
بسم الله الرحمن الرحیم

Angular Momentum Flux Non-Invariance in Asymptotically Flat Spacetimes

Based on RJ and Massimo Porrati, Phys. Rev. Lett. 132 .151604.

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Outline

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4. Conclusion

Angular momentum problem

1. Asymptotically flat metrics in the Bondi gauge $(x^A = (\theta, \phi))$,

$$ds^2 = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B + \frac{2m(u,x)}{r} du^2$$
$$+ rC_{AB}(u,x) dx^A dx^B + D^B C_{AB} du dx^A$$
$$\frac{C_{AB}C^{AB}}{16r^2} dudr + \left(\frac{4N_A(u,x)}{3} - \frac{1}{8} D_A(C_{CB}C^{CB})\right)\frac{dudx^A}{r}$$
$$+ \ldots .$$

Angular momentum problem

1. Asymptotically flat metrics in the Bondi gauge $(x^A = (\theta, \phi))$,

$$ds^2 = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B + \frac{2m(u,x)}{r} du^2 + rC_{AB}(u,x) dx^A dx^B + D^B C_{AB} du dx^A \frac{C_{AB}C^{AB}}{16r^2} dudr + \left(\frac{4N_A(u,x)}{3} - \frac{1}{8} D_A(C_{CB}C^{CB})\right) \frac{dudx^A}{r} + \dots .$$

2. The functions $m(u, x^A)$, $N_A(u, x^A)$ and C_{AB} are the mass aspect, the angular momentum aspect and the shear tensor.

Angular momentum problem

1. An element ξ of the BMS algebra is a vector field (ξ) allowed by the Bondi gauge condition and asymptotic form of the metric

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$$\xi^u = f(x) + \frac{u}{2} D.Y,$$
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2. $f(x)$ is a "supertranslation" and Y^As are Lorentz generators. Shear tensor C_{AB}, changes as $C_{AB} - (2D_A D_B - q_{AB} D^2)f$.

Electric shear

1. Shear tensor is a symmetric traceless tensor and has two polarity,

$$D^2(D^2+2)C(u,x) = D^A D^B C_{AB}(u,x),$$

$C(u,x)$ is called "electric shear".

2. The electric shear can be expanded in terms of spherical harmonics,

$$C(u,x) = \sum_{l=0}^{\infty} \mathcal{C}_l(u) Y_l$$

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3.
$$\delta_f C = f \quad , \quad \delta_Y C = Y^A D_A C - \frac{1}{2} D.YC.$$

Electric shear

1. Equation of motion for the mass aspect,

$$\Delta m(\Theta) = \frac{1}{4} D^2 (D^2 + 2) \Delta C(\Theta) - \int_{-\infty}^{+\infty} du T_{uu}(u, \Theta)$$

2. Again one can see the first two modes $\mathcal{C}_{l=0}(u), \mathcal{C}_{l=1}(u)$ are absent, even more they do not appear at all in the asymptotic metric and hence their time dependence is completely arbitrary even if the initial value of all harmonics of C is given at $u = -\infty$.

Invariant Lorentz charge

With the help of electric shear we can construct an intrinsic Lorentz charge,

$$J_Y(u)=\frac{1}{8\pi G}\int_{S^2} d^2\, Y^A\Big[N_A(u,x^A)-3m(u,x^A)D_A C(u,x^A)-D_A m(u,x^A)C(u,x^A)\Big],$$

$$J_Y(u)=J_Y^{can}(u)-j_Y[m(u),C(u)].$$

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j_Y term can be written as,

$$j_Y[m(u),C(u)]=\frac{1}{4\pi G}\int_{S^2} m\delta_Y C=-\frac{1}{4\pi G}\int_{S^2} C\delta_Y m.$$

Energy

1. Needless to say there is no problem with the energy (flux) because it is already supertranslation invariant,

$$E(u) = \frac{1}{4\pi G}\int d^2\Theta\sqrt{h}m(u,\Theta),$$

$$\Delta m(\Theta) = \frac{1}{4}D^2(D^2+2)\Delta C(\Theta) - \int_{-\infty}^{+\infty} du T_{uu}(u,\Theta).$$

Intrinsic charge

$$\delta_Y \, \delta_f \, \Delta J_Z - \delta_f \, \delta_Y \, \Delta J_Z = \delta_{[Y,f]} \, \Delta J_Z.$$

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2. On the other hand, $[Y,f]$ is a supertranslation BUT not a proper one,

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   A proper supertranslation in a frame is an improper supertranslation in a boosted frame.
3. A covariant supertranslation invariant charge must be an intrinsic charge.

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3. The question is how to fix these coefficients in a covariant way.

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These conditions remove the arbitrariness in choosing the origin of the coordinates.

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4. The final form of the charge is

$$\mathfrak{J}^-_Y = J^-_Y - j_Y[m^-, C^-]. \tag{1}$$

Angular momentum flux

To define a flux we also have to fix the $\mathcal{C}_{l=0}, \mathcal{C}_{l=1}$ at $u = \infty$. There are many possibilities, two simple choices are:

i) Set $C^+|_{l\leq 1} = C^-|_{l\leq 1}$.

ii) Solve $J^+_{\tilde{Y}} - j_{\tilde{Y}}[m^+, C^+] = 0$, where $\tilde{Y}^A$ is defined by Lorentz-transforming pure boosts defined in the *final* center of mass rest frame.

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2. Prescription *i* simply corresponds to measuring $\mathfrak{J}^+_Y$ with respect to the initial CMRF.
3. By construction both $\mathfrak{J}^\pm_Y$ are invariant under supertranslations so we can compute them by changing coordinates in the non-radiative far past and far future regions to set $C^\pm_{AB} = 0$.

$$\Delta\mathfrak{J}_Y = J_Y(+\infty)|_{C^+|_{l>1}=0} - J_Y(-\infty)|_{C^-|_{l>1}=0}.$$

Angular momentum flux

4. There are arguments showing that $h_{AB} + O(1/r^2)$ is the frame where the Bondi charge reduces to the ADM charge. Therefore the flux is the canonical charge measured after a gravitational scattering in a "round metric" frame minus the initial canonical charge, also measured in a "round metric" frame.
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   This procedure "forgets" the initial frame-fixing.
5. Prescription *i* includes a term due to the motion of the final CMRF, namely $\Delta\vec{J} = \Delta\vec{J}^{intrinsic} + \vec{a} \times \Delta\vec{P}$, with $\vec{a}$ the displacement of the origin of the final CMRF with respect to the initial CMRF. Prescription *ii* instead gives $\Delta\vec{J} = \Delta\vec{J}^{intrinsic}$. The difference between the two prescriptions amounts to a term proportional to $\Delta\vec{P}$, i.e. the change of the center of mass momentum due to gravitational radiation.

Summary

- Supertranslation and covariance makes the charge invariant under solid translations.
- $l = 0, 1$ modes of the electrical shear do not appear in the asymptotic metric nor they are fixed by EOM. However they are essential for covariance.
- We fix these coefficients by fixing the origin of coordinates.
- To calculate flux we also need to know $l = 0, 1$ modes at the far future. Two important prescriptions are, first measuring the initial and final charge with respect to the initial CMRF or measuring the final charge with respect to the final CMRF.
- The second one coincides with the ADM flux.