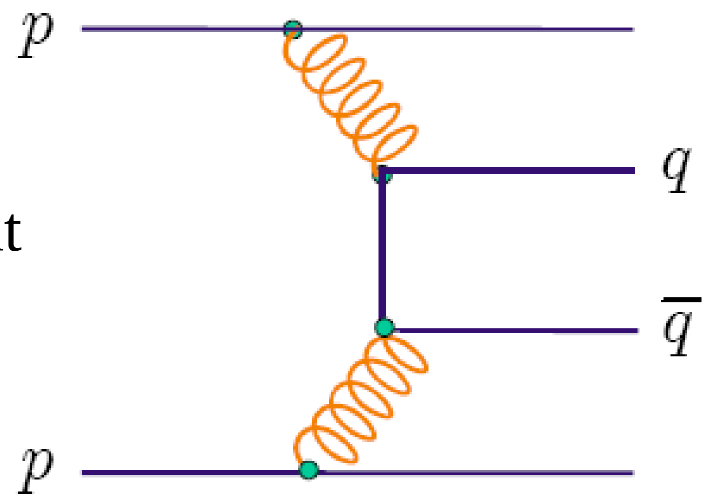
Why structure Functions and Parton densities are Important?

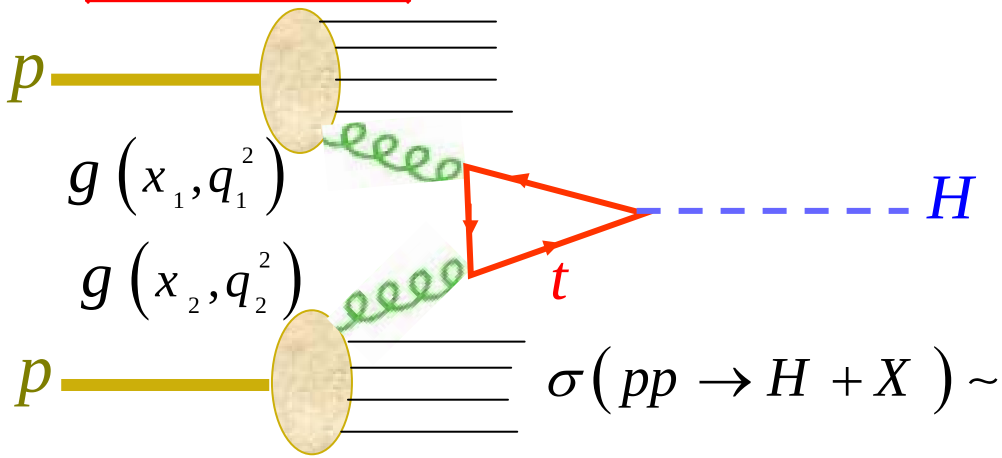
Nucleon in a collider is a beam of partons. We use them to search for new physics. Take the LHC:

LHC is a "gluon collider"



Knowledge about the gluon content of proton at low x is essential, e.g. for Higgs, W/Z production

$$g + g \to H$$



Good knowledge of PDF allow us to predict cross sections at coliders.

$$\sigma(pp \to H + X) \sim \int_0^1 dx_1 \int_0^1 dx_2\, g(x_1) g(x_2) \sigma(gg \to H)$$

Deep Inelastic Scattering reveals internal structure of hadrons (mainly nucleon) and gives insight into the structure of other particles.

It shows that nucleon is composed of quarks (and gluons). It has determined that quarks are spin ½ particles and attempts to determine their distributions, which turns out to be so important in search and discovery of new physics.

you may all have heard of all these. In these lectures I intend to shed some light on these topics and refresh your memory. In the process, I will also show some details.

Outline of the talk

- Quick introduction to Form Factors: The nucleus
- Elastic e-p Scattering and Nucleon form factors
- Deep Inelastic Scattering (DIS) – mainly e-p, but polarization is also included
  - Parton Model, Scaling, scaling violation and QCD improved P.M.
  - Structure functions, parton distributions, and extraction
  - DGLAP evolution equation
  - Polarized structure functions

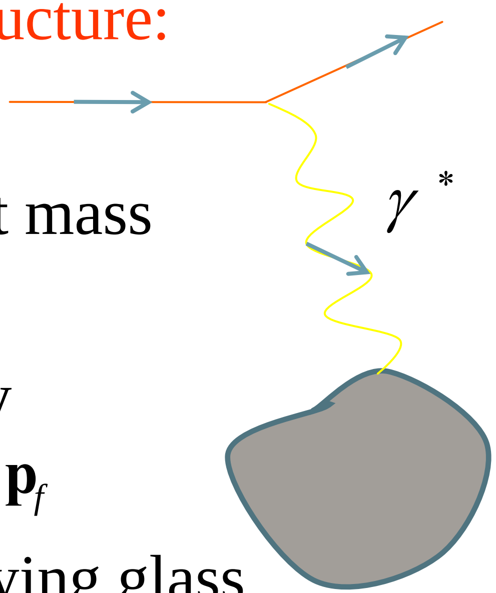
Looking inside a Hadron

Hadrons (mesons and baryons) are bound state of quarks. Like any composi system their spatial extend is complicated

To "see" inside of them you need a probe. Light (photon) is a familiar tool to see things. But ordinary light is no good. Its wavelength is way too big. Need to make a magnifying glass with very high resolution.

Where to get it? Use electron, and neutrino

Leptons are the best probes: they don't have structure:



Begin with E&M Probes:

Photon has a 4-momentum q_μ and an invariant mass

$$q^2 = q_\mu q^\mu = q_\circ^2 - |\mathbf{q}|^2 \neq 0$$

We can change the invariant mass of photon by changing its energy, or scattering angle. $\mathbf{q} = \mathbf{p}_i - \mathbf{p}_f$

$\mathbf{q} \sim 1/\lambda$ Acts as magnification of the magnifying glass

Increasing $\mathbf{q}$ decreases the wavelength, e.g:

I. Nucleus Form Factor

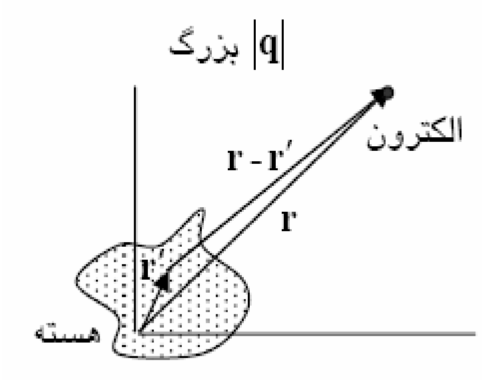
Suppose $|\mathbf{q}| = 20\,\text{KeV} \sim \lambda \approx 10^{-9}\text{cm}$, nucleous appears as a point charge ze

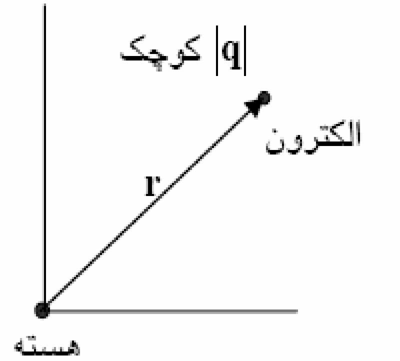
Nuclear size: $10^{-12}\,\text{cm}$

Increase $|\mathbf{q}|$ to $|\mathbf{q}| = 20\,\text{MeV} \sim \lambda \approx 10^{-12}\text{cm}$

Nucleus appears as a charge distribution $\rho(\mathbf{r}')$

YOU (PHOTON) gets to see protons inside the nucleus





$$V = \frac{1}{4\pi\varepsilon_0}\int d^3r \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \qquad \longleftarrow \qquad V_p = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{|\mathbf{r}|}$$

These potentials are in position space. Lets go to the momentum space by a Fourier Transform, see what happens

$$V_p(\mathbf{q}) = \frac{Ze}{4\pi\varepsilon_\circ}\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{1}{r} \quad ; \quad V(\mathbf{q}) = \frac{1}{4\pi\varepsilon_\circ}\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}\left(\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)$$

I want to show that these two are related to each other by a measurable quantity, $F_N(q^2)$ The Form Factor (FF)

Let us evaluate these potentials:

$$V_p(\mathbf{q}) = \frac{Ze}{4\pi\varepsilon_\circ}\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{1}{r} = \frac{Ze}{4\pi\varepsilon_\circ}\lim_{\mu\to\circ}\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{e^{-\mu r}}{r} = \frac{Ze}{4\pi\varepsilon_\circ}\lim_{\mu\to\circ}\int r^2\sin\theta\, dr\, d\theta\, d\phi\, e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{e^{-\mu r}}{r}$$

$$= \frac{Ze}{4\pi\varepsilon_\circ}(2\pi)\lim_{\mu\to\circ}\int_\circ^\infty \frac{e^{-\mu r}}{r} r^2 dr \int_{-1}^{1} e^{-|\mathbf{q}| r\cos\theta} d(\cos\theta) = \frac{Ze}{4\pi\varepsilon_\circ}\lim_{\mu\to\circ}\frac{4\pi}{\left(|\mathbf{q}|^2+\mu^2\right)} = \frac{Ze}{4\pi\varepsilon_\circ}\frac{4\pi}{|\mathbf{q}|^2}$$

And for $V(\mathbf{q})$ we get:

Everything is in terms of 3-vector $\mathbf{q}$

$$V(\mathbf{q}) = \frac{1}{4\pi\varepsilon_\circ}\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}\left(\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) \Rightarrow$$

$$\Rightarrow V(\mathbf{q}) = \frac{1}{4\pi\varepsilon_\circ}\int d^3R\, \frac{e^{-i\mathbf{q}.\mathbf{R}}}{|\mathbf{R}|}\underbrace{\int d^3r' e^{-i\mathbf{q}\cdot\mathbf{r}'}\rho(\mathbf{r}')}_{\equiv F_N(q^2)} = \frac{1}{\varepsilon_\circ}\frac{1}{|\mathbf{q}|^2}F_N(q^2)$$

Not in Covariant form either

$$F_N\left(q^2=-|\mathbf{q}|^2\right)\equiv\int d^3r' e^{-i\mathbf{q}\cdot\mathbf{r}'}\rho(\mathbf{r}')$$

Is the Fourier Transform of charged proton distribution, $\rho(\mathbf{r}')$ in nucleus

Obviously, $F_N(0)=Ze$ and that is the Normalization we take.

$$\Rightarrow \qquad V\left(q^2\right)=V_P\left(q^2\right)\frac{F_N\left(q^2\right)}{F_N(\circ)}=-\frac{F_N\left(q^2\right)}{q^2}$$

Compare With $V_p\left(q^2\right)=-\dfrac{Ze}{\varepsilon_\circ q^2}$

FF is equal to the Fourier transform of potential due to charge Ze

Use the expansion

$$e^{-i\mathbf{q}\cdot\mathbf{r}'}=1-i\mathbf{q}\cdot\mathbf{r}'-\frac{1}{2}|\mathbf{q}|^2 r^2\cos^2\theta+\cdots$$

And integrate over d^3r' we get

$$F(\mathbf{q})=\int\left(1-i\mathbf{q}\cdot\mathbf{r}'-\frac{1}{2}|\mathbf{q}|^2 r^2\cos^2\theta+\cdots\right)\rho(r')d^3r'=F_N(\circ)\left[1-\frac{1}{6}|\mathbf{q}|^2\langle r^2\rangle+\cdots\right]$$

$$\langle r^2\rangle\equiv\int r^2\rho(r)d^3r=6\left|\frac{dF_N\left(q^2\right)}{dq^2}\right|_{q^2=0}$$

Integration of linear term vanishes, by spatial symmetry

The derivative of FF gives the mean size of the nucleus.

The greater the object is, the faster its form factor decreases.
For Point objects, FF is INDEPENDENT of q^2

- FF can be measured in Rutherford or Mott scattering.

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2}\frac{m\mathbf{p}}{\nu}|V_{fi}|^2 = \left|\frac{m}{2\pi}\int d\mathbf{r}\,V(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\right|^2 = \frac{4z^2\alpha^2 m^2}{\mathbf{q}^4}$$

For Point charge

If Charge is spread: $\Rightarrow V(\mathbf{r}) \to V'(\mathbf{r}) = \int d\mathbf{r}' V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')$

And the matrix element:

$$V_{fi} \to V'_{fi} = \int d\mathbf{r}\,V'(\mathbf{r})e^{i\mathbf{q}.\mathbf{r}} = \left[\int d(\mathbf{r}-\mathbf{r}')V(\mathbf{r}-\mathbf{r}')e^{i\mathbf{q}.(\mathbf{r}-\mathbf{r}')}\right]\times\int d\mathbf{r}'\rho(\mathbf{r}')e^{i\mathbf{q}.\mathbf{r}} = V_{fi}F(\mathbf{q}^2)$$

So, the cross- section for an extended object is:

$$d\sigma(\theta) = d\sigma_{pt}\left|F(\mathbf{q}^2)\right|^2$$

- Inverse Fourier Transform of FF gives the charge (proton) distribution of The nucleus:

$$\rho(\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3q\, e^{+i\mathbf{q}\cdot\mathbf{r}'}F_N\left(q^2 = -|\mathbf{q}|^2\right)$$

$$V(\mathbf{q}) = \frac{1}{\varepsilon_\circ}\frac{1}{|\mathbf{q}|^2} F_N(q^2)$$

Not in Covariant form

So , lets make it covariant, using Lorentz invariance

$$q^2 = q_\circ^2 - |\mathbf{q}|^2$$

In Fourier transform of static coulomb potential, energy does not get transformed , only momentum is transformed , so $q^2 = -|\mathbf{q}|^2$ and we can replace $|\mathbf{q}|^2 \rightarrow -q^2$

Static, No recoil

So , we get for the point charge potential

$$V_p(q^2) = -\frac{Ze}{\varepsilon_\circ q^2}$$

♣ The point is : NONRELATIVISTIC LIMIT OF $1/|\mathbf{q}|^2$ is the propagator $-\frac{1}{q^2}$ of massless boson Interchanged between electron and the nucleus.

→ Exchange of a meson with non-relativistic propagator is the source of interaction potential between the two particles.

II. The Nucleon

We can also use the concept of FF for the nucleon and learn that nucleon is an extended object, with structure like proton.

Anomalous Magnetic Moment of nucleon is another implication that nucleon has an internal structure. (let us see this)

Take a **point like** fermion current $\bar{u}(p')\gamma_\mu u(p)$, interacting with electromagnetic field A^μ

$$\bar{u}(p')\gamma_\mu u(p)A^\mu = \frac{e}{2m}\bar{u}(p')\left[(p'+p)_\mu + i\sigma_{\mu\nu}(p'-p)^\nu\right]u(p)A^\mu$$ Gordon Decomposition

Non-relativistic limit: second term reduces to:

$\mathbf{B}$: External Field

$\boldsymbol{\sigma}/2$: proton's Spin

$$(e/2m)\bar{u}(p')\boldsymbol{\sigma}\cdot\mathbf{B}\,u(p)$$

Proof: Non-Relativistic Limit means $p_\mu = p'_\mu = (M, \mathbf{0})$

$$\Rightarrow \ (p'+p)_\mu A^\mu \to 2m\phi, \quad i\sigma_{\mu\nu}q_\mu A^\mu \to i\sigma_{ij}q_j A^i \sim \boldsymbol{\sigma}\cdot\nabla\times\mathbf{A} = \boldsymbol{\sigma}\cdot\mathbf{B}$$

This shows that the magnetic moment of a point particle of charge e, mass m, is one Bohr Magneton: $\mu_B \equiv (e/2m)$

Experimentally: Proton and neutron Magnetic moments are: $\mu_p = 2.79\mu_B, \quad \mu_n = -1.91\mu_B$

CONCLUSION: Nucleon is not point fermion, they have internal structure $\Delta\mu_{\text{electron}} = 0.0016$

Large magnetic moment ⟹ Quarks!

II. The Nucleon: Electromagnetic form factore

Can't use the same argument to tackle the Nucleon FF as we did for Nucleus. Because

1. It is not static charge distribution. Upon collision with electron, the nucleon will recoil.
2. In addition to electric charge, it also posses magnetic moment which contributes to electromagnetic interaction.

If Proton had no structure and no recoil, i.e. it could be considered as a heavy point particle. we know the *e- p* cross section: Mott scattering.

If proton is considered as a point particle of charge *e* and magnetic moment $e/2m_p$

still we know the cross- section: replace muon mass with proton mass in $e\mu \to e\mu$

$$\left.\frac{d\sigma}{d\left(\cos\theta\right)}\right|_{\text{lab}} = \left(\frac{\alpha^2\pi}{2E_1^2\sin^4\left(\theta_{lab}/2\right)}\right)\frac{E_3}{E_1}\left(\cos^2\left(\theta_{lab}/2\right) - \frac{q^2}{2m_p^2}\sin^2\left(\theta_{lab}/2\right)\right)$$

$$\frac{E_3}{E_1} = \left(1 + \frac{2}{m_p}\sin^2\left(\theta/2\right)\right)^{-1}$$

Effect of recoil

Mag. Moment contribution

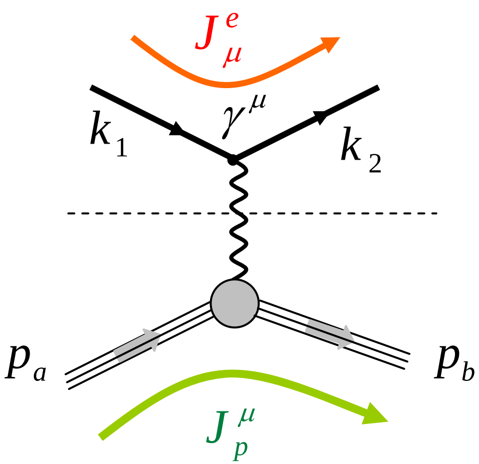
If muon was spinless (no mag. Moment) this term would not be there

This is an important formula for our story

Transition amplitude

$$T_{fi} = -i\int j_\mu^e\left(-\frac{1}{q^2}\right)J_p^\mu d^4x$$

$$j_\mu^e = -e\bar{u}\left(k_2\right)\gamma^\mu u\left(k_1\right)e^{i\left(k_2-k_1\right)\cdot x}$$



But Cannot use γ_μ for proton vertex

Yet, initial and final proton can be treated ad Dirac particle

Recall: for scatt. of electron from a point charge:

$$\left.\frac{d\sigma}{d\left(\cos\theta\right)}\right|_{\text{lab}} = \left(\frac{\alpha^2\pi}{2E_1^2\sin^4\left(\theta_{lab}/2\right)}\right)\frac{E_3}{E_1}\left(\cos^2\left(\theta_{lab}/2\right) - \frac{q^2}{2m_p^2}\sin^2\left(\theta_{lab}/2\right)\right)$$

Due to coulomb scattering

Due to magnetic moment

Since proton is not point charge, we need to introduce a Form factor for charge distribution, and one more to describe the distribution of the magnetic moment.

Two Form factors. What are they?

Or what is the most general form of J_p^μ

Well, J_p^μ is a Lorentz 4- vector, and can use the general considerations to construct it from 4-vectors at the proton vertex: p_a, p_b, q And Dirac γ Matrices

So, only γ^μ, $\left(p_b - p_a\right)$ and $\left(p_b + p_a\right)$ But No $q = p_a - p_b$ It is not independent

The most general 4-vector

$$J_p^\mu = e\bar{u}(p_b)\left[\gamma^\mu k_1 + i\sigma^{\mu\nu}(p_b - p_a)_\nu k_2 + i\sigma^{\mu\nu}(p_b + p_a)_\nu k_3 + (p_b - p_a)^\mu k_4 + (p_b + p_a)^\mu k_5\right]u(p_a)$$

k_i 's Functions of scalar variables, and we got only one: q^2

Note: no γ_5 Parity conservation

Note: $p_a \cdot q$ and $p_b \cdot q$ Are also scalars, but are not independent they can be written in terms of q^2

The most general 4-vector

Use Gordon decomposition and simplify it

Write out $(p_b + p_a)^\mu$ In terms of γ^μ and $\sigma^{\mu\nu}(p_b - p_a)_\nu$

$$\frac{1}{2m}\bar{u}(p_b)\left[(p_b + p_a)^\mu\right]u(p_a) = \bar{u}(p_b)\gamma^\mu u(p_a) - \frac{1}{2m}\bar{u}(p_b)\left[i\sigma^{\mu\nu}(p_b - p_a)_\nu\right]u(p_a)$$

And plug into

$$J_p^\mu = e\bar{u}(p_b)\begin{bmatrix}\gamma^\mu k_1 + i\sigma^{\mu\nu}(p_b - p_a)_\nu k_2 + i\sigma^{\mu\nu}\left[2m\gamma^\mu - i\sigma^{\mu\nu}(p_b - p_a)_\nu\right]k_3 \\ +(p_b - p_a)^\mu k_4 + \underbrace{2m\gamma^\mu k_5 - i\sigma^{\mu\nu}(p_b - p_a)_\nu k_5}_{(p_b + p_a)^\mu k_5}\end{bmatrix}u(p_a)$$

$$= e\bar{u}(p_b)\begin{bmatrix}\gamma^\mu(k_1 + 2m\,k_5) + (k_2 - k_5)i\sigma^{\mu\nu}(p_b - p_a)_\nu \\ +(p_b - p_a)^\mu(12k_2 I + g_{\mu\nu}k_4)\end{bmatrix}u(p_a)$$

Note $\sigma^{\mu\nu}\sigma_{\mu\nu} = 12I$

Collect and rearrange. We get

$$J_p^\mu = e\bar{u}(p_b)\left[\gamma^\mu F_1(q^2) + \frac{i\kappa}{2m_p}F_2(q^2)\sigma^{\mu\nu}q_\nu + q^\mu F_3(q^2)\right]u(p_a)$$

Use proton current conservation $\partial_\mu J_p^\mu = q_\mu J_p^\mu = 0$

$$q_\mu J_p^\mu = e\bar{u}(p_b)\left[\gamma^\mu q_\mu F_1(q^2) + \frac{i\kappa}{2m_p}F_2(q^2)q_\mu\sigma^{\mu\nu}q_\nu + q_\mu q^\mu F_3(q^2)\right]u(p_a) = \circ$$

=0, by Dirac Eq.

=0, since $\sigma^{\mu\nu}$ is anti-symmetric

The third term $= q^2 F_3 \neq 0$ since $q^2 \neq m = 0$ $\Rightarrow$ $\boxed{F_3 = 0}$

We are left with two FFs

$$J_p^\mu = e\bar{u}(p_b)\left[F_1(q^2)\gamma^\mu + \frac{i\kappa}{2m_p}F_2(q^2)\sigma^{\mu\nu}q_\nu\right]u(p_a)$$

When $q^2 \to 0$

That is photon wavelenght larger than a Fermi, it sees proton as a point charge, and its crrent also reduces to the current of a point charge. This requires that we set

$$F_1(q^2 = 0) = F_2(q^2 = 0) = 1$$

Now we can write the transition amplitude

$$\mathcal{M}_{i\to f} = \frac{e^2}{q^2}\left[\bar{u}(k_2)\gamma^\mu u(k_1)\right]\left\{\bar{u}(p_b)\left[\left(F_1+F_2'\right)\gamma^\mu - \left(K_\mu/2m_p\right)F_2'\right]u(p_a)\right\}$$

Where, $F_2' \equiv \kappa F_2, \quad K \equiv p_a + p_b$

And calculate the cross section. Summing over final state spins and averaging over the initial state particles spin, redefining some symbols, we arrive at

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{\text{spins}} \mathcal{M}^*_{i\to f}\,\mathcal{M}_{i\to f} = \left(\frac{e^2}{q^2}\right)^2 \ell_e^{\mu\nu} L^p_{\mu\nu}$$

Evaluating Traces, using linear combinations of F_1 and F_2

$$G_E\left(q^2\right) = F_1\left(q^2\right) + \frac{q^2}{4M^2}F_2\left(q^2\right), \quad G_M\left(q^2\right) = F_1\left(q^2\right) + F_2\left(q^2\right)$$

$$G_E^p(0) = 1, \quad G_E^n(0) = 0, \quad G_M^p(0) = 2.97, \quad G_M^n(0) = -1.91$$

$$F_1^2 - \frac{q^2}{4M^2}F_2^2 = \frac{G_E^2 - \left(q^2/4M^2\right)G_M^2}{1 - \left(q^2/4M^2\right)}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{lab} = \frac{\left(d\sigma/d\Omega\right)_{Mott}}{1+\frac{2E}{M}\sin^2\theta/2}\left\{\frac{G_E^2\left(q^2\right)+\tau G_M^2\left(q^2\right)}{1+\tau}+2\tau G_M^2\left(q^2\right)\tan^2\theta/2\right\}$$

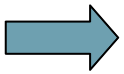
$$=\left(\frac{d\sigma}{d\Omega_{lab}}\right)_{NS}\left\{\frac{G_E^2\left(q^2\right)+\tau G_M^2\left(q^2\right)}{1+\tau}+2\tau G_M^2\left(q^2\right)\tan^2\theta/2\right\}$$

Rosenbluth Cross-section

The Coefficient $\tan^2\theta/2$ is due to magnetic moment and reflects the spin of the proton.

If The target was spinless, like a pion, this term would not be there.

Experimentally: for a number of q^2 plot the measured values of cross section as a function of $\tan^2\theta/2$ Since it is linear, i.e $d\sigma/d\Omega \propto A\left(q^2\right)+B\left(q^2\right)\tan^2\theta/2$



Determine The Form Factors

Exercise: electromagnetic FF of $\pi^\pm$ cab be measured in $e^-+\pi^\pm \to e^-+\pi^\pm$

Dcattering Amplitude is $\pm e\varepsilon_\mu(q)T^\mu\left(k',k\right)$ with $T^\mu\left(k',k\right)\equiv\left\langle\pi^\pm\left(k'\right)\left|J_{em}^\mu(0)\right|\pi^\pm(k)\right\rangle$

Show that $T^\mu\left(k',k\right)=\left(k'+k\right)^\mu F_\pi\left(q^2\right)$

Experiment shows that as q^2 increases, The Form Factors $G_E\left(q^2\right)$, $G_M\left(q^2\right)$ Behave as

$$\frac{G_E^p\left(q^2\right)}{G_E^p\left(0\right)}=\frac{G_M^p\left(q^2\right)}{G_M^p\left(0\right)}=\frac{G_M^n\left(q^2\right)}{G_M^n\left(0\right)}=\frac{1}{\left(1-\frac{q^2}{\Lambda^2}\right)^2}\ ;\ G_E^n\left(q^2\right)=0$$

A Dipole Form

$\Lambda = 0.84\ \mathrm{GeV}$

Inverse Fourier Transform of this gives proton charge distribution (quarks) $\rho\left(r=|\mathbf{x}|\right)$ and its mean square radius

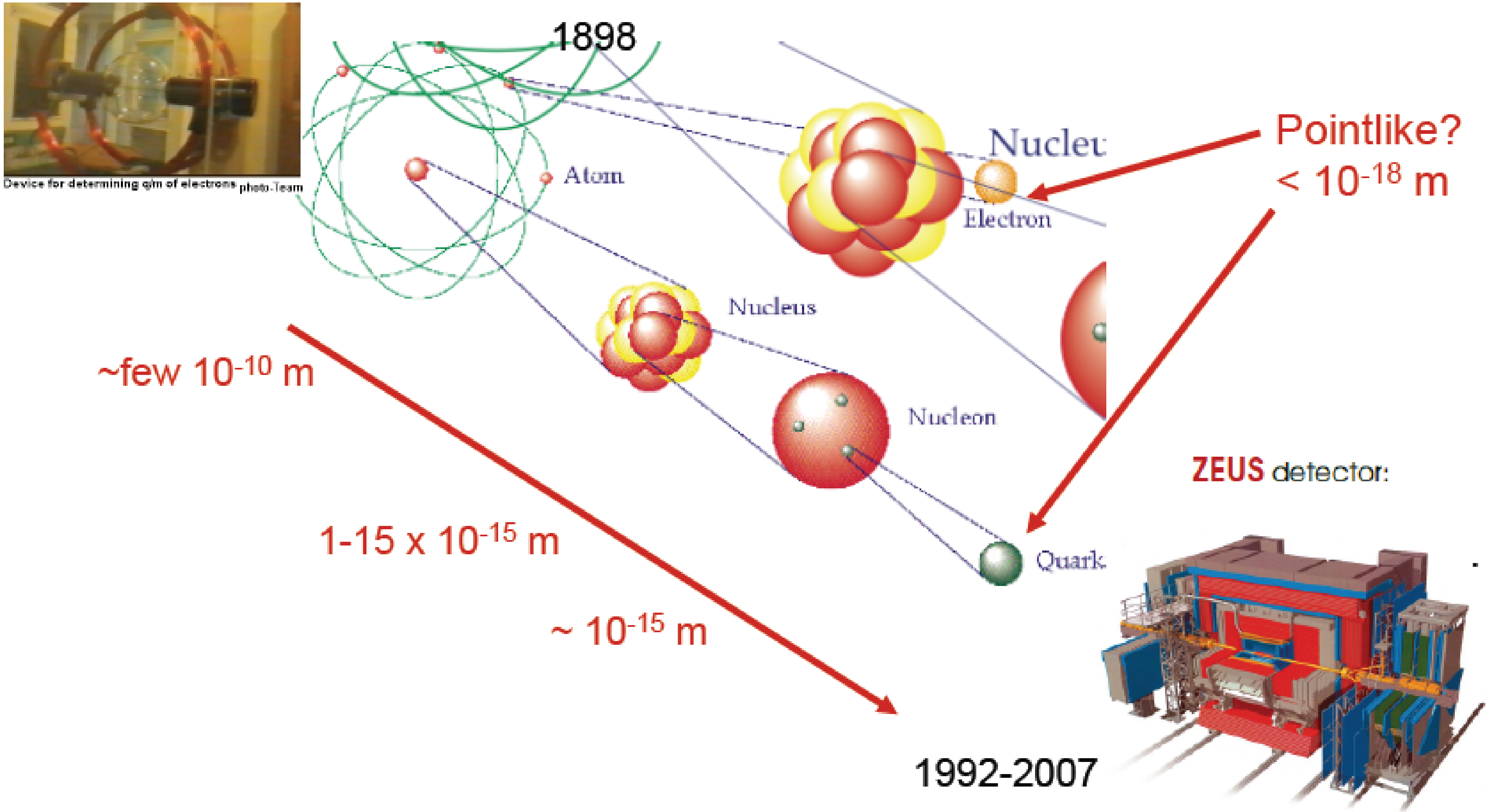
$$\rho\left(r\right)=\int\frac{d^3q}{\left(2\pi\right)^3}\frac{e^{i\,\mathbf{q}\cdot\mathbf{x}}}{\left(1+\frac{|\mathbf{q}|^2}{\Lambda^2}\right)^2}=\frac{\Lambda^3}{8\pi}e^{-\Lambda r},\quad \int d^3x\,\rho\left(r\right)=1$$

$$\left\langle r^2\right\rangle=6\left(dG_E\left(q^2\right)/dq^2\right)\Big|_{q^2=0}=6\frac{d}{dq^2}\left(1-\frac{q^2}{0.84}\right)^{-2}\approx\left(0.81\times10^{-13}\mathrm{cm}\right)^2$$

Form factors fall off rapidly as q^2 increases. Remember this

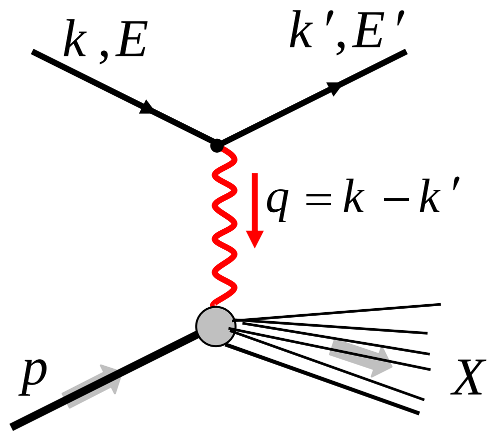
II Deep Inelastic Scattering (DIS)

Inelastic Scattering
– Probing the Structure of Matter



II DEEP INELASTIC SCATTERING (DIS)

Incoming beam of lepton with Energy E scatters off a fixed hadron target. The Energy and the direction (scattering angle) of the scattered lepton is measured.



Basic diagram for DIS

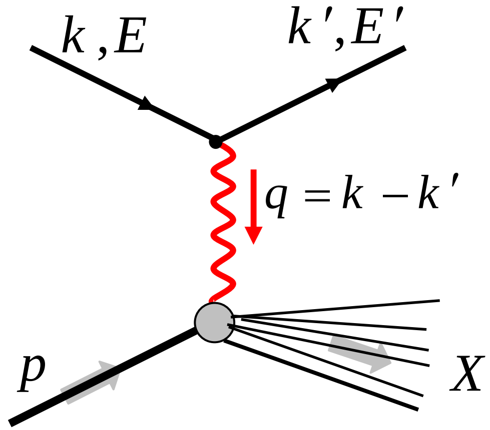
No Final state hadron (denoted by X) is measured.

Lepton interacts with hadron through the exchange of a virtual photon, Z (or W, if lepton is a neutrino). The target hadron absorbs the virtual photon, to produce the final state hadrons X .

X may be the hadron itself (Elastic Scatt.) or an excited state of it. If q is large the initial hadron breaks up

Warning! various kinematic variables are used. I will chose z-axes along the incident lepton beam direction. The kinematic variables are:

Kinematic Variables



Basic diagram for DIS

M: Mass of target

k: momentum of initial lepton

$k = (E, 0, 0, E)$ if lepton mass neglected.

Ω the solid angle into which final lepton scattered.

k' Momentum of scattered lepton

$$k' = (E', E'\sin\theta\cos\phi, E'\sin\theta\sin\phi, E'\cos\theta)$$

p: Momentum of target. For fixed target, $p = (M, 0, 0, 0)$

q: momentum transfer , *i.e* the momentum of virtual photon $q = k - k'$

ν The energy loss of the lepton $\nu = E - E' = q \cdot p / M$

y : The fractional energy loss of lepton $y = \frac{\nu}{E} = \frac{p \cdot q}{p \cdot k}$

$$Q^2 \equiv -q^2 = 2EE'(1 - \cos\theta) = 4EE'\sin^2\frac{\theta}{2}$$

$$x = \frac{Q^2}{2M\nu} = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{2MEy}$$

$$\omega = \frac{1}{x}$$

x is called Bjorken variable. It is crucial in understanding of DIS, because QCD predicts that the structure functions are functions of x and independent of Q^2 in the leading order

DEFINITION: DIS is the study of lepton-hadron scattering in the Region of kinematics that $Q^2 \to \infty,\ \nu \to \infty$, But x is Fixed and limited

The INVARIANT mass of final state hadronic system X is

$$M_X^2 = (p+q)^2 = M^2 + 2p.q + q^2 \qquad (1)$$

The invariant mass of X system must be at least equal to the mass of target nucleon. WHY?

$$\Rightarrow \quad M_X^2 \geq M^2 \ \Rightarrow \ M^2 + 2p.q - Q^2 \geq M^2 \ \Rightarrow \ \boxed{x \leq 1} \ \star$$

Since Q^2 and ν are both positive, x must also be positive.

$$x = \frac{Q^2}{2\underbrace{p \cdot q}_{M\nu}}$$

The lepton energy loss $\nu = E - E'$ is between zero and E, so, the physically allowed kinematic region is

$$0 \leq x \leq 1, \qquad 0 \leq y \leq 1$$

(1) can be written as

$$x = \frac{Q^2}{2p \cdot q} = 1 - \frac{M_x^2 - M^2}{2p \cdot q}$$

The value x=1 implies that $M_x^2 = M^2$ and so,

$x = 1$ corresponds to Elastic Scatt.

Any fixed hadron state X with invariant mass M_X^2 Contributes to the cross section at the value of x

$$x = \frac{1}{1+\left(M_X^2 - M^2\right)/Q^2}$$

In DIS limit $\left(Q^2 \to \infty\right)$ any state X with fixed mass M_X gets driven to $x = 1$

In particular, all nucleon resonances such as N* gets pushed to $x = 1$

The experimental measurements give the cross section as a function of final lepton energy and scattering angle $d^2\sigma/dE'd\Omega$ the results often are presented instead by giving the differential cross section as a function of $\left(x, Q^2, \phi\right)$ or $\left(x, y, \phi\right)$

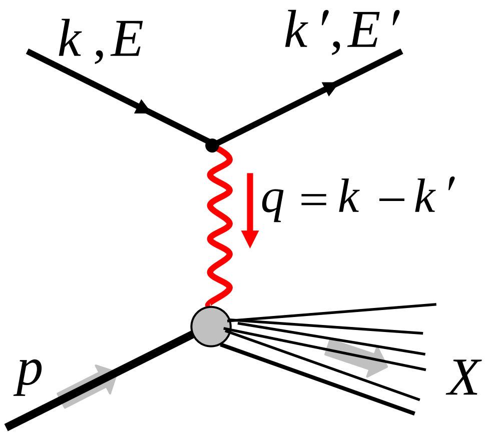
The Jacobian for converting between thses cases is

$$\frac{\partial\left(x, Q^2\right)}{\partial\left(x, y\right)} = \begin{vmatrix} 1 & 0 \\ 2MEy & 2MEx \end{vmatrix} = 2MEx = \frac{Q^2}{y}$$

$$\frac{\partial\left(x, y\right)}{\partial\left(E', \cos\theta\right)} \begin{vmatrix} 1 & \frac{-2EE'}{2M\nu} \\ -\frac{1}{E} & 0 \end{vmatrix} = \frac{E'}{M\nu}$$

The basic Feynman graph for DIS shown. The scattering Amplitude $\mathcal{M}$ is given by

$$\mathcal{M} = (-ie)^2 \left(\frac{-ig_{\mu\nu}}{q^2} \right) \langle k' | j_\ell^\mu(0) | k, s_\ell \rangle \langle X | j_h^\nu(0) | p, \lambda \rangle$$



s_ℓ : is the polarization of lepton and

λ : is the polarization of the initial hadron

For spin $1/2$ target, $\lambda = \pm 1/2$

The differential cross section is obtained from $\mathcal{M}$ by squaring it and multiplying by the phase space factors

λ : Can be chosen to be the value of spin in arbitrary direction, usually beam direction

$$d\sigma = \sum_X \int \frac{d^3k'}{(2\pi)^3 2E'} (2\pi)^4 \delta^4(k + p - k' - p_x) \frac{|\mathcal{M}|^2}{(2E)(2M)(v_{rel} = 1)}$$

$$= \sum_x \int \frac{d^3k'}{(2\pi)^3 2E'} \frac{(2\pi)^4 \delta^4(k + p - k' - p_x)}{(2E)(2M)} \frac{e^4}{Q^4}$$

$$\times \langle p, \lambda | j_h^\mu(0) | X \rangle \langle X | j_h^\nu(0) | p, \lambda \rangle \langle k, s_\ell | j_{\ell\mu}(0) | k' \rangle \langle k' | j_{\ell\mu}(0) | k, s_\ell \rangle$$