta).$$

[a] See, for instance, S. M. Leach, M. Sasaki, D. Wands and A. R. Liddle, Phys. Rev. D **64**, 023512 (2001).

Criterion for amplification of perturbations

Taking $\alpha + \beta = 1$ without loss of generality, $\mathcal{R}_k$ and $\mathcal{R}'_k$ at $\eta = \eta_k$ are given by

$$\mathcal{R}_k(\eta_k) = u(\eta_k)$$
$$\mathcal{R}'_k(\eta_k) = u'(\eta_k) - \frac{3(1-\alpha)\mathcal{H}_k u(\eta_k)}{D_k}$$

where $D_k = D(\eta_k)$ then α can be expressed as

$$\alpha = 1 + \frac{D_k}{3\mathcal{H}_k}\left[\frac{\mathcal{R}'_k}{\mathcal{R}_k} - \frac{u'}{u}\right]_{\eta=\eta_k} .$$

The long wavelength approximation[a]

In the long-wavelength approximation *i.e.* when $k^2 \ll |z''/z|$, the general solution for $\mathcal{R}_k$ is given approximately by

$$\mathcal{R}_k(\eta) \approx A + B \int_{\eta_*}^{\eta} \frac{d\eta'}{z^2(\eta')} .$$

So the lowest order solutions for u and v are

$$u_0 = \text{const.}, \quad v_0 = u_0 \frac{D(\eta)}{D_k} .$$

where $D(\eta)$ is given by

$$D(\eta) \approx 3\mathcal{H}_k \int_{\eta}^{\eta_*} d\eta' \, \frac{z^2(\eta_k)}{z^2(\eta')} .$$

If there is an epoch at which the slow-roll conditions are violated then the corrections to the growing mode u due to the effect of finite wavenumber k becomes important.

[a] See, for instance, S. M. Leach, M. Sasaki, D. Wands and A. R. Liddle, Phys. Rev. D **64**, 023512 (2001).

The long wavelength approximation

The growing mode solution can be written in the form

$$u(\eta) = \sum_{n=0}^{\infty} u_n(\eta) k^{2n}.$$

then the equation for $\mathcal{R}_k$ requires that

$$u''_{n+1} + 2\frac{z'}{z} u'_{n+1} = -c_{\rm S}^2 u_n.$$

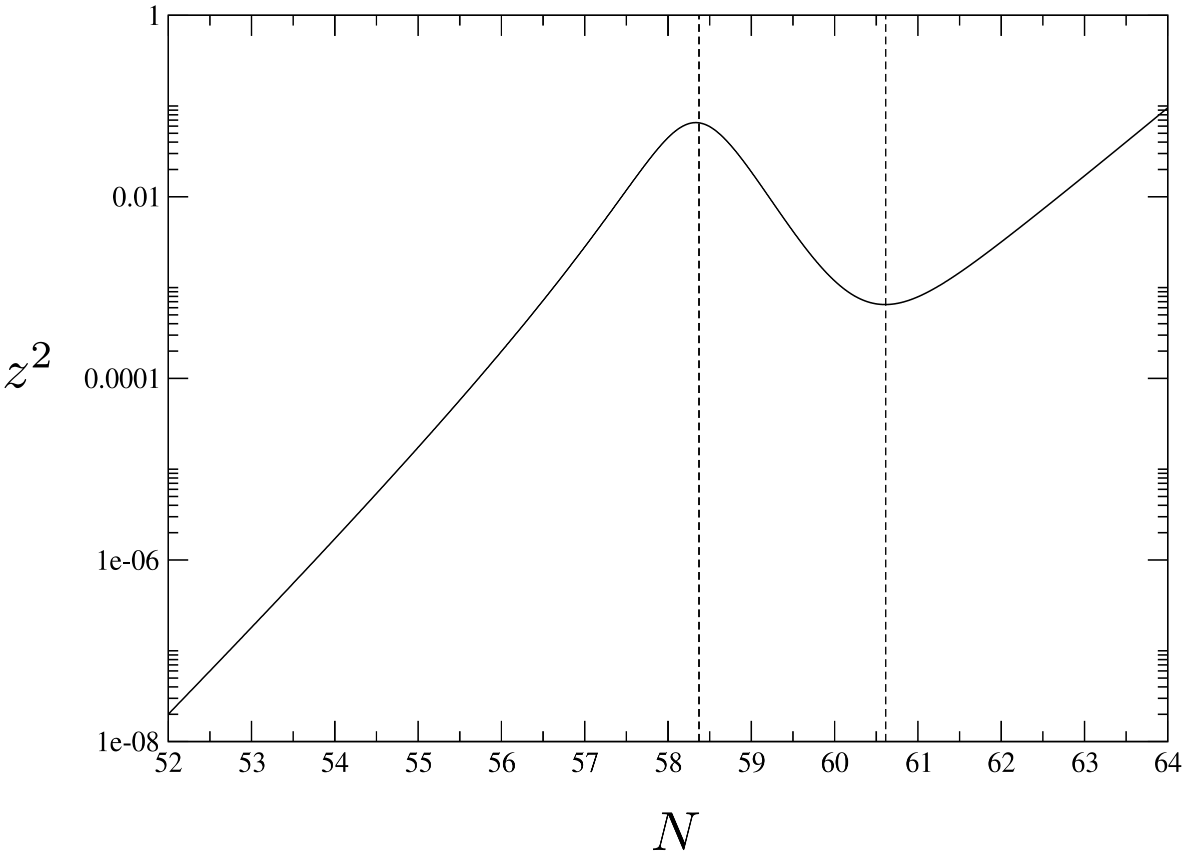
In particular, the $\mathcal{O}(k^2)$ correction to u_0 can be written as

$$u \approx u_0 + [C_1 + C_2 D(\eta) + F(\eta)]\, u_0\,,$$

where

$$F(\eta) = k^2 \int_{\eta}^{\eta_*} \frac{d\eta'}{z^2(\eta')} \int_{\eta_k}^{\eta'} c_{\rm S}^2 z^2(\eta'') d\eta''\,.$$

Criterion for amplification–long wavelength approx....



The evolution of the quantity z^2 plotted as a function of the number of e-folds N. Note the dip during the fast roll regime which in turn leads to the amplification of the curvature perturbation during this regime.

Criterion for amplification–long wavelength approx....

Using the above equations, α can be approximated as

$$\alpha \approx 1 + \frac{D_k}{3\mathcal{H}_k}\frac{\mathcal{R}'_k}{\mathcal{R}_k} - F_k \,.$$

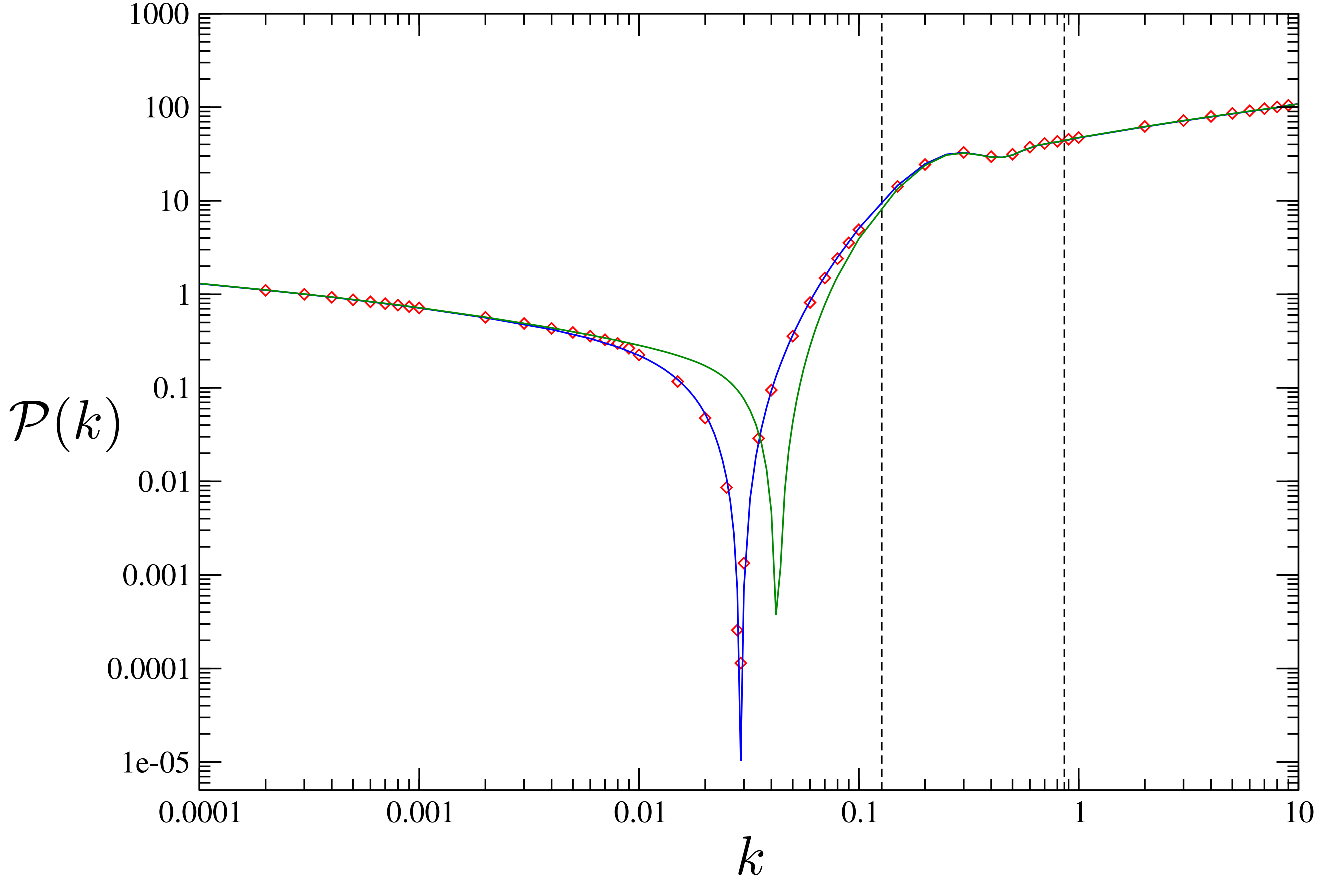
Then at late times, we have

$$\mathcal{R}_k(\eta_*) = \alpha u(\eta_*) \approx \alpha u(\eta_k) = \alpha \mathcal{R}_k(\eta_k)\,.$$

Thus the final amplitude will be enhanced by a factor $|\alpha|$, which can be large if $D_k \gg 1$ or $F_k \gg 1$.

Mode k	$\mathcal{R}_k(\eta_*)$	$\alpha\mathcal{R}_k(\eta_k)$
$k = 0.5$	75.4	66.2
$k = 0.1$	397.1	380.2
$k = 0.05$	348.313	348.318
$k = 0.01$	282.5	295.1

The power spectrum

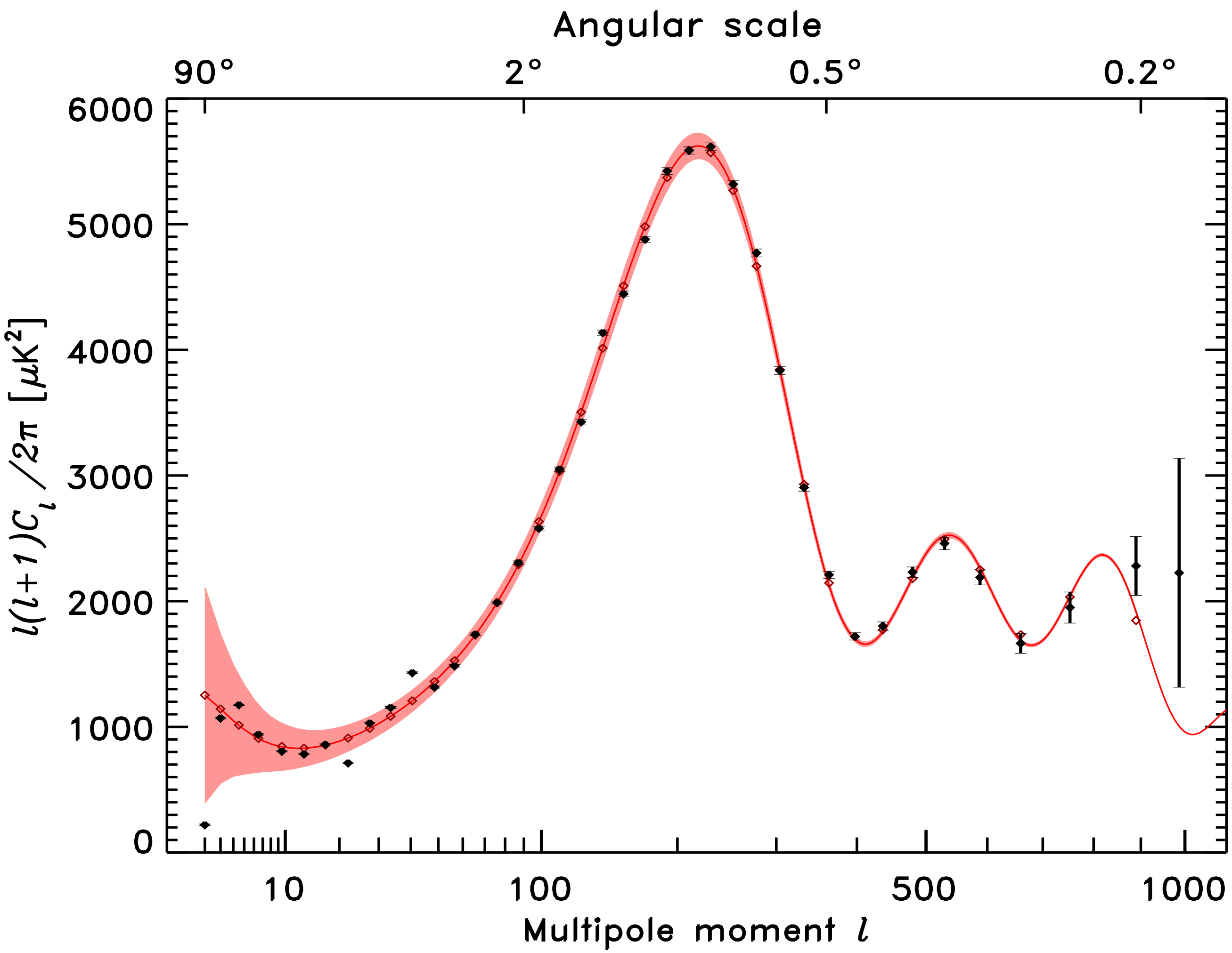


The scalar power spectrum evaluated at the end of inflation through the mode by mode integration.

Conclusions

- We have shown, with the help of a specific example, that the tachyonic curvature perturbations can be amplified at super-Hubble scales during a period of deviations from slow roll inflation.
- The rapid growth of entropy perturbations during such a transition is responsible for the amplifications of perturbations.
- Such transitions from slow roll to fast roll inflation lead to specific features in the power spectrum which may be required to fit the CMB observations better.

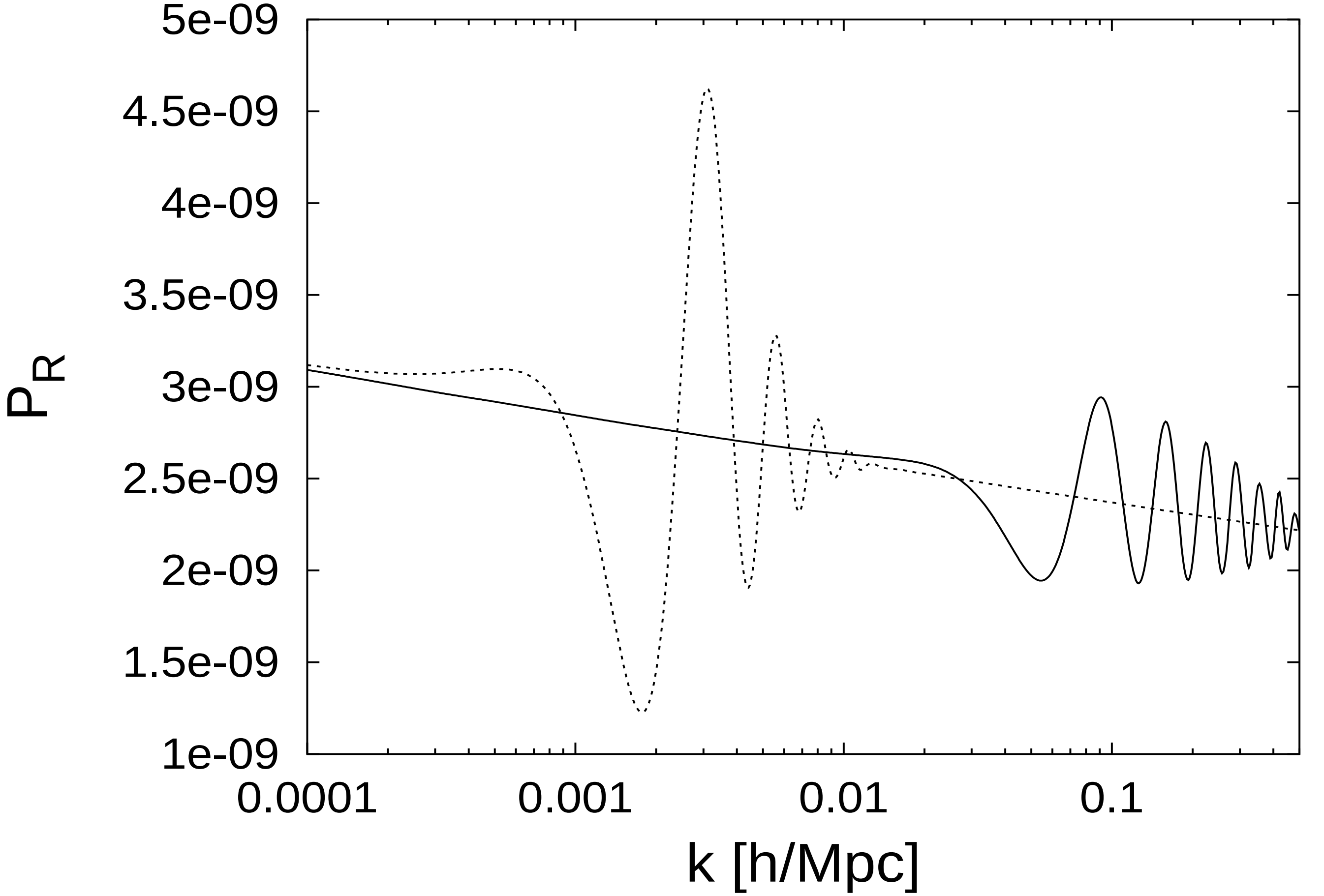
Angular power spectrum from recent WMAP data[a]



The three-year angular power spectrum (in black) for $l = 2$–1000. The red curve is the best-fit ΛCDM model, fit to WMAP data only.

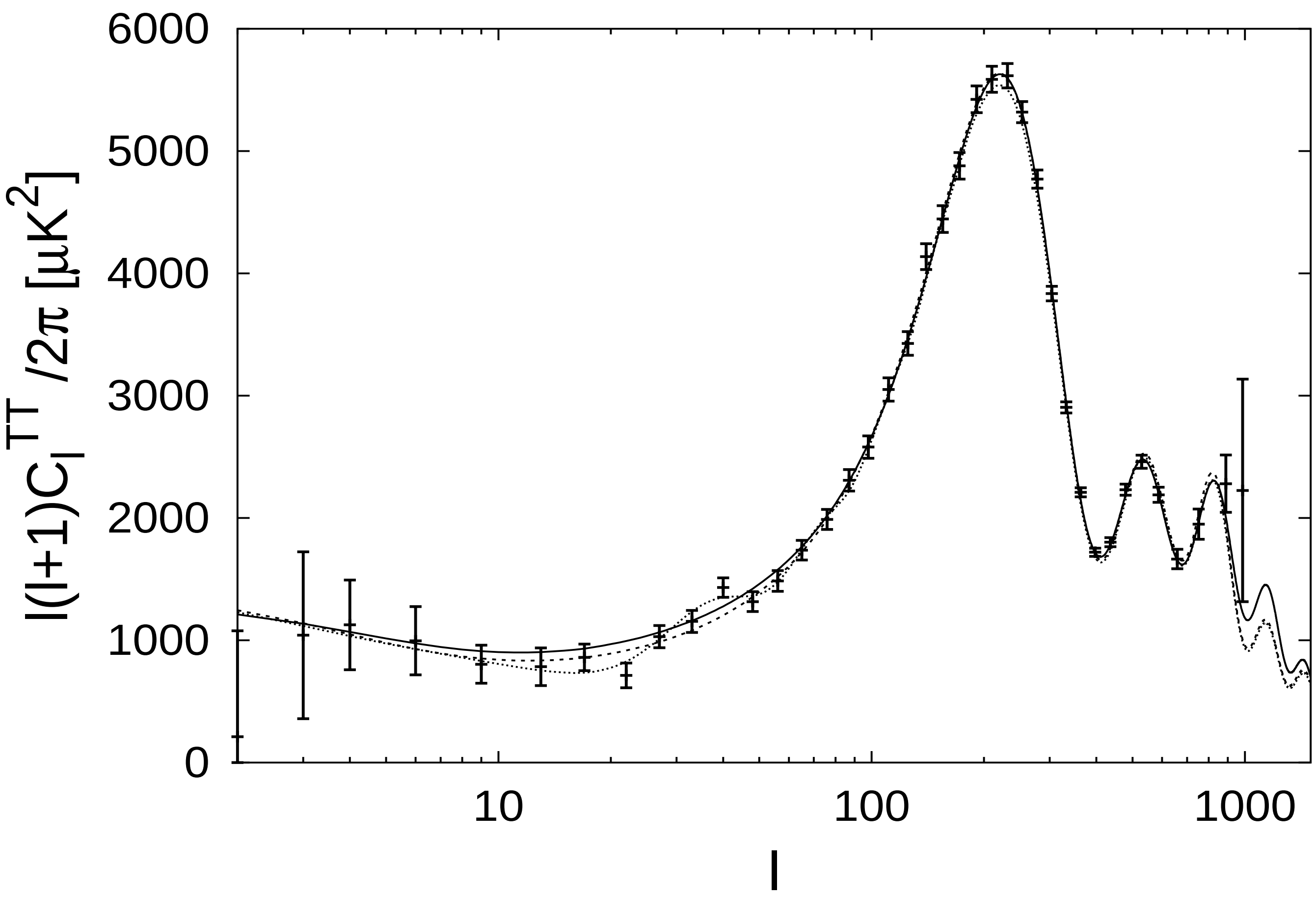
[a] G. Hinshaw et. al., astro-ph/0603451.

Power spectrum for the step model[a]



[a] L. Covi, J. Hamann, A. Melchiorri, A. Slosar and I. Sorbera, Phys. Rev. D **74** (2006) 083509, J. Hamann, L. Covi, A. Melchiorri and A. Slosar, arXiv:astro-ph/0701380.

Angular power spectrum for best fit step model[a]



[a] L. Covi, J. Hamann, A. Melchiorri, A. Slosar and I. Sorbera, Phys. Rev. D **74** (2006) 083509, J. Hamann, L. Covi, A. Melchiorri and A. Slosar, arXiv:astro-ph/0701380.

The talk is based upon...

- **Rajeev Kumar Jain**, Pravabati Chingangbam and L. Sriramkumar, *Amplification of tachyonic perturbations at super-Hubble scales*, arXiv:astro-ph/0703762.

Thank you