A Covariant Approach to Spatial Averaging in Cosmology

Aseem Paranjape

Tata Institute of Fundamental Research, Mumbai

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  - Assume the Universe is almost homogeneous and isotropic.
  - Hence its metric must be FLRW, with small perturbations.
  - Solve Einstein's equations for the FLRW background sourced by a homogeneous and isotropic perfect fluid

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \ ; \ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ . \tag{1}$$

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- Explicit averaging, however, can lead to nontrivial effects in GR. Since Einstein's equations are non-linear, in general $G[\langle g \rangle] \neq \langle G[g] \rangle$. (G. F. R. Ellis, *Gen. Rel. and Grav.*, 1984.)

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- Only two averaging schemes at present can deal with inhomogeneities *nonperturbatively* – Buchert's Spatial Averaging of scalars, and Zalaletdinov's covariant Macroscopic Gravity.

Plan

1. Macroscopic Gravity (MG)
   - Brief Overview
   - Spatial Averaging and Gauge Choices in MG
   - Deriving Cosmological Equations

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The Averaging Operation

Arbitrary tensors can be averaged in MG. For example, given a tensor $P^a_b(x)$ on a manifold $\mathcal{M}$,

$$\left\langle \widetilde{P}^a_b \right\rangle_{ST} = \frac{1}{V_\Sigma} \int_\Sigma d^4x' \sqrt{-g'}\, \widetilde{P}^a_b(x', x) \;\; ; \;\; V_\Sigma = \int_\Sigma d^4x' \sqrt{-g'} \, , \qquad (2)$$

Here $\widetilde{P}^a_b(x', x)$ is the bilocal extension of the tensor $P^a_b(x)$.

V_Σ is kept constant while averaging on the manifold $\mathcal{M}$.

▸ Details

The Averaged Manifold

- The average of the connection Γ^a_{bc} on $\mathcal{M}$ is by definition taken to be the connection $\bar{\Gamma}^a_{bc}$ of the averaged manifold $\bar{\mathcal{M}}$.
- The averaged metric $G_{ab} = \langle \widetilde{g}_{ab} \rangle_{ST}$ can be chosen as the metric on $\bar{\mathcal{M}}$.
- In general $G^{ab} \neq \langle \widetilde{g}^{ab} \rangle_{ST}$, although for a highly symmetric $\bar{\mathcal{M}}$ these two are equal.

The Averaged Equations

After tedious algebra, and some simplifying assumptions, the averaged Einstein equations reduce to

$$E^a_b = -8\pi G_N T^a_b + C^a_b \,. \tag{3}$$

E^a_b : Einstein tensor on $\bar{\mathcal{M}}$,
T^a_b : averaged energy-momentum tensor,
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$$C^a_b = \left(Z^a{}_{ijb} - \frac{1}{2}\delta^a_b Z^m{}_{ijm} \right) G^{ij} \,, \tag{4}$$

where $Z^a{}_{ijb}$ is the tensorial correlation term defined by

$$Z^a{}_{b[m}{}^i{}_{\underline{j}n]} = \left\langle \widetilde{\Gamma}^a_{b[m} \widetilde{\Gamma}^i_{\underline{j}n]} \right\rangle - \left\langle \widetilde{\Gamma}^a_{b[m} \right\rangle \left\langle \widetilde{\Gamma}^i_{\underline{j}n]} \right\rangle \quad ; \quad Z^a{}_{ijb} = 2Z^a{}_{ik}{}^k{}_{jb} \,. \tag{5}$$

Plan

The Averaged Geometry
1. Homogeneity and Isotropy

Assume an averaging on length scales $L_{\rm FLRW} \gtrsim 100h^{-1}$Mpc. This is assumed to satisfy $L_{\rm FLRW} \ll L_{\rm Hubble}$.

Also, the physically relevant *time scale* of averaging T satisfies $T \ll L_{\rm FLRW}$, since measurements are made over short time intervals – this characterises the spatial averaging limit $\langle(...)\rangle_{ST} \to \langle(...)\rangle$.
[This limit is important since we want $\langle g_{\rm FLRW}\rangle = g_{\rm FLRW}$.]

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Assume $\bar{\mathcal{M}}$ to admit homogeneous and isotropic spatial sections.

Namely, $\bar{\mathcal{M}}$ admits a unit timelike vector field $\bar{v}^a$, which is orthogonal to spacelike hypersurfaces of constant curvature, and is tangent to trajectories of “observers” (effectively pointlike averaging domains) who see an isotropic CMB radiation.

The Averaged Geometry

2. Comoving Coordinates and Energy-Momentum Tensor

Choose coordinates $(t, x^A), A = 1, 2, 3$, so that spatial metric becomes

$$^{(\bar{\mathcal{M}})}ds^2_{\text{spatial}} = a^2(t)\delta_{AB}dx^A dx^B \,. \tag{6}$$

The spatial coordinates are comoving with the preferred observers, $\bar{v}^a = (\bar{v}^t, 0, 0, 0)$. Proper time τ is defined by $\partial_\tau = \bar{v}^a\partial_a = \bar{v}^t\partial_t$.

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Assume the averaged energy-momentum tensor to take the perfect fluid form

$$T^a_b = \rho\bar{v}^a\bar{v}_b + p\pi^a_b\,, \qquad (7)$$

where $\pi^a_b = \delta^a_b + \bar{v}^a\bar{v}_b$ and

$$\rho \equiv T^a_b\bar{v}^b\bar{v}_a \;\; ; \;\; p \equiv \frac{1}{3}\pi^b_a T^a_b\,. \qquad (8)$$

The Averaged Geometry

3. Constraints on C^a_b

Since E^a_b has an identical structure $E^a_b = j_1(x)\bar{v}^a\bar{v}_b + j_2(x)\pi^a_b$, the correlation tensor C^a_b is constrained to also have the form

$$C^a_b = c_1(x)\bar{v}^a\bar{v}_b + c_2(x)\pi^a_b\,, \tag{9}$$

with the remaining components forced to vanish. In comoving coordinates, this translates to

$$C^0_A = 0 \;\; ; \;\; C^A_B - (1/3)\delta^A_B(C^J_J) = 0 \;\; ; \;\; A, B, ... = 1, 2, 3 \tag{10}$$

These are conditions on the *inhomogeneous* geometry.

C^a_b also satisfies the "conservation criterion" $C^a_{b;a} = 0$.

Gauge choices

Comoving coordinates on $\bar{\mathcal{M}}$ $\leftrightarrow$ unknown gauge in $\mathcal{M}$.

For a general lapse function $N(t, \mathbf{x})$ and shift vector $N^A(t, \mathbf{x})$, the unaveraged metric is

$$^{(\mathcal{M})}ds^2 = -\left(N^2 - N_A N^A\right) dt^2 + 2N_B dx^B dt + h_{AB} dx^A dx^B \,. \tag{11}$$

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Leaving the gauge choice formally unspecified, our assumptions require (shifting to the spatial averaging limit)

$$G_{00} = \langle \widetilde{g}_{00} \rangle = -f^2(t) \;\; ; \;\; G_{0A} = \langle \widetilde{g}_{0A} \rangle = 0$$
$$G_{AB} = \langle \widetilde{g}_{AB} \rangle = a^2(t)\delta_{AB} \,. \tag{12}$$

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Computationally, a gauge specification is crucial to the assumption of large scale homogeneity.

Gauge choices – VPC system

The algebra simplifies considerably if
Comoving coordinates on $\bar{\mathcal{M}}$ $\leftrightarrow$ volume preserving gauge on $\mathcal{M}$:

$$^{(\mathcal{M})}ds^2 = -\frac{d\bar{t}^2}{h(\bar{t},\mathbf{x})} + h_{AB}(\bar{t},\mathbf{x})dx^A dx^B \,. \tag{13}$$

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In this gauge, the averaging operation (and the spatial averaging limit) is most transparent,

$$\left\langle \widetilde{p}^{\,i}_{\,j} \right\rangle_{ST}(\bar{t},\mathbf{x}) = \left\langle p^{\,i}_{\,j} \right\rangle_{ST}(\bar{t},\mathbf{x}) = \frac{1}{TL^3}\int_{\Sigma} d^4x' \left[p^i_j(t',\mathbf{x}') \right] . \tag{14}$$

We will display results for this gauge choice.

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The Averaged Metric

For the volume preserving gauge, the averaged metric also reduces to the volume preserving form

$$^{(\bar{\mathcal{M}})}ds^2 = -\frac{d\bar{t}^2}{\bar{a}^6(\bar{t})} + \bar{a}^2(\bar{t})\delta_{AB}dx^A dx^B \,, \tag{15}$$

with $G^{ab} = \langle g^{ab} \rangle$. The vector field $\bar{v}^a$ defining the preferred FLRW observers is

$$\bar{v}^a = \left(\bar{a}^3, 0, 0, 0\right) \quad ; \quad \bar{v}_a = G_{ab}\bar{v}^b = \left(-\frac{1}{\bar{a}^3}, 0, 0, 0\right) \,. \tag{16}$$

One can transform to cosmic time τ using

$$\tau = \int^{\bar{t}} \frac{dt}{\bar{a}^3(t)} \,. \tag{17}$$

The Correlation Tensor
1. Scalar terms in the Cosmological equations

Recall that the Friedmann and Raychaudhuri equations are scalar equations.

Details

Relevant scalar terms in C^a_b are $C^a_b\bar{v}^b\bar{v}_a$ and $C^a_b(\bar{v}^b\bar{v}_a + \pi^b_a)$, given by

$$C^a_b\bar{v}^b\bar{v}_a = \frac{1}{2}\left[\mathcal{Q} + \mathcal{S}\right] ,$$

$$C^a_b(\bar{v}^b\bar{v}_a + \pi^b_a) = 2\mathcal{Q} , \tag{18}$$

Details

The Correlation Tensor
2. The Modified Equations

The correlation terms were evaluated in the VPC system. But being spacetime scalars, they can be expressed in, e.g., synchronous coordinates.

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The modified equations, in standard form, are given by

$$\left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{1}{6}\left[\mathcal{Q}+\mathcal{S}\right] , \tag{19a}$$

$$\frac{1}{a}\frac{d^2a}{d\tau^2} = -\frac{4\pi G_N}{3}(\rho+3p) + \frac{1}{3}\mathcal{Q} , \tag{19b}$$

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$$\frac{1}{a}\frac{d^2a}{d\tau^2} = -\frac{4\pi G_N}{3}(\rho+3p) + \frac{1}{3}\mathcal{Q} , \tag{19b}$$

$\mathcal{Q}$ and $\mathcal{S}$ also satisfy the differential identity

$$\left(\partial_\tau\mathcal{Q} + 6\mathcal{Q}\partial_\tau(\ln a)\right) + \left(\partial_\tau\mathcal{S} + 2\mathcal{S}\partial_\tau(\ln a)\right) = 0 , \tag{20}$$

which follows from $C^a_{b;a} = 0$.

The Correlation Tensor
3. Additional Constraints

Recall the constraints imposed by the structure of E^a_b and T^a_b. These take the form

$$\delta^{JK}\left[\left\langle\sqrt{h}\Theta_{JB}\,{}^{(3)}\Gamma^B_{AK}\right\rangle - \left\langle\sqrt{h}\Theta_{JK}\,{}^{(3)}\Gamma^B_{AB}\right\rangle\right] = 0\,, \tag{21a}$$

$$\delta^{JK}\left\langle\frac{1}{\sqrt{h}}\Theta^B_K\,{}^{(3)}\Gamma^A_{JB}\right\rangle - \delta^{AJ}\left\langle\frac{1}{\sqrt{h}}\Theta\,{}^{(3)}\Gamma^B_{JB}\right\rangle = 0\,, \tag{21b}$$

$$\delta^{JK}\left\langle{}^{(3)}\Gamma^A_{JC}\,{}^{(3)}\Gamma^C_{KB}\right\rangle - \delta^{AJ}\left\langle{}^{(3)}\Gamma^C_{JC}\,{}^{(3)}\Gamma^K_{BK}\right\rangle = \frac{1}{3}\delta^A_B\left(\bar{a}^2\mathcal{S}\right)\,. \tag{21c}$$

These are nontrivial in general and are absent in, e.g., Buchert's approach which only deals with scalars.

Summary

- Zalaletdinov's MG and the Spatial Averaging Limit, lead to spacetime scalar corrections to the standard Cosmological equations.
- Nontrivial conditions are imposed on the inhomogeneous geometry due to the assumed structure of the homogeneous Einstein tensor and energy-momentum tensor. These conditions do not arise in approaches which deal only with scalar quantities.
- Reference : A. Paranjape and T. P. Singh, arXiv:gr-qc/0703106.

- Still to do!
  - Test these equations in realistic settings, e.g. – in the perturbative FLRW framework and in N-body simulations.

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Thank you.

The averaging is sophisticated, involving a Lie dragging of the averaging domain along chosen vector fields and bilocal extensions of tensorial objects which satisfy a bilocal exterior calculus.

The bilocal extension $\widetilde{P}^a_b(x', x)$ of the tensor $P^a_b(x)$ is defined as

$$\widetilde{P}^a_b(x', x) = \mathcal{W}^{b'}_b(x', x)\mathcal{W}^a_{a'}(x, x')P^{a'}_{b'}(x') \,. \tag{22}$$

Here $\mathcal{W}^{b'}_b(x', x)$ is a bilocal operator which also defines the Lie dragging of the averaging domain $\mathbf{\Sigma}$. $\mathcal{W}^a_{a'}(x, x')$ is its inverse.

Back

$\mathcal{W}^{a'}_{j}(x', x)$ takes the form

$$\mathcal{W}^{a'}_{j}(x', x) = f^{a'}_{m}(x') {f^{-1}}_{j}^{m}(x) , \quad (23)$$

with the tetrad $f^{a}_{m}(x)$ given by

$$f^{a}_{m}(x(\phi^{n})) = \frac{\partial x^{a}}{\partial \phi^{m}} \ ; \ {f^{-1}}_{j}^{m}(\phi(x^{k})) = \frac{\partial \phi^{m}}{\partial x^{j}} , \quad (24)$$

where the functions $\{\phi^{m}(x)\}$ form a volume preserving coordinate (VPC) system in which $\det g(\phi^{m}) =$ constant.

Back

To Averaged Manifold

For FLRW, $E^a_b = j_1(x)\bar{v}^a\bar{v}_b + j_2(x)\pi^a_b$.

In the VPC system, with $H \equiv d(\ln \bar{a})/d\bar{t}$

$$j_1(x) = -3\bar{a}^6 H^2 \ ; \ j_2(x) = \bar{a}^6 \left[2\left(\frac{\ddot{\bar{a}}}{\bar{a}} + 3H^2 \right) + H^2 \right] . \tag{25}$$

Terms in E^a_b relevant for the (effective) Friedmann and Raychaudhuri equations are $j_1(x)$ and $3j_2(x) + j_1(x)$ respectively.

Back to Correlation tensor

Scalar terms in the Cosmological equations

$$\mathcal{Q} = \bar{a}^6 \left[\frac{2}{3} \left(\left\langle \frac{1}{h} \Theta^2 \right\rangle - \frac{1}{\bar{a}^6} ({}^{\mathrm{F}}\Theta^2) \right) - 2 \left\langle \frac{1}{h} \sigma^2 \right\rangle \right] ,$$

$$(1/\bar{a}^6)({}^{\mathrm{F}}\Theta^2) = (3H)^2 ,$$

$$\mathcal{S} = \frac{1}{\bar{a}^2} \delta^{AB} \left[\left\langle {}^{(3)}\Gamma^{J}_{AC} \, {}^{(3)}\Gamma^{C}_{BJ} \right\rangle - \left\langle \partial_A (\ln \sqrt{h}) \partial_B (\ln \sqrt{h}) \right\rangle \right] ,$$

◂ Back to Correlation tensor