Canonical wave packets in quantum cosmology

Classical ⟷ Quantum

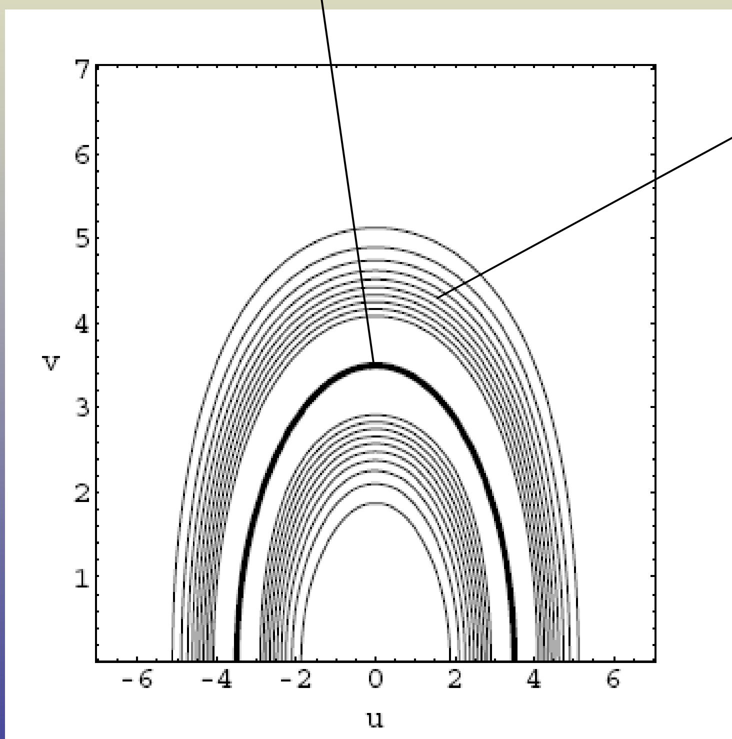
$$H = 0$$

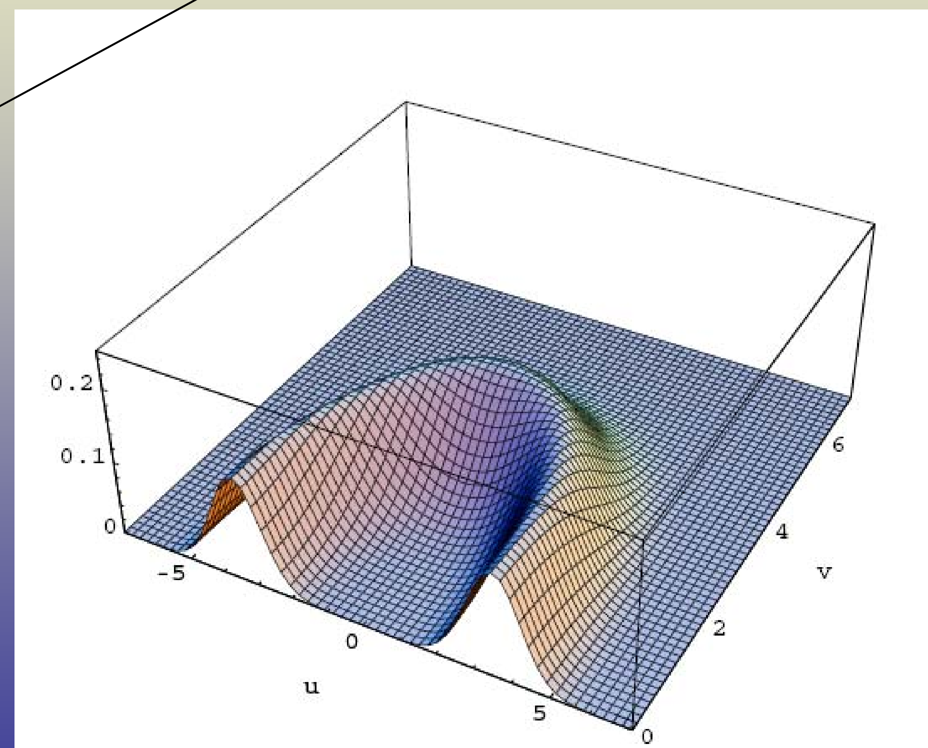
$$H \,|\, \psi >= 0$$

IP ⟶ IM

IPD ⟶(?) IMD

Classical ⟷ Quantum



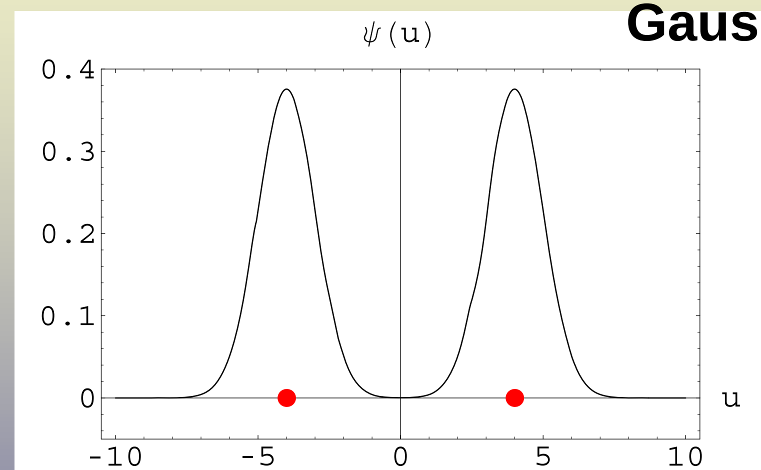


Initial Conditions

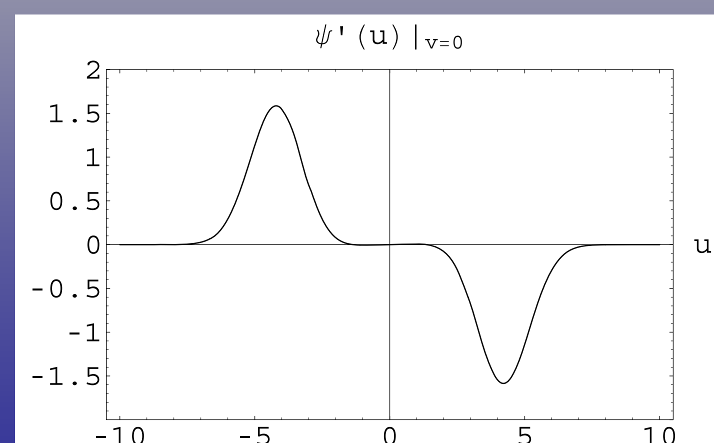
Classical ⟷ Quantum

Gaussian

Initial position



Initial velocity



FRW Universe

with scalar field

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}(\phi),$$

$$\Delta^2 \phi - \frac{\partial U}{\partial \phi} = 0.$$

$$g = -dt^2 + R^2(t) \frac{\sum dr^i dr^i}{(1 + kr^2/4)^2}.$$

Field equations

$$3\left[\left(\frac{\dot{R}}{R}\right)^2+\frac{k}{R^2}\right]=\frac{\dot{\phi}^2}{2}+U(\phi),$$

$$2\left(\frac{\ddot{R}}{R}\right)+\left(\frac{\dot{R}}{R}\right)^2+\frac{k}{R^2}=-\frac{\dot{\phi}^2}{2}+U(\phi),$$

$$\ddot{\phi}+3\frac{\dot{R}}{R}\dot{\phi}+\frac{\partial U}{\partial \phi}=0.$$

$$L = -3R\dot{R}^2 + 3kR + R^3\left[\frac{\dot{\phi}^2}{2} - U(\phi)\right]$$

$$U(\phi) = \Lambda + \frac{m^2}{2\alpha^2}\sinh^2(\alpha\phi) + \frac{b}{2\alpha^2}\sinh(2\alpha\phi)$$

$$L(u,v)=\frac{4}{3}\left[(\dot{u}^2-\omega_1^2u^2)-(\dot{v}^2-\omega_2^2v^2)-\frac{9}{4}k(u^2-v^2)^{1/3}\right]$$

$$\ddot{u}+\omega_1^2u+\frac{3k}{2}\frac{u}{(u^2-v^2)^{2/3}}=0,$$

$$\ddot{v}+\omega_2^2v+\frac{3k}{2}\frac{v}{(u^2-v^2)^{2/3}}=0,$$

$$\dot{u}^2+\omega_1^2u^2-\dot{v}^2-\omega_2^2v^2+\frac{9}{4}k(u^2-v^2)^{1/3}=0.$$

Hyperbolic PDEs

$$H\psi(u,v) = \left\{ -\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \omega_1^2 u^2 - \omega_2^2 v^2 + \frac{9}{4} k (u^2 - v^2)^{1/3} \right\} \psi(u,v) = 0$$

$$\left\{ -\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \widehat{f}(u,v) \right\} \psi(u,v) = 0$$

Spectral Method

$$\psi(u,v) = \sum_{i,j=1}^{2} \sum_{m,n} A_{m,n,i,j}\, g_i\left(\frac{m\pi u}{L}\right) g_j\left(\frac{n\pi v}{L}\right)$$

$$\begin{cases} g_1\left(\frac{m\pi u}{L}\right) = \sqrt{\frac{2}{R_m L}} \sin\left(\frac{m\pi u}{L}\right) \\ g_2\left(\frac{m\pi u}{L}\right) = \sqrt{\frac{2}{R_m L}} \cos\left(\frac{m\pi u}{L}\right) \end{cases} \quad \text{and} \quad R_m = \begin{cases} 1, & m \neq 0 \\ 2, & m = 0 \end{cases}$$

Spectral Method

$$\left[\left(\frac{m\pi}{L}\right)^2-\left(\frac{n\pi}{L}\right)^2\right]A_{m,n,i,j}+B_{m,n,i,j}=0,$$

$$B_{m,n,i,j}=\sum_{m',n',i',j'}\left[\int\int_{-L}^{L} g_i\left(\frac{m\pi u}{L}\right)g_j\left(\frac{n\pi v}{L}\right)\hat{f}'(u,v)g_{i'}\left(\frac{m'\pi u}{L}\right)g_{j'}\left(\frac{n'\pi v}{L}\right)dudv\right]A_{m',n',i',j'}$$

$$=\sum_{m',n',i',j'}C'_{m,n,i,j,m',n',i',j'}A_{m',n',i',j'}$$

$$\left[\left(\frac{m\pi}{L}\right)^2-\left(\frac{n\pi}{L}\right)^2\right]A_{m,n,i,j}+\sum_{m',n',i',j'}C'_{m,n,i,j,m',n',i',j'}A_{m',n',i',j'}=0,$$

Spectral Method

$$DA' = 0$$

$$\psi(u,v) = \sum_k \lambda^k \psi^k(u,v) = \sum_k \lambda^k \sum_{m,n,i,j} A^k{}_{m,n,i,j}\, g_i\left(\frac{m\pi u}{L}\right) g_j\left(\frac{n\pi v}{L}\right)$$

Prescription

Near v=0

$$\left\{-\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}+\omega_1^2u^2+\frac{9}{4}k(u^2)^{1/3}\right\}\psi(u,v)=0$$

$$\psi(u,v)=\varphi(u)\chi(v)$$

$$\frac{d^2\chi_n(v)}{d^2v^2}+E_n\chi_n(v)=0,$$

$$-\frac{d^2\varphi_n(u)}{du^2}+\left(\omega_1^2u^2+\frac{9}{4}ku^{2/3}\right)\varphi_n(u)=E_n\varphi_n(u).$$

Prescription

$$\chi_n(v) = \alpha_n \cos\left(\sqrt{E_n}\ v\right) + i\beta_n \sin\left(\sqrt{E_n}\ v\right),$$

$$\psi(u,v) = \sum_{n=\mathrm{even}} (A_n \cos(\sqrt{E_n} v) + iB_n \sin(\sqrt{E_n} v))\varphi_n(u)$$

$$+ \sum_{n=\mathrm{odd}} (C_n \cos(\sqrt{E_n} v) + iD_n \sin(\sqrt{E_n} v))\varphi_n(u).$$

$$\psi(u,0) = \sum_{even} A_n \varphi_n(u) + \sum_{odd} C_n \varphi_n(u)$$

$$\psi'(u,0) = i \sum_{even} B_n \sqrt{E_n}\ \varphi_n(u) + i \sum_{odd} D_n \sqrt{E_n} \varphi_n(u),$$

Prescription

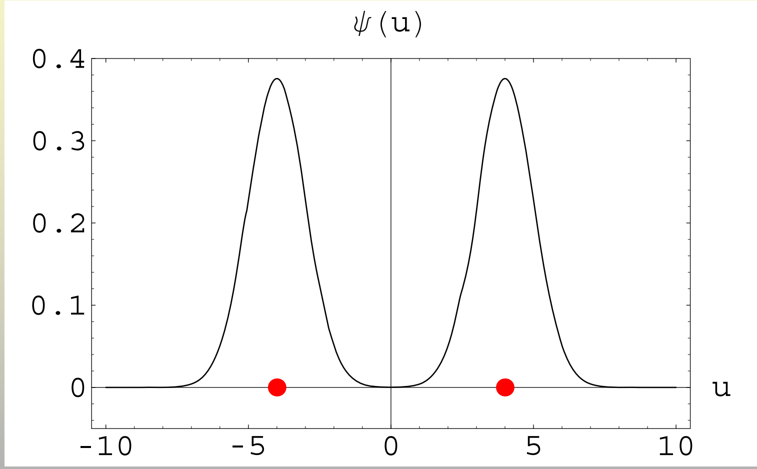
$$\psi(u,v) = \sum_{n=\text{even}} (A_n \cos(\sqrt{E_n} v) + iB_n \sin(\sqrt{E_n} v))\varphi_n(u)$$

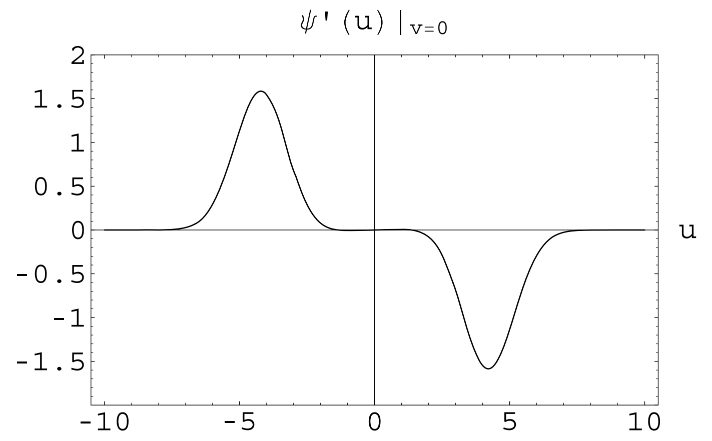
$$+ \sum_{n=\text{odd}} (C_n \cos(\sqrt{E_n} v) + iD_n \sin(\sqrt{E_n} v))\varphi_n(u).$$

$$B_n = C_n \quad \text{for } n \text{ even} \qquad D_n = A_n \quad \text{for } n \text{ odd}$$

$$[\langle p_v \rangle]_{v=0} = \int_0^\infty \left[\psi(u,v)^* \left(-i \frac{\partial}{\partial v} \right) \psi(u,v) \right]_{v=0} du.$$

k=0



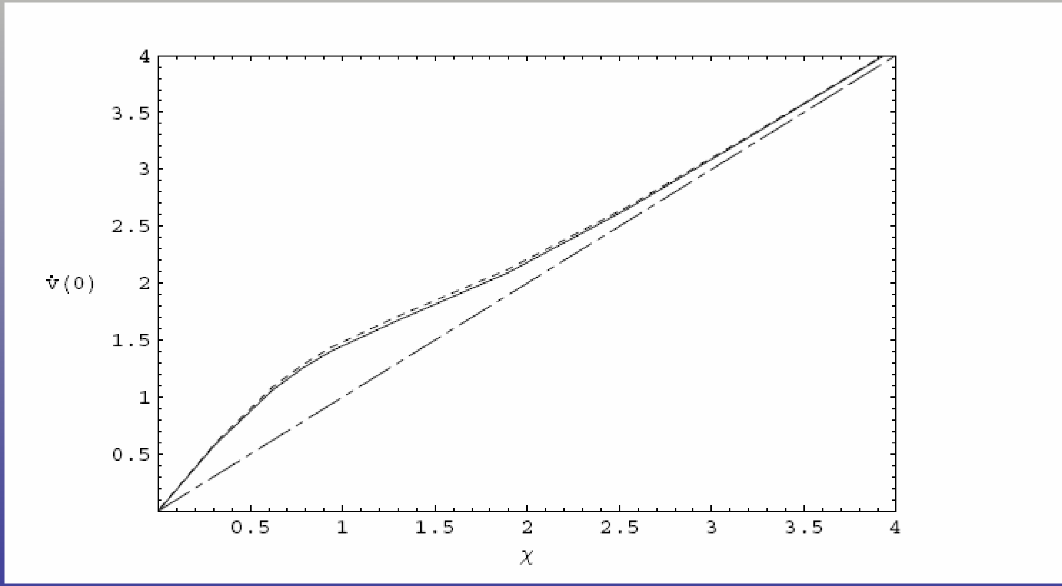


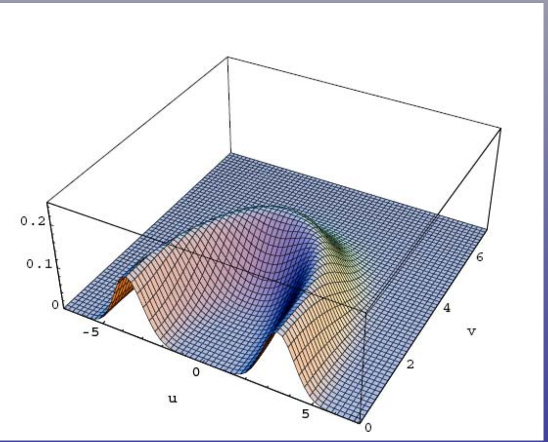
$$
\begin{aligned}
\psi(u,0) &= \sum_{even} c_n \varphi_n^0(u) \\
\psi'(u,0) &= i \sum_{odd} c_n \sqrt{(2n+1)\omega_1}\; \varphi_n^0(u),
\end{aligned}
$$

Initial velocity

$$[\langle p_v \rangle]^{\text{approx.}}_{v=0} = 4 \sum_{m_{\text{odd}}} \sum_{n_{\text{even}}} \frac{c_n c_m}{\sqrt{2^n n! \sqrt{\pi}} \sqrt{2^m m! \sqrt{\pi}}} \left[i\sqrt{(2m+1)\omega} \right]$$

$$n!m! \sum_{k=0}^{\text{Min}\{m,n\}} \frac{2^k H(0)_{m+n-2k-1}}{k!(n-k)!(m-k)!}$$





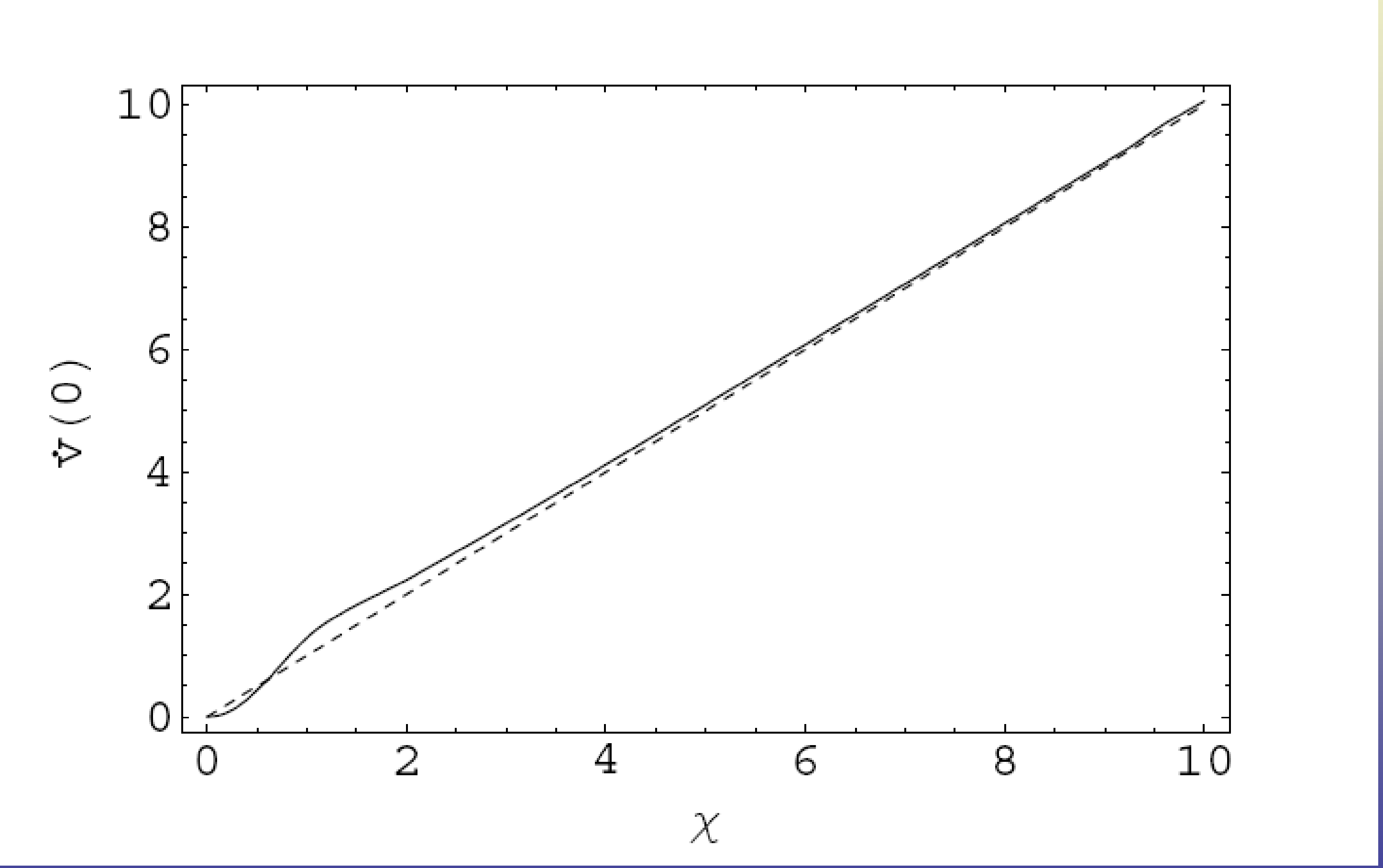
de-Broglie Bohm Interpretation

$$\psi = R\, e^{iS} = x + iy$$

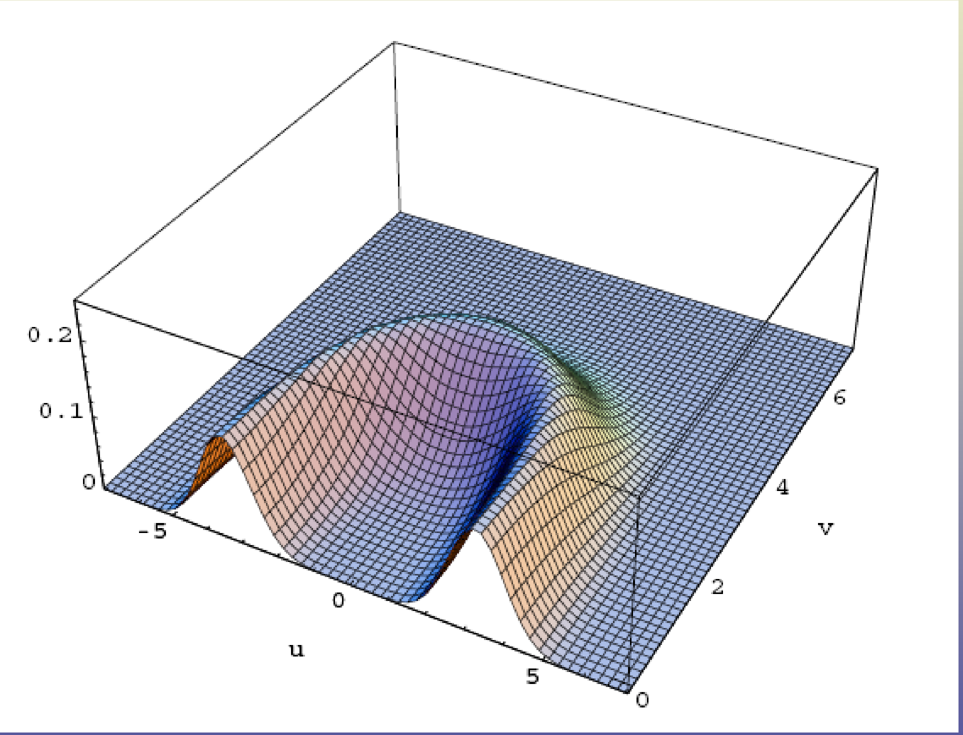
$$\dot{u} = \frac{1}{2} \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{d}{du} \left(\frac{y}{x}\right),$$

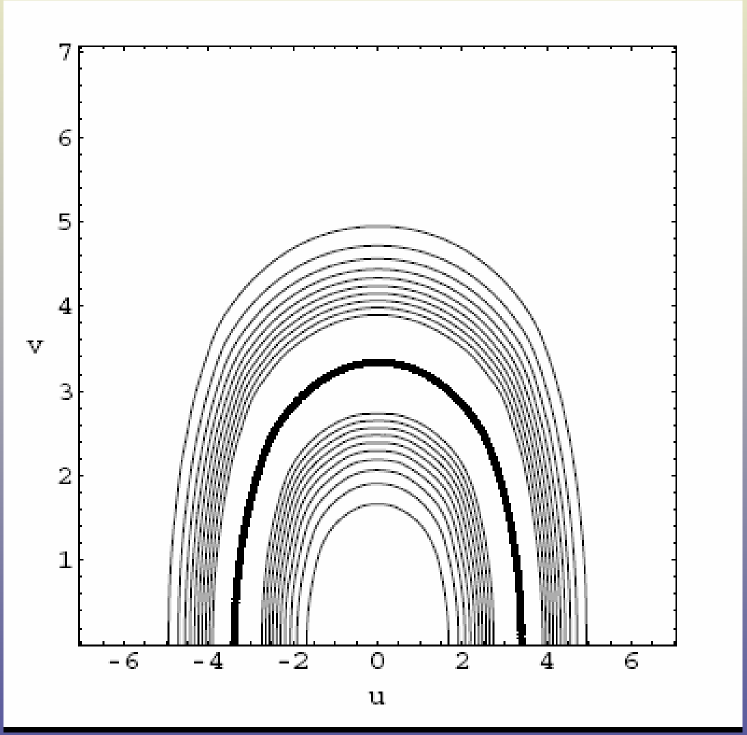
$$\dot{v} = -\frac{1}{2} \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{d}{dv} \left(\frac{y}{x}\right).$$

Initial velocity

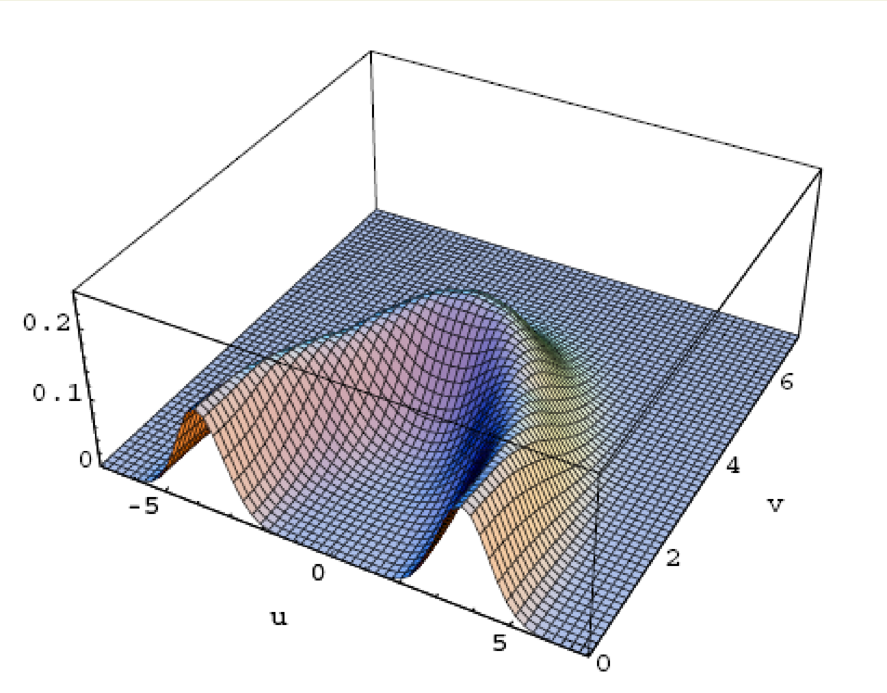


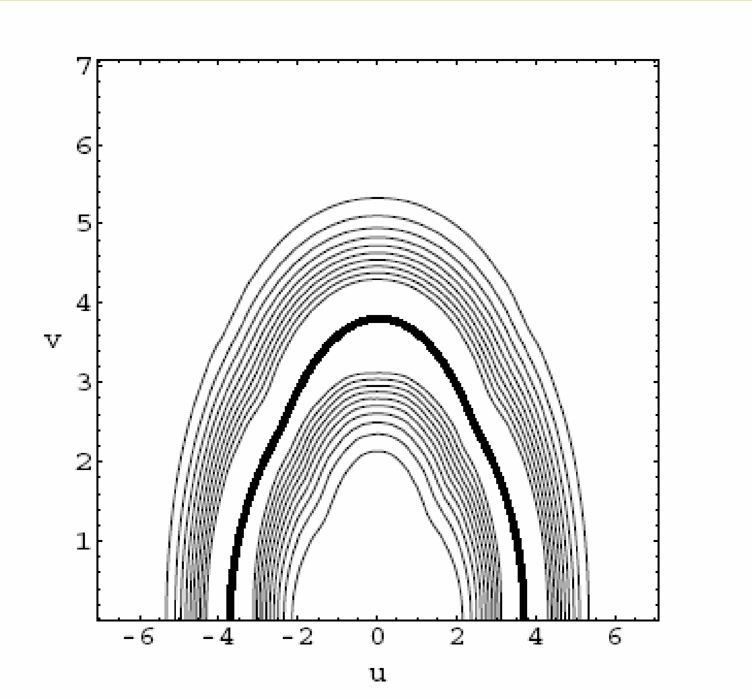
k=+1



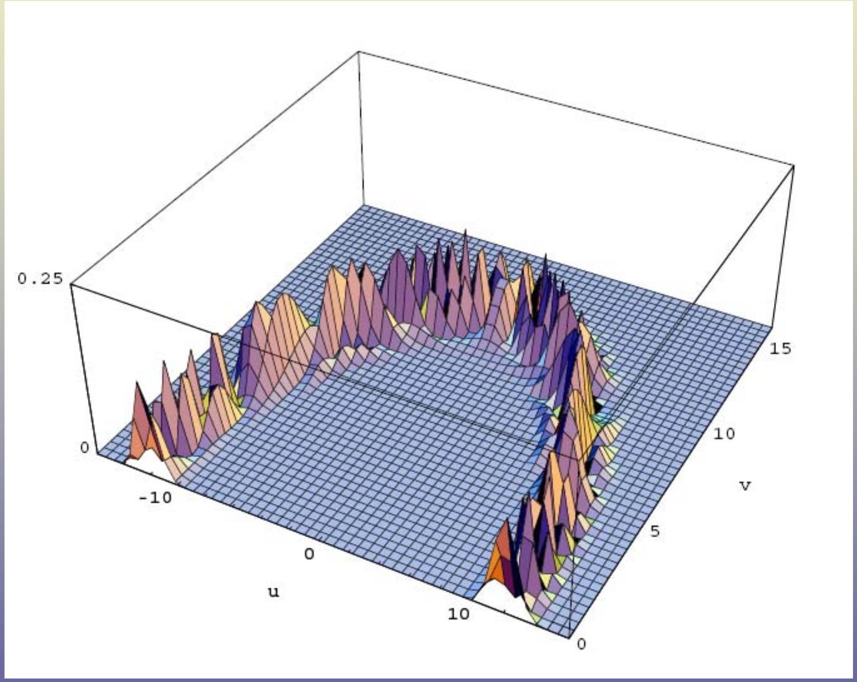


k=-1

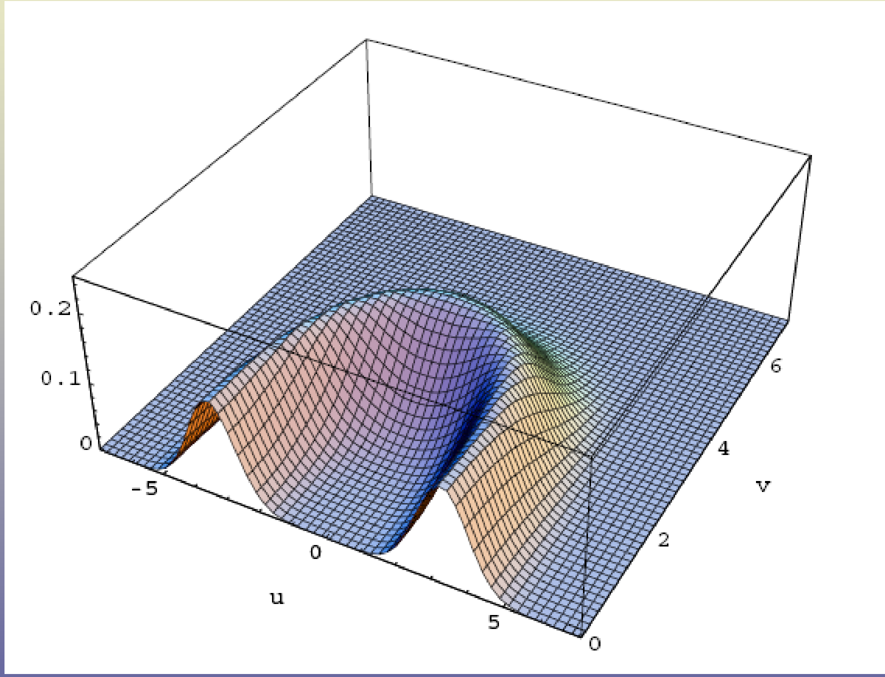




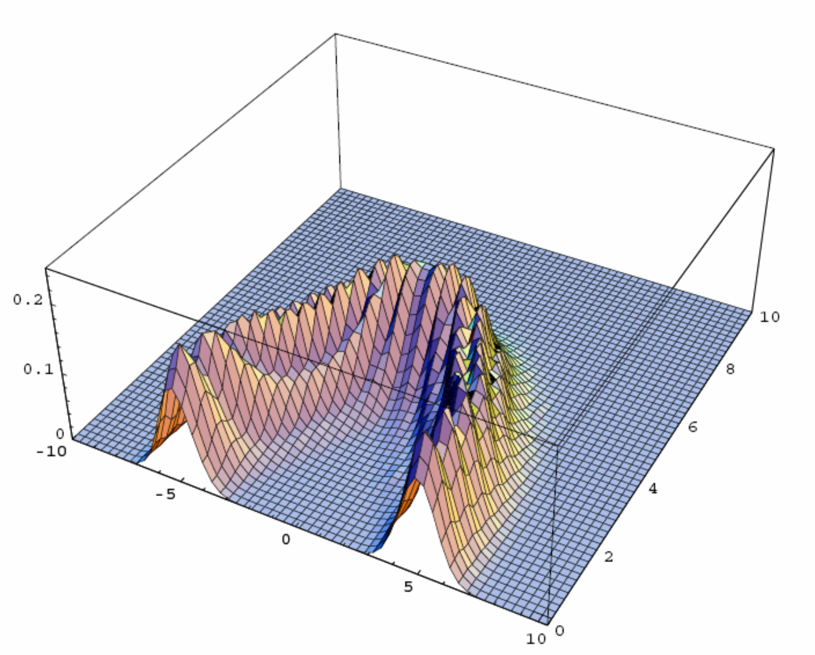
k=0

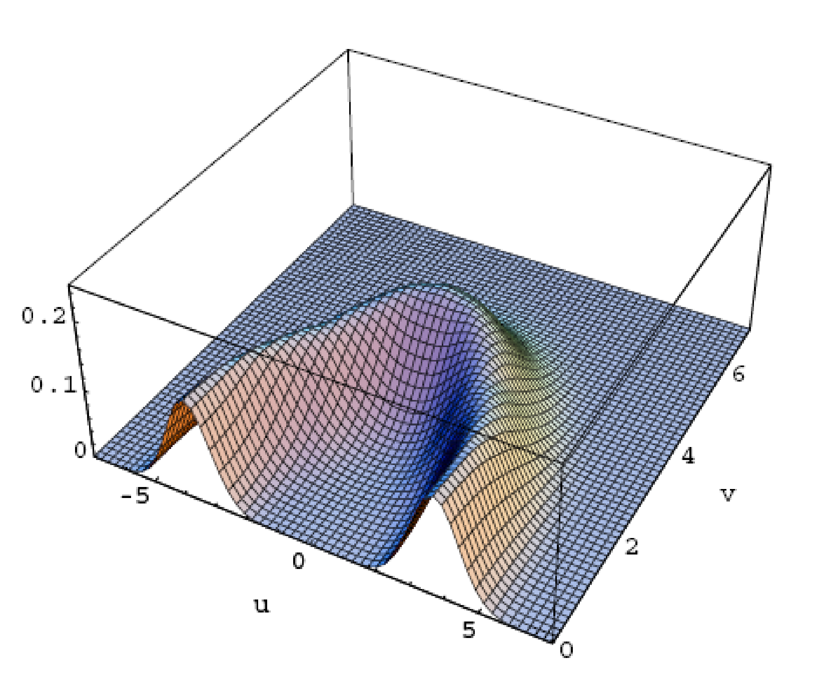


$$(\mathrm{Re}(\psi(u,v)))^2$$



$$|\psi(u,v)|^2$$


0.2
0.1
0
-10
-5
0
5
10
0
2
4
6
8
10


0.2
0.1
0
-5
0
5
u
0
2
4
6
v

$$L(u,v) = \dot{u}^2 - u^2 - \frac{u^4}{10} - \dot{v}^2 + v^2 + \frac{v^4}{10}$$