Timelike Entanglement Entropy

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IPM, IRCHEP (1402)

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- An early motivation: Concerns from dS/CFT;
  Is it possible to understand about the entanglement structure of theories dual to dS geometries via weak rotation of well-known results in AdS?
- The content of this talk:
  - A reminder: Definition of pseudo entanglement
  - Definition of timelike entanglement in QFT
  - TEE is an example of pseudo entanglement
  - Holographic prescription to calculate TEE (v.1)
  - A comment about: TEE in AdS $\Leftrightarrow$ EE in dS/CFT

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$$\boxed{S(\tau_A) = -\mathrm{Tr}_A\left[\tau_A \log \tau_A\right] \quad , \quad \tau_A = \mathrm{Tr}_B\left[\tau_{AB}^{\psi_1|\psi_2}\right]}$$

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- **Weak value** [Aharonov-Albert-Vaidman '88]

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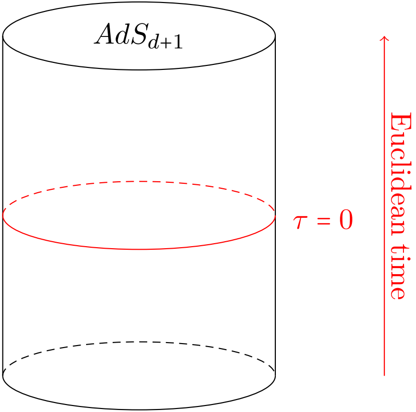
Start with $|\psi_1\rangle$, perform a measurement O_A, discard all outcomes except those which the final state is $|\psi_2\rangle$

Holographic Pseudo Entanglement Entropy



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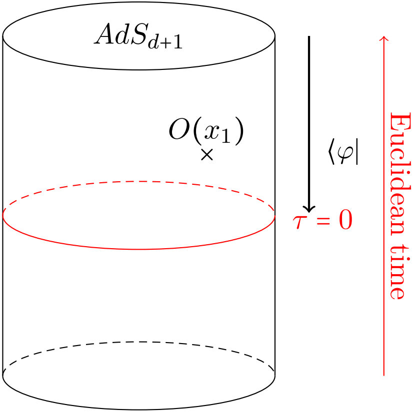

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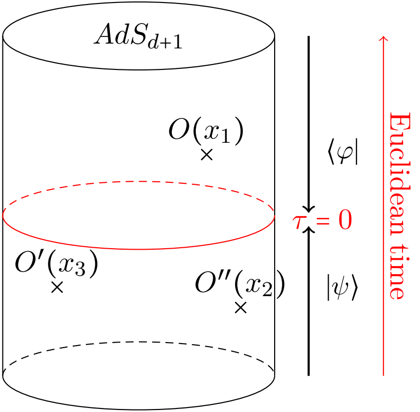

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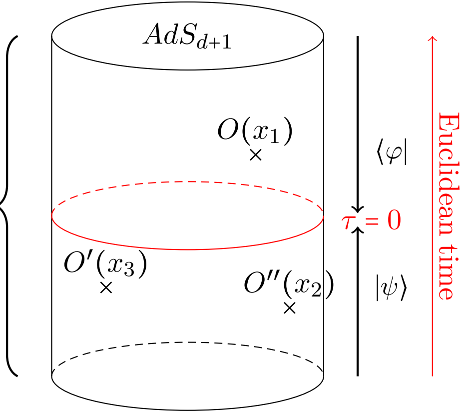

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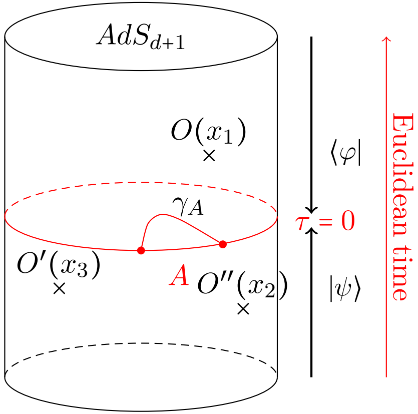

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What is γ_A in CFT?

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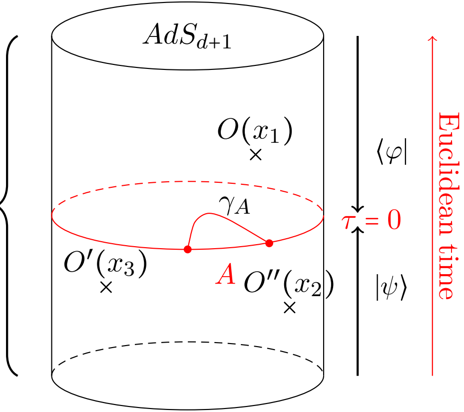

[Nakata-Takayanagi-Taki-Tamaoka-Wei ’20]

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- For A: a single interval

$$\mathcal{Z}_n \propto \langle \sigma_n \bar{\sigma}_n \rangle_{\mathbb{C}} \quad , \quad \Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

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- This leads to

$$\mathrm{Tr}\left[\rho_A^n\right] = \frac{\mathcal{Z}_n}{\mathcal{Z}_1^n} \propto \left(\frac{L_A}{\epsilon}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}$$

Timelike EE in QFT I: Replica Method

- Reminder: For spacelike regions

$$
\begin{aligned}
S_A^{(n)} &= \frac{1}{1-n}\log\langle\sigma_n(P)\bar{\sigma}_n(Q)\rangle \\
&= \frac{1}{1-n}\log\left[\left(\frac{\epsilon}{\sqrt{(x_P-x_Q)^2-(t_P-t_Q)^2}}\right)^{2\Delta_n}\right]
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- Continue to $(\Delta x)^2 - (\Delta t)^2 < 0$, for pure timelike region T_0

$$S_A^{(\mathrm{T})} = \frac{c}{3} \log \frac{T_0}{\epsilon} + \frac{c\pi}{6} i$$

Timelike EE in QFT I: Replica Method

- Finite Size: CFT on a circle (R),
  For a pure timelike region T_0

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- Finite Temperature: CFT at temperature ($1/\beta$),
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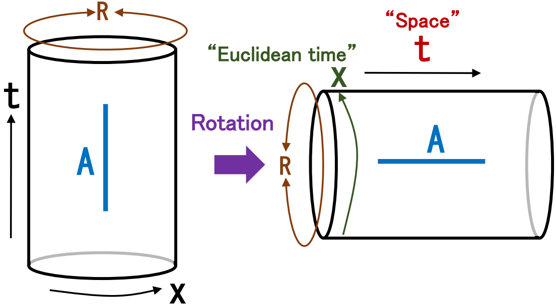
- Consider t: spatial direction, $T = -ix$: real-time



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- Prescription: With $\beta_S \to -iR$ and $m \to -im$, we can find TEE from EE

Timelike EE in QFT II: Wick Rotation of Coordinates

- Finite temperature EE to finite size TEE

$$S_A = \frac{c}{3}\log\left[\frac{\beta_S}{\pi\tilde{\epsilon}}\sinh\frac{\pi X_0}{\beta_S}\right] \xrightarrow[\tilde{\epsilon}\to -i\epsilon]{\beta_S\to -iR}$$

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- Tracing over one copy

$$\mathrm{Tr}_2 |\Psi\rangle\langle\Psi^*| = e^{i(R+i\delta)\tilde{H}}$$

TEE is naturally expressed in terms of **pseudo entropy**

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- Similar to the standard formulation of KG theory

$$H = -i \int d^{d-1} k_y \, dk \, \Omega_{\mathbf{k}} \, a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$$

- In $2d$ case ($\Omega_k = \sqrt{k^2 - m^2}$) the structure of $\mathrm{Tr}_A \left[\tau_A \mathcal{O}_A \right]$

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Numerical Method: Finite Size

- The relevant correlation functions (similarly for $\Pi_{tt'}$)

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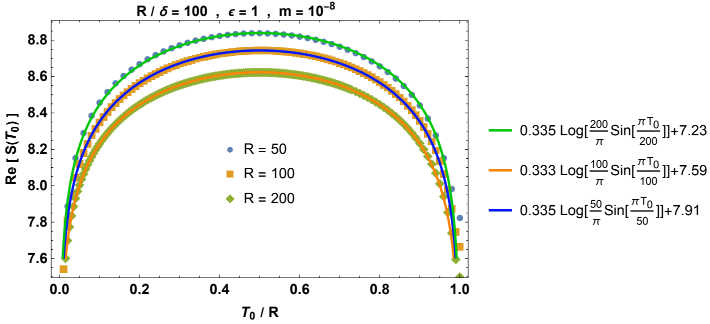

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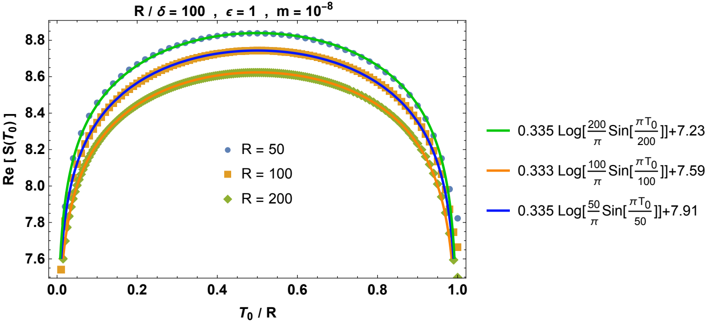

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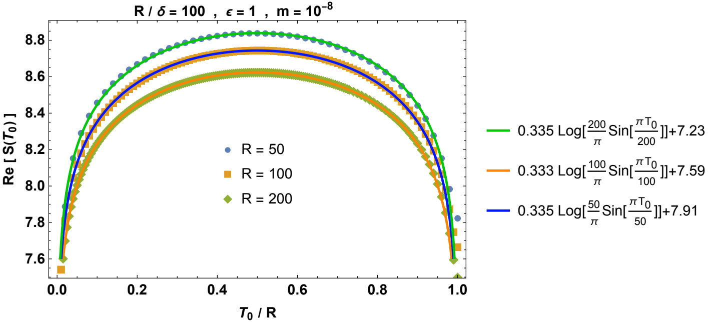

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Holographic Timelike EE in Pure AdS_3

- Poincare AdS_3

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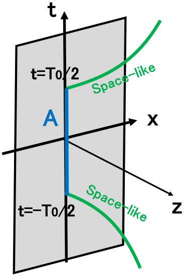

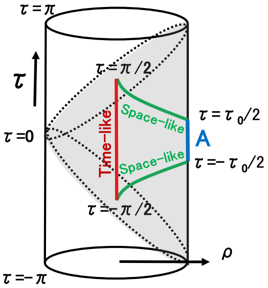

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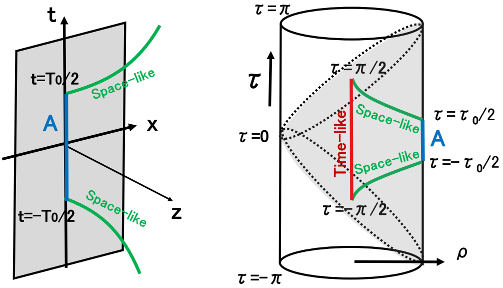

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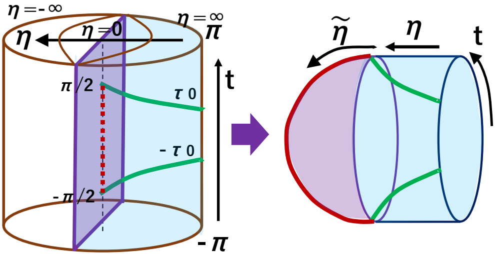

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- Prescription summary:
  1. Construct candidates of Γ_A from a union of spacelike and timelike geodesics such that $\partial\Gamma_A = \partial A$
  2. (Considering Wick rotated geometry for the timelike geodesics), require all variations of the joining points to be stationary

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- Putting three pieces together

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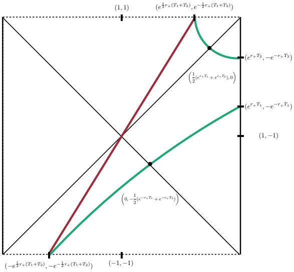

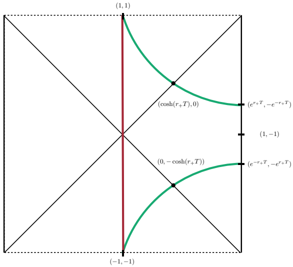

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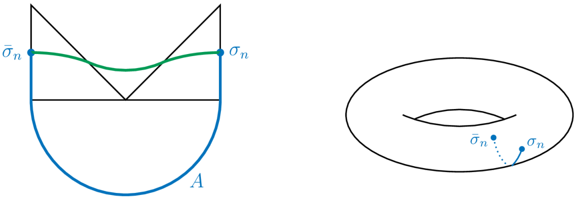

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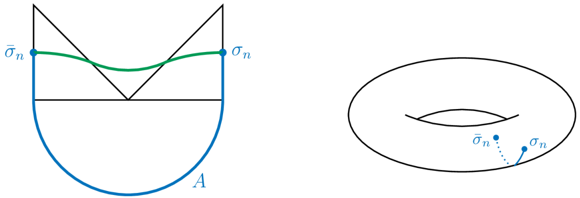

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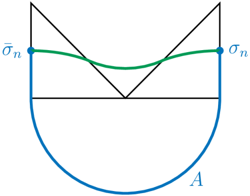

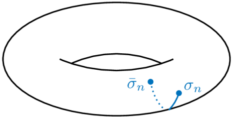

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Holographic Pseudo Entropy in dS_3/CFT_2

- In global dS_3

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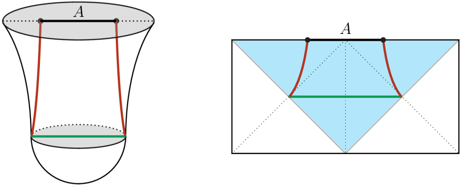

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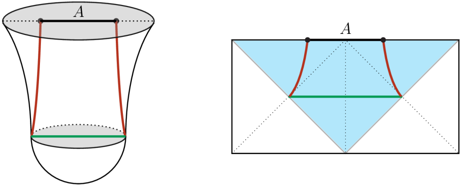

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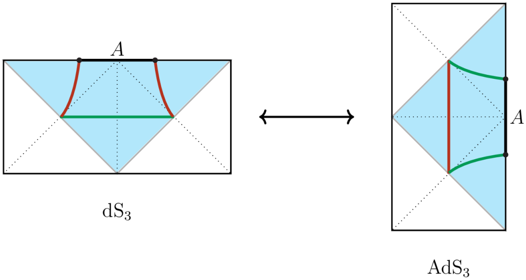

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- For holographic theories, there is an *extermination* procedure justified by extermination the geodesics with real and imaginary values
- Pseudo Entropy in field theories dual to dS spacetime followed by a "*double Wick rotation*" is the same as TEE in field theories dual to AdS

Holographic TEE: Geodesics in Global AdS_3

- Geodesic length between $(\rho_\infty, \frac{T_0}{2}, \phi_0)$ and $(\rho_\infty, -\frac{T_0}{2}, \phi_0)$

$$
\begin{aligned}
D &= \cosh^{-1}\left(\cosh^2\rho_\infty \cos T_0 - \sinh^2\rho_\infty\right)\\
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  same as CFT result with $R = 2\pi$

Numerical Results (Imaginary Part): Finite Size