Strings and Sandwich Virasoro Conditions

Arjun Bagchi, Aritra Banerjee, **Ida Mehin Rasulian**, M.M. Sheikh-Jabbari

School of Physics
Institute for Research in Fundamental Sciences

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Quick Review of Textbook Worldsheet Theory

The Polyakov action with $T = 1/2\pi\alpha' = 1$ is

$$S_P = \int d^2\sigma \sqrt{\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}. \quad (1)$$

Using Diff $\oplus$ Weyl gauge symmetry we can choose

$$\gamma^{ab} = \eta^{ab}, \quad (2)$$

The solution to the equations of motion for a closed string $(\sigma \sim \sigma + 2\pi)$ is

$$X^\mu = x^\mu + \tilde{x}^\mu + \alpha_0^\mu \sigma_+ + \tilde{\alpha}_0^\mu \sigma_- + \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{in\sigma_+} + \tilde{\alpha}_n^\mu e^{in\sigma_-}\right). \quad (3)$$

Quick Review of Textbook Worldsheet Theory

Consistency of fixing the gauge requires that the worldsheet energy-momentum tensor should also vanish on-shell:

$$T_{ab} := -\frac{2}{\sqrt{\gamma}} \frac{\delta S_P}{\delta \gamma^{ab}} = 0. \tag{4}$$

Therefore physical states satisfy

$$\langle phys' | T_{ab} | phys \rangle = 0 \qquad \forall |phys\rangle, |phys'\rangle \in \mathcal{H}_{\text{P}} \ . \tag{5}$$

Quick Review of Textbook Worldsheet Theory

Nonzero components of T_{ab} are

$$T_{++} = \sum_n \mathcal{L}_n^+ e^{in\sigma_+}, \ \mathcal{L}_n^+ = \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m - a\, \delta_{n,0}$$
$$T_{--} = \sum_n \mathcal{L}_n^- e^{in\sigma_-}, \ \mathcal{L}_n^- = \sum_{m=-\infty}^{\infty} \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m - a\, \delta_{n,0}. \quad (6)$$

$\mathcal{L}_n^\pm$ generate two copies of the Virasoro algebra and $a = \frac{D-2}{12}$ is the normal ordering constant, the zero point energy.

Quick Review of Textbook Worldsheet Theory

The residual symmetries of the worldsheet symmetry after gauge fixing (2) is the 2d conformal algebra,

$$[\mathcal{L}_n^{\pm}, \mathcal{L}_m^{\pm}] = (n-m)\mathcal{L}_{n+m}^{\pm} + \frac{c}{12}\delta_{n+m,0}(n^3-n), \;\; c = D-2 \tag{7}$$

with $[\mathcal{L}_n^{\pm}, \mathcal{L}_m^{\mp}] = 0$.

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Sandwich Virasoro Conditions

The above constraints yield the Sandwich Virasoro conditions (SVC), for all states $|phys\rangle, |phys'\rangle \in \mathcal{H}_{\text{P}}$

$$\langle phys|\mathcal{L}_n^+|phys'\rangle = 0 = \langle phys|\mathcal{L}_n^-|phys'\rangle \qquad \forall n \in \mathbb{Z}. \tag{8}$$

In almost all string theory textbooks, the above condition has been solved through the right-action,

$$\mathcal{L}_n^{\pm}|phys\rangle = 0, \quad \forall n \geq 0. \tag{9}$$

But, as we emphasize below, all solutions of the SVC don't necessarily obey this right action constraint.

Sandwich Virasoro Conditions

For all physical states $|\psi\rangle, |\psi'\rangle$, we would have,

$$\mathcal{L}_n^{\pm}|\psi\rangle \neq 0, \quad \text{but} \quad \langle\psi'|\mathcal{L}_n^{\pm}|\psi\rangle = 0. \tag{10}$$

All states $|\phi_i\rangle$ that satisfy $\mathcal{L}_0^{\pm}|\phi_i\rangle = 0$, automatically satisfy $\langle\phi_i|\mathcal{L}_n^{\pm}|\phi_j\rangle = 0$, by virtue of the Virasoro algebra and hermiticity of $\mathcal{L}_0^{\pm}$.

We henceforth consider the behaviour of $\mathcal{L}_0$ constraint in what follows.

We find the most general solutions of $\mathcal{L}_0$ sandwich condition fall into either of the following *four* classes:

I. $(\mathcal{L}_0^+ - \mathcal{L}_0^-)|phys\rangle = 0, \qquad (\mathcal{L}_0^+ + \mathcal{L}_0^-)|phys\rangle = 0;$

II. $(\mathcal{L}_0^+ - \mathcal{L}_0^-)|phys\rangle \in \mathcal{H}_C, \qquad (\mathcal{L}_0^+ + \mathcal{L}_0^-)|phys\rangle = 0;$

III. $(\mathcal{L}_0^+ - \mathcal{L}_0^-)|phys\rangle = 0, \qquad (\mathcal{L}_0^+ + \mathcal{L}_0^-)|phys\rangle \in \mathcal{H}_C;$

IV. $(\mathcal{L}_0^+ - \mathcal{L}_0^-)|phys\rangle \in \mathcal{H}_C, \qquad (\mathcal{L}_0^+ + \mathcal{L}_0^-)|phys\rangle \in \mathcal{H}_C;$

Solutions of the right action constraint are all covered in **Class I**, while **Class II, III** and **IV** do not have any counterparts in the usual closed strings Hilbert space.

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Construction of general solutions

We consider positive and negative eigenvalues of $\mathcal{L}_0^+ \pm \mathcal{L}_0^-$ as $\lambda_\pm$ with

$$(\mathcal{L}_0^+ \pm \mathcal{L}_0^-)|\lambda_+, \mathfrak{s}_+; \lambda_-, \mathfrak{s}_-\rangle = \mathfrak{s}_\pm \lambda_\pm |\lambda_+, \mathfrak{s}_+; \lambda_-, \mathfrak{s}_-\rangle, \quad \lambda_\pm \geq 0 \tag{11}$$

For **Class I** we have $\lambda_\pm = 0$.

For **Class II** we have $\lambda_+ = 0$, $\lambda_- \neq 0$.

For **Class III** we have $\lambda_- = 0$, $\lambda_+ \neq 0$

and for **Class IV** we have $\lambda_\pm \neq 0$.

Construction of general solutions

To solve for the $(\mathcal{L}_0^+ \pm \mathcal{L}_0^-)|phys\rangle \in \mathcal{H}_{\mathrm{C}}$ conditions, consider a full set of states of the form

$$\begin{aligned} |\lambda_\pm; s_1, s_2\rangle := \mathcal{N}\Big[&|\lambda_+, +; \lambda_-, +\rangle + s_1|\lambda_+, -; \lambda_-, +\rangle \\ &+ s_2|\lambda_+, +; \lambda_-, -\rangle + s_1 s_2|\lambda_+, -; \lambda_-, -\rangle \Big], \end{aligned} \tag{12}$$

where s_1, s_2 may be chosen to be -1 or 1.

Construction of general solutions

We have

$$(\mathcal{L}_0^+ - \mathcal{L}_0^-)|\lambda_\pm; s_1, s_2\rangle = \lambda_- \; |\lambda_\pm; s_1, -s_2\rangle,$$
$$(\mathcal{L}_0^+ + \mathcal{L}_0^-)|\lambda_\pm; s_1, s_2\rangle = \lambda_+ \; |\lambda_\pm; -s_1, s_2\rangle, \tag{13}$$

Therefore we can choose explicitly

$$\textbf{Class I}: \quad |0, +; 0, +\rangle \tag{14a}$$

$$\textbf{Class II}: \quad \frac{1}{\sqrt{2}}\left(|0, +; \lambda_-; +\rangle + |0, +; \lambda_-; -\rangle\right) \tag{14b}$$

$$\textbf{Class III}: \quad \frac{1}{\sqrt{2}}\left(|\lambda_+, +; 0; +\rangle + |\lambda_+, -; 0; +\rangle\right) \tag{14c}$$

$$\textbf{Class IV}: \quad \frac{1}{2}\Big[|\lambda_+, +; \lambda_-, +\rangle + |\lambda_+, -; \lambda_-, +\rangle + |\lambda_+, +; \lambda_-, -\rangle + |\lambda_+, -; \lambda_-, -\rangle\Big]. \tag{14d}$$

Construction of general solutions

We will see that these classes can be associated with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry that divides the total Hilbert space $\mathcal{H}_{\mathrm{T}}$ into four sectors, as summarized in the Table.

States	$\mathbb{Z}_2(\mathrm{P})$	$\mathbb{Z}_2(\mathrm{T})$
Class I	✓	✓
Class II	×	✓
Class III	✓	×
Class IV	×	×

Table: A summary of the characteristics for four classes of SVC solutions.

Construction of general solutions

We can see that

$$(\mathcal{L}_0^+ \pm \mathcal{L}_0^-)^2 |\lambda_\pm; s_1, s_2\rangle = \lambda_\pm^2 \, |\lambda_\pm; s_1, s_2\rangle. \tag{15}$$

That is $|\lambda_\pm; s_1, s_2\rangle$, while not eigenstates of $\mathcal{L}_0^+ \pm \mathcal{L}_0^-$, are eigenstates of $(\mathcal{L}_0^+ \pm \mathcal{L}_0^-)^2$.

For either of the four Classes we have

$$(\mathcal{L}_0^+ \pm \mathcal{L}_0^-)^2 |\psi^p\rangle \in \mathcal{H}_\text{P}, \qquad (\mathcal{L}_0^+ \pm \mathcal{L}_0^-)^2 |\psi^c\rangle \in \mathcal{H}_\text{C}. \tag{16}$$

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Example of Quantized String Theory

For tensile string theory we have

$$\begin{aligned} \mathcal{L}_0^+|r,s;k^\mu\rangle &= (r - \frac{m^2}{4} - a)|r,s;k^\mu\rangle, \\ \mathcal{L}_0^-|r,s;k^\mu\rangle &= (s - \frac{m^2}{4} - a)|r,s;k^\mu\rangle, \end{aligned} \tag{17}$$

where a is the zero point energy, r, s are non-negative integers, k^μ denotes the momentum of center of mass of strings and $m^2 := -k^2$.

$$\lambda_- = |r - s|, \qquad \lambda_+ = |r + s - \frac{m^2}{2} - 2a|. \tag{18}$$

Example of Quantized String Theory

Class I states have $\lambda_\pm = 0$ are the usual level-matched string states of the form

$$|r, r; k^\mu\rangle, \qquad m^2 = 4(r - a) \tag{19}$$

and are hence labeled by a single non-negative integer.

Class II states have $\lambda_+ = 0$ and $\lambda_- \neq 0$:

$$|r, s; k^\mu\rangle + |s, r; k^\mu\rangle, \qquad m^2 = 2(r + s) - 4a, \qquad r > s \tag{20}$$

So they are labeled by two non-negative integers $r > s$.

Example of Quantized String Theory

Class III states have $\lambda_- = 0$ and $\lambda_+ \neq 0$, states of the form

$$|r, r; k^\mu\rangle + |s, s; k^\mu\rangle, \qquad m^2 = 2(r+s) - 4a, \qquad r > s \tag{21}$$

So they are labeled by two non-negative integers $r > s$.

Class IV states have $\lambda_+, \lambda_- \neq 0$ and are generically quantified by three independent ordered integers $r > s > |q|$:

$$\begin{aligned} &\frac{1}{2}(|r_1, s_1; k^\mu\rangle + |r_2, s_2; k^\mu\rangle + |s_1, r_1; k^\mu\rangle + |s_2, r_2; k^\mu\rangle), \\ &r_1 = r + q, \quad s_1 = s + q, \quad r_2 = r - q, \quad s_2 = s - q, \end{aligned} \tag{22}$$

or $2r = r_1 + r_2, 2s = s_1 + s_2, 2q = r_1 - r_2 = s_1 - s_2$, with $m^2 = 2(r+s) - 4a$.

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$\mathbb{Z}_2 \times \mathbb{Z}_2$ as Non-inertial Worldsheet Symmetry

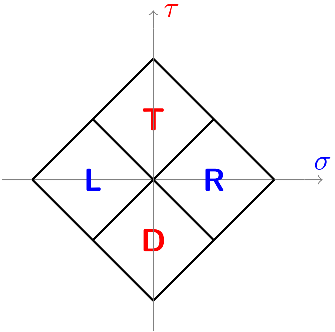

Figure: Penrose diagram of 2d Minkowski space as a 4-fold cover of R (or L or T or D) sectors. Each of the quadrants is conformal to Minkowski. L and R quadrants are two copies of what a Rindler observer have access to and T and D are two copies of a Milne space. String worldsheet Rindler or Milne observers hence have access to Minkowski$/\mathbb{Z}_2 \times \mathbb{Z}_2$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$ as Non-inertial Worldsheet Symmetry

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2, && x, t \in \mathbb{R}, && \text{Minkowski} \\
&= e^{\pm 2\kappa\sigma}(-d\tau^2 + d\sigma^2), && \sigma, \tau \in \mathbb{R}, && \text{Right/Left Rindler,} \\
&= e^{\pm 2\kappa\tau}(-d\tau^2 + d\sigma^2), && \sigma, \tau \in \mathbb{R}, && \text{Top/Down Milne,}
\end{aligned}
$$

The four quadrants R, T, L and D are related by $\mathbb{Z}_2 \times \mathbb{Z}_2$ null orientifoldings: $\tau \pm \sigma \to 2\pi - (\tau \pm \sigma)$. Explicitly,

$$
\begin{aligned}
R \to T: &\ \tau + \sigma \to 2\pi - (\tau + \sigma), && \tau - \sigma \to \tau - \sigma \\
T \to L: &\ \tau + \sigma \to \tau + \sigma, && \tau - \sigma \to 2\pi - (\tau - \sigma) \\
L \to D: &\ \tau + \sigma \to 2\pi - (\tau + \sigma), && \tau - \sigma \to \tau - \sigma \\
D \to R: &\ \tau + \sigma \to \tau + \sigma, && \tau - \sigma \to 2\pi - (\tau - \sigma)
\end{aligned}
\tag{23}
$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ and SVC

Under worldsheet parity P and time-reversal T,

$$\mathcal{L}_n^{\pm\mathrm{M}} \xrightarrow{x\to -x} \mathcal{L}_n^{\mp\mathrm{M}}, \qquad \mathcal{L}_n^{\pm\mathrm{M}} \xrightarrow{t\to -t} -\mathcal{L}_{-n}^{\mp\mathrm{M}}, \tag{24}$$

whereas the combinations

$$J_n^{\mathrm{M}} := \mathcal{L}_n^{+\mathrm{M}} + \mathcal{L}_n^{-\mathrm{M}}, \quad P_n^{\mathrm{M}} := \mathcal{L}_n^{+\mathrm{M}} - \mathcal{L}_{-n}^{-\mathrm{M}} \tag{25}$$

are the only ones that do not mix under P and T,

$$(J_n^{\mathrm{M}}, P_n^{\mathrm{M}}) \xrightarrow{x\to -x} (J_n^{\mathrm{M}}, -P_{-n}^{\mathrm{M}}), \;\; (J_n^{\mathrm{M}}, P_n^{\mathrm{M}}) \xrightarrow{t\to -t} (-J_{-n}^{\mathrm{M}}, P_n^{\mathrm{M}}). \tag{26}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ and SVC

J_n^{M} and P_n^{M} are respectively invariant under P and T.

• **Class I** states are invariant under both P, T, **Class II** (or **III**) states that keep T (or P), are zero-eigenstates of J_0^{M} (or P_0^{M}) and **Class IV** states keep neither of P and T.

Mapping R, T, L and D Virasoro generators to each other, leads to the full set of identifications:

$$\mathcal{L}_n^{\pm\mathrm{T}} = \mp\mathcal{L}_{\mp n}^{\pm\mathrm{R}}, \qquad \mathcal{L}_n^{\pm\mathrm{L}} = -\mathcal{L}_{-n}^{\pm\mathrm{R}}, \qquad \mathcal{L}_n^{\pm\mathrm{D}} = \pm\mathcal{L}_{\pm n}^{\pm\mathrm{R}}, \tag{27}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ and SVC

The combinations

$$\begin{aligned} P_n^{\mathrm{R}} &= \mathcal{L}_n^{+\mathrm{R}} + \mathcal{L}_{-n}^{-\mathrm{R}} = \mathcal{L}_n^{+\mathrm{R}} - \mathcal{L}_n^{-\mathrm{L}}, \\ J_n^{\mathrm{R}} &= \mathcal{L}_n^{+\mathrm{R}} - \mathcal{L}_{-n}^{-\mathrm{R}} = \mathcal{L}_n^{+\mathrm{R}} + \mathcal{L}_n^{-\mathrm{L}} \end{aligned} \tag{28}$$

do not mix under P and T and are invariant under either of P or T.

• This analysis suggests that to describe the same physics as seen by Minkowski observer with **Class I** Hilbert space (textbook string theory), a Rindler observer should use **Class I** and **III** Hilbert spaces and a Milne observer that of **Class I** and II.

$\mathbb{Z}_2 \times \mathbb{Z}_2$ and SVC

• One can consider a more general case of non-uniform acceleration for which the Weyl factor is $e^{-2W(\tau,\sigma)}$, breaking both of the discrete symmetries. This gives rise to **Class IV** states.

• The case with constant $\partial_a W$ keep either (or both) of the $\mathbb{Z}_2$'s and yield **Class I** states (for $\partial_a W = 0$), **Class II** states ($\partial_a W$ a constant timelike vector), **Class III** states ($\partial_a W$ a constant spacelike vector).

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Beyond **Class I** states

Bogoliubov transformations relating a worldsheet Rindler observer with acceleration κ and a Minkoswki one are given in terms of the unitary *squeezing operator*

$$\beta_n(\kappa) = e^{iG(\kappa)}\alpha_n e^{-iG(\kappa)}, \quad \tilde{\beta}_n(\kappa) = e^{iG(\kappa)}\tilde{\alpha}_n e^{-iG(\kappa)}, \tag{29a}$$

$$G(\kappa) = 2\kappa\left[\alpha_0\cdot x - \tilde{\alpha}_0\cdot\tilde{x} - \sum_{n=1}^{\infty}\frac{i}{n}\left(\alpha_{-n}\tilde{\alpha}_{-n} - \alpha_n\tilde{\alpha}_n\right)\right]. \tag{29b}$$

The associated vacua are related as $|0(\kappa); k^\mu\rangle_\beta = e^{iG(\kappa)}|0; k^\mu\rangle_\alpha$. Minkowski and Rindler Virasoro generators are then related as $\mathcal{L}_n^{\pm\mathcal{R}} = e^{iG(\kappa)}\mathcal{L}_n^{\pm\mathrm{M}}e^{-iG(\kappa)}$.

Beyond **Class I** states

$$J_n^{\mathcal{R}} = e^{iG(\kappa)}\ J_n^{\mathrm{M}}\ e^{-iG(\kappa)}, \qquad P_n^{\mathcal{R}} = e^{iG(\kappa)}\ P_n^{\mathrm{M}}\ e^{-iG(\kappa)}. \tag{30}$$

In particular,

$$P_0^{\mathcal{R}} = P_0^{\mathrm{M}} = N - \tilde{N}, \quad J_0^{\mathcal{R}} = \cosh 2\kappa\ J_0^{\mathrm{M}} - \sinh 2\kappa\ \mathcal{T}_0, \tag{31}$$

with $J_0^{\mathrm{M}} = 2\alpha_0^2 + N + \tilde{N} - 2a,\ N = 2\sum_{m>0}\alpha_{-m}\cdot\alpha_m\ ,\tilde{N} = 2\sum_{m>0}\tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m,$ and

$$\mathcal{T}_0 = 2\sum_{m>0}\alpha_{-m}\cdot\tilde{\alpha}_{-m} + h.c. \tag{32}$$

Considering a **Class I** state $|r, r; k^\mu\rangle$, we can see that $J_0^{\mathcal{R}}|r, r; k^\mu\rangle$ is nonzero and belongs to H_c. Therefore these are like **Class III** states for the Rindler observer.

Conclusions

• While 2d Rindler and Minkowski metrics are (locally) the same up to a Weyl factor, the Rindler covers only a quarter of Minkowski space. There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry associated with worldsheet parity P and time-reversal T and even while considering the Weyl factor, Rindler and Minkowski worldsheets are *distinguishable* through their behavior under this $\mathbb{Z}_2 \times \mathbb{Z}_2$.

• **Class II, III** or **IV** states are not level-matched or on-shell in the usual string theory terminology. Our explicit construction exhibits that these states are necessitated with horizons on the worldsheet.

• It has been argued that worldsheet Rindler observers are ultimately apt to study strings probing black hole horizons in the target spacetime, or when one heats up a gas of strings. Thus, our analysis and results here, besides adding a couple of pages to string theory textbooks, is crucial to study strings probing black holes.

Thank you.