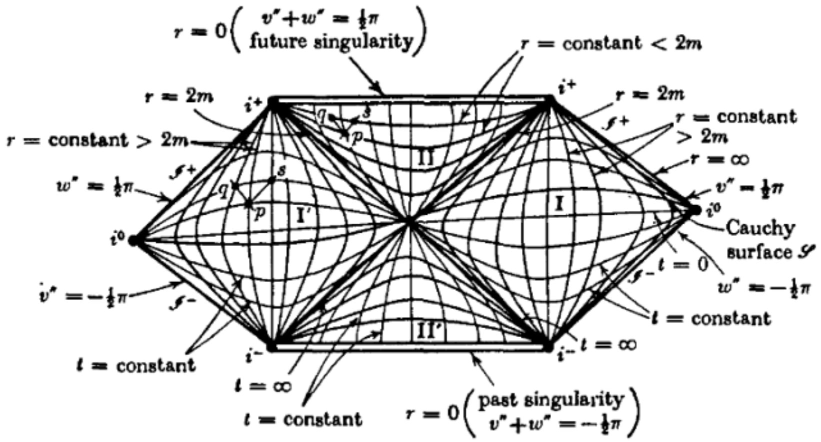
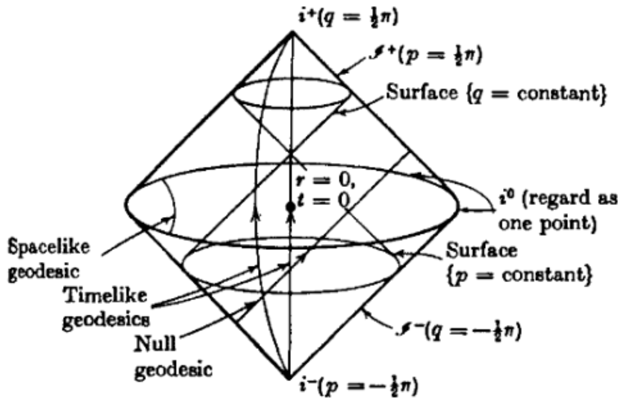
Iranian Conference on High Energy Physics 1402
21/11/2023
Deciphering the Universe Ciphers
School of Physics IPM

Lessons from DLCQ for gravity





Hawking & Ellis

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Motivation & Contents

1) Quantization on null surfaces - Front form of dynamics

2) Simplest example : (Massless) boson in 1+1 dimension

3) Dirac algorithm & characteristic initial value problem

4) Puzzles in single front formulation

5) Double front formulation & matching conditions

6) Decomposition of conformal transformations

with Majumdar, Speziale, Tan
in progress

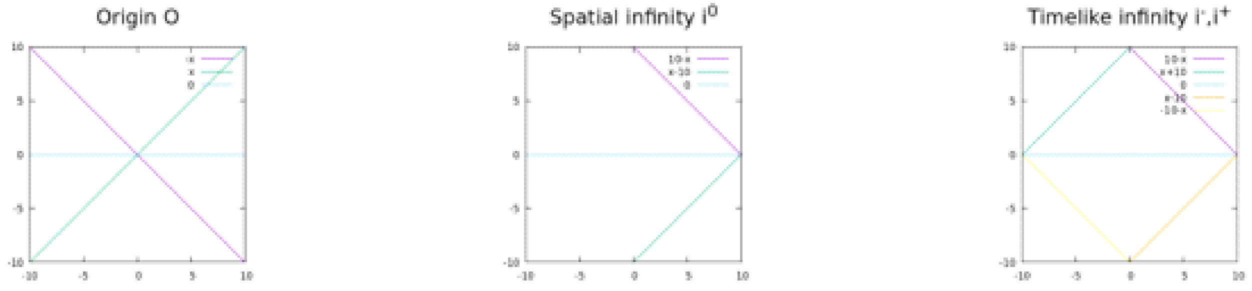
Light-cone Lagrangian analysis

Simplest system: massless boson $\qquad S=\frac{1}{2}\int dx^0 dx^1\, \partial_\mu\phi\, \partial^\mu\phi \qquad ds^2=(dx^0)^2-(dx^1)^2$

Light-cone coordinates: $\qquad x^\pm=\frac{x^0\pm x^1}{\sqrt{2}} \qquad S=\int dx^+ dx^-\, \partial_+\phi\, \partial_-\phi\,, \qquad \partial_+\partial_-\phi=0$

Solution: $\quad \phi=\phi^S_+(x^+)+\phi^S_-(x^-)$ $\qquad$ initial conditions $\quad \phi^S_+(x^+)=\phi(x^+,c^-) \qquad \phi^S_-(x^-)=\phi(c^+,x^-)$

matching condition $\quad \phi^S_+(c^+)=\phi^S_-(c^-)\quad$ at $\left\{\begin{array}{l}\text{intersection}\\ \text{corners}\end{array}\right.$



(a) (b) (c)

Figure 2: Intersecting initial value null lines

Lesson 0:

$\left\{\begin{array}{l} u, r \\ v, r \end{array}\right.$ at $\begin{array}{l}\mathcal{J}^+\\ \mathcal{J}^-\end{array} \longleftrightarrow$ u, v

Kehrberger

Symmetries & currents

$$\frac{\delta \mathcal{L}}{\delta \phi}\,\delta\phi + \partial_\mu j^\mu = 0$$

$$\left\{\begin{array}{l} \delta\phi = \xi^\rho \partial_\rho \phi \qquad j^\mu = T^\mu{}_\rho\, \xi^\rho \\ T_{\mu\rho} = \partial_\mu\phi\,\partial_\rho\phi - \frac{1}{2}\eta_{\mu\rho}\,\partial_\nu\phi\,\partial^\nu\phi \\ \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \partial_\rho \xi^\rho\, \eta_{\mu\nu} \end{array}\right.$$

- constant shift $\quad \delta\phi = c \qquad j^\mu = \partial^\mu \phi$

- conformal trsf

light-cone coordinates
$$\left\{\begin{array}{l} \partial_\pm \xi^\mp = 0 \\ T_{\pm\pm} = (\partial_\pm \phi)^2 , \quad T_{\pm\mp} = 0 \qquad j^\pm = \xi^\mp\, T_{\mp\mp} \end{array}\right.$$

- chiral shifts $\quad \delta^\pm \phi = \epsilon^\pm (x^\pm)$

$$\left\{\begin{array}{l} j^+_{\epsilon^+} = \partial_-\phi\, \epsilon^+ , \quad j^-_{\epsilon^+} = \partial_+\phi\, \epsilon^+ - \phi\, \partial_+ \epsilon^+ \\ j^+_{\epsilon^-} = \partial_-\phi\, \epsilon^- - \phi\, \partial_- \epsilon^- , \quad j^-_{\epsilon^-} = \partial_+\phi\, \epsilon^+ \end{array}\right.$$

∞ # of global symmetries

Single front Hamiltonian analysis

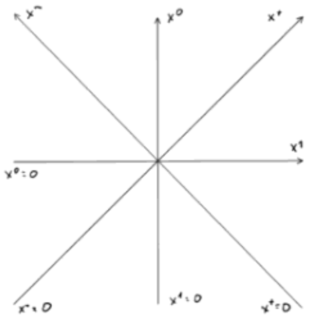


Figure 1: Coordinate axes

Conventions x^+ : "time" evolution direction (Kogut-Spencer)

$$\det\left(\frac{\partial x^+ \partial x^-}{\partial x^0 \partial x^1}\right) = -1 \quad \begin{cases} \text{reversal of orientation} \\ \left(\text{rotation of } -\frac{\pi}{4} + \text{reflection of } x^-\right) \end{cases}$$

primary constraint $\quad g^+ = \pi^+ - \partial_-\phi \,, \qquad H_c \approx 0$

first order action $\quad S_H = \int dx^+ \int_0 dx^- \, \mathcal{L}^+_H \,, \qquad \mathcal{L}^+_H = \left[\pi^+ \partial_+\phi - \lambda^+(\pi^+ - \partial_-\phi)\right]$

$$\partial_+ \pi^+ + \partial_- \lambda^+ = 0 \,, \qquad \pi^+ = \partial_-\phi \,, \qquad \lambda^+ = \partial_+\phi$$

right mover $\qquad$ left mover

fixed on-shell $\qquad$ determined on-shell

NB: A_0 : Lagrange multiplier for Gauss law $\quad A_0 = \frac{1}{\Delta}\left[\partial_0 \vec{\nabla}\cdot\vec{A} + j^0\right]$ from unphysical dof.

Dirac analysis $\quad \{\phi(x^+,x^-), \pi^+(x^+,y^-)\}_+ = \delta(x^-,y^-)$

constraints $\quad G^+[\lambda^+] = \int_0^{L_-} dx^- \, g^+(\lambda^+) \qquad \{G^+[\lambda^{+1}], G^+[\lambda^{+2}]\}_+ = \int_0^{L_-} dx^- \left(\lambda^{+2}\partial_-\lambda^{+1} - (1\leftrightarrow 2)\right)$

$H_c \approx 0 \quad$ no secondary constraints but restrictions on Lagrange multipliers

$$\{g^+, G^+[\lambda^+]\} = -2\partial_-\lambda^+ \approx 0 \Rightarrow \partial_-\lambda^+ = 0 \Rightarrow \lambda^+ = \bar{\lambda}^+_+(x^+)$$

$\Rightarrow$ zero mode of constraint $\bar{g}^+_+ = \int_0^{L_-} dy^- \, g^+$, $\quad G^+[\bar{\lambda}^+] = \bar{g}^+_+ \bar{\lambda}^+$ is first class

$$\begin{cases} \partial_+\phi = \{\phi, G^+(\bar{\lambda}^+)\}_t = \bar{\lambda}^+_+ \\ \partial_+\pi^+ = \{\pi, G^+(\bar{\lambda}^+)\}_t = 0 \end{cases} \Rightarrow \begin{cases} \phi(x^+,x^-) = \int_{c^+}^{x^+} dy^+ \, \bar{\lambda}^+(y^+) + \phi(c^+,x^-) \\ \pi^+ = \pi^+(x^-) \end{cases} \Bigg\| \quad \partial_-\phi = \pi^+ \Rightarrow \phi(x^+,x^-) = \int_{c^-}^{x^-} dy^- \, \pi^+(y^-) + \phi(x^+,c^-)$$

Matching at $x^+ = c^+$ or $x^- = c^-$ $\qquad \phi(x^+,x^-) = \int_{c^+}^{x^+} dy^+ \, \bar{\lambda}^+(y^+) + \int_{c^-}^{x^-} dy^- \, \pi^+(y^-) + \phi(c^+,c^-)$

$$\boxed{\text{Lesson 1: Free data } \bar{\lambda}^+_+(x^+) \text{ at } x^- = c^-, \ \pi^+(x^-) \text{ at } x^+ = c^+, \ \phi(c^+,c^+) \text{ at corner}}$$

Characteristic initial value problem

Puzzle 1: first class constraint $G[\epsilon^+]$, $\epsilon^+(x^+)$ generates global but not a gauge symmetry!

$$\delta_{\epsilon^+}\Phi = \{\Phi, G[\epsilon^+]\}_t = \epsilon^+ , \qquad \delta_{\epsilon^+}\pi^+ = \{\pi^+, G[\epsilon^+]\}_t = 0$$

$$\delta_{\epsilon^+} a^+ = \partial_t \epsilon^+$$

Resolution:

REVIEWS OF MODERN PHYSICS VOLUME 21, NUMBER 3 JULY, 1949

Forms of Relativistic Dynamics

P. A. M. Dirac
St. John's College, Cambridge, England

A similar difficulty arises, in a less serious way, with the front form of theory. Waves moving with the velocity of light in exactly the direction of the front cannot be described by physical conditions on the front, and some extra variables must be introduced for dealing with them.

Dirac 1949 "Front form of dynamics"

Henneaux & Teitelboim

1.2. FIRST-CLASS CONSTRAINTS AS GENERATORS OF GAUGE TRANSFORMATIONS

1.2.1. Transformations That Do Not Change the Physical State. Gauge Transformations

The presence of arbitrary functions v^a in the total Hamiltonian tells us that not all the q's and p's are observable. In other words, although the physical state is uniquely defined once a set of q's and p's is given,

the converse is not true—*i.e.*, there is more than one set of values of the canonical variables representing a given physical state. To see how this conclusion comes about, we notice that if we give an initial set of canonical variables at the time t_1 and thereby completely define the physical state at that time, we expect the equations of motion *to fully determine the physical state at other times.* Thus, by definition, any ambiguity in the value of the canonical variables at $t_2 \neq t_1$ should be a physically irrelevant ambiguity.

(Primary) first class constraints generate gauge symmetries in instant form under the assumption that initial data uniquely fix the physical state.

not the case in front form

left mover $\phi^S(x^+)$ does not intersect $x^+ = c^+$

Assumption: fields periodic in x^-, $\forall x^+$

Zero mode & chiral boson sectors

$$\phi(x^+,x^-) = \bar{\Phi}_+(x^+) + \tilde{\Phi}(x^+,x^-)$$

$$J^+ = \bar{J}^+_+(x^+) + \tilde{J}^+_+ \;, \quad \begin{cases} \bar{J}^+_+ = \frac{1}{L_-}\int_0^{L_-} dx^- \, J^+(x^+,x^-) \\ \int_0^{L_-} dx^- \, \tilde{J}^+_+ = 0 \end{cases}$$

idem for $\quad \pi^+(x^+,x^-) = \frac{\bar{\pi}^+_+(x^+)}{L_-} + \tilde{\pi}^+(x^+,x^-)$

$$\{\bar{\Phi}_+, \bar{\pi}^+_+\}_+ = 1, \quad \{\tilde{\Phi}_+(x^-), \tilde{\pi}^+_+(y^-)\}_+ = \delta(x^-,y^-) - \frac{1}{L_-}$$

constraints $\quad \bar{g}^+_+ = \bar{\pi}^+_+ \quad$ first class $\qquad \tilde{g}^+_+ = \tilde{\pi}^+_+ - \partial_-\tilde{\Phi}_+ \quad$ second class

solve in the action

$$S_R = \int_{-\infty}^{+\infty} dx^+ \, L^+_R \;, \quad L^+_R = \bar{\pi}^+_+ \partial_+\bar{\Phi}_+ - \bar{J}^+_+ \bar{\pi}^+_+ + \int_0^{L_-} dx^- \, \tilde{\mathcal{L}}^+ \;, \quad \tilde{\mathcal{L}}^+ = \partial_-\tilde{\Phi}_+ \partial_+\tilde{\Phi}_+$$

looks like pure gauge dof

but information on left mover

finite volume-analog of principal value prescription

Dirac brackets $\quad \{\tilde{g}^+_+(x^-), \tilde{g}^+_+(y^-)\}_+ = 2\,\partial^x_-\,\delta(x^-,y^-) \;\Rightarrow$

$$\begin{cases} \{\tilde{\Phi}_+(x^-), \tilde{\Phi}_+(y^-)\}^* = -\frac{1}{4}\epsilon(x^- - y^-) + \frac{x^- - y^-}{2L_-} & (x) \\ \{\tilde{\Phi}_+(x^-), \tilde{\pi}^+_+(y^-)\}^* = \frac{1}{2}\left[\delta(x^-,y^-) - \frac{1}{L_-}\right] & (xx) \\ \{\tilde{\pi}^+_+(x^-), \tilde{\pi}^+_+(y^-)\}^* = \frac{1}{2}\partial^x_-\,\delta(x^-,y^-) & (xxx) \end{cases}$$

(x) primitive of (xx) without zero-mode

Maskawa & Yamakawa 1976

Progress of Theoretical Physics, Vol. 56, No. 1, July 1976

The Problem of $P^+=0$ Mode in the Null-Plane Field Theory and Dirac's Method of Quantization

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**Research Institute for Fundamental Physics, Kyoto University, Kyoto 606*

(Received January 7, 1976)

The null-plane quantization is studied with the emphasis on the $P^+=0$ mode, by using Dirac's quantization for constrained systems. This mode is eliminated from the Hilbert space and the physical vacuum can be defined in a kinematical way. It enables us to construct the physical Fock space kinematically. Poincaré invariance is also studied in detail.

N·H ELSEVIER

PHYSICS REPORTS

Physics Reports 301 (1998) 299–486

Quantum chromodynamics and other field theories on the light cone

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Received October 1997; editor: R. Petronzio

Dynamics $\quad H_R = \tilde{\partial}^+_+ \,\bar{\pi}^+_+$ only in the zero-mode sector = left movers

conformal symmetries : left chiral half $\quad Q^{+*}_{\xi^+} = \bar{\pi}^+_+ \tilde{\partial}^+_+ \xi^+$ act only on zero-modes

$$\delta_{\xi^+} \tilde{\Phi}_+ = \{\tilde{\Phi}_+ , Q^{+*}_{\xi^+}\}^*_+ = 0$$

right chiral half $\quad Q^{-+}_{\xi^-} = \int_0^{L_-} dx^- \, \xi^- \left[(\partial_- \tilde{\Phi}_+)^2 + \bar{\pi}^+_+ \partial_- \tilde{\Phi}_+ \right]$

$$\delta_{\xi^-} \tilde{\Phi}_+ = \xi^- \partial_- \tilde{\Phi}_+ - \frac{1}{L_-} \int_0^{L_-} dy^- \, \xi^- \partial_- \tilde{\Phi}_+ \, , \quad \delta_{\xi^-} \bar{\Phi}_+ = \frac{1}{L_-} \int_0^{L_-} dy^- \, \xi^- \partial_- \tilde{\Phi}_+$$ mixes sectors !

Puzzle 2: No representation of conformal algebra

$$\{ Q^{+*}_{\xi_1^+} , Q^{+*}_{\xi_2^+} \}^*_+ = 0 \neq Q^{+*}_{[\xi_1^+ , \xi_2^+]}$$

$$\{ Q^{+*}_{\xi_1^-} , Q^{+*}_{\xi_2^-} \}^*_+ = Q^{+*}_{[\xi_1^- , \xi_2^-]}$$ only if there is no zero mode sector

Preliminary attempt at quantization $Z(\beta,\alpha) = \mathrm{Tr}\, e^{-\beta \hat{H} + i\alpha \hat{P}}$

Boundary conditions x^- :

periodic $x^- \sim x^- + L_-$ (finite-volume analog of Christodoulou Klainerman)

Discrete light-cone quantization (DLCQ)

box in a null coordinate

clean separation of zero-mode and oscillator sector

Mode expansion $\Phi_+(x^-) = \sum_{n>0} \frac{1}{\sqrt{2k_- L_-}} \left(a_{k_-} e^{-ik_- x^-} + c.c. \right) = \phi_R(x^-)$, $k_- = \frac{2\pi n}{L_-}$

$$\{ a_{k_-}, a^+_{k'_-} \}^*_+ = -i\,\delta_{n,n'} \iff (*), (**), (***)$$

$$-\beta\, Q^+_{J_0} - i\alpha\, Q^+_{J_1} = \frac{i\tau L}{\sqrt{2}} H^{+R} - \frac{i\bar{\tau} L}{\sqrt{2}} H^{+*} \qquad \tau = \frac{\alpha + i\beta}{L}$$

$$H^{+R} = \int_0^{L_-} dx^- \left(\partial_- \Phi_+\right)^2 = \frac{1}{2} \sum_{n>0} k_- \left(a^*_{k_-} a_{k_-} + a_{k_-} a^*_{k_-} \right), \qquad H^{+*} = \bar{J}^+_+ \bar{\pi}^+_+$$

no contribution if quantized as a pure gauge dof

$$\hat{H}^{+R} = E_0^{+R} + \sum_{n>0} k_- \hat{a}^{\dagger}_{k_-} \hat{a}_{k_-} \qquad E_0^{+R} = \frac{\pi}{L_-} \sum_{n>0} n = -\frac{2\pi}{24 L_-}$$ Casimir energy

partition function $Z(\tau,\bar\tau) = \dfrac{1}{\eta\left(q\left(\frac{L\tau}{\sqrt{2}L_-}\right)\right)}$

the contribution from the left mover & particle zero mode is missing

Results on the other front "time" x^- exchange the roles of left $(+)$ and right $(-)$

$$S = \int dx^- \int_0^{L_+} dx^+ \, \mathcal{L}^-_H \,, \qquad \mathcal{L}^-_H = \pi^- \partial_- \phi - \mathcal{H}^-(\pi^-, \partial_+\phi)$$

$\boxed{\bar{\mathcal{J}}^-_-(x^-) \text{ analog of news}}$

$$\tilde{\phi}_-(x^+) = \sum_{n>0} \frac{1}{\sqrt{2k_+L_+}} \left(\tilde{a}_{k_+} e^{-ik_+x^+} + c.c. \right) = \phi^L(x^+) \qquad \{\tilde{a}_{k_+}, \tilde{a}^*_{k'_+}\} = -i\,\delta_{n,n'} \qquad k_+ = \frac{2\pi}{L_+} n$$

other chiral half of partition function $Z(\tau,\bar\tau) = \dfrac{1}{\eta\left(q\left(\frac{L\tau}{\sqrt{2}L_+}\right)\right)}$

Double front Hamiltonian analysis

renaming Lagrange multipliers $\lambda^+ = \pi^-$, $\lambda^- = \pi^+$

$$S_H = \int dx^+ dx^- \left[\pi^+ \partial_+ \phi + \pi^- \partial_- \phi - \pi^- \pi^+\right] \qquad \pi^+ = \partial_- \phi \,, \quad \pi^- = \partial_- \phi \,, \quad \partial_+ \pi^+ + \partial_- \pi^- = 0$$

standard instant form periodicity $\Longleftrightarrow$ entangled periodicities in 2 null coord.

$$x^1 \sim x^1 + L \quad \Longleftrightarrow \quad (x^+, x^-) \sim (x^+ + L_+ \,, x^- - L_-) \qquad L_\pm = \frac{L}{\sqrt{2}}$$

Lesson 2

Sectors $\quad \phi(x^+,x^-) = \bar{\phi}_\pm(x^\pm) + \tilde{\phi}_\pm(x^+,x^-) \;; \quad \pi^\pm(x^+,x^-) = \frac{1}{L_\mp}\, \bar{\pi}^\pm_\pm(x^\pm) + \tilde{\pi}^\pm_\pm(x^+,x^-)$

not independent $\quad \frac{1}{L_+}\int_0^{L_+} dx^+\, \bar{\phi}_+(x^+) = \frac{1}{L_-}\int_0^{L_-} dx^-\, \bar{\phi}_-(x^-) \qquad \int_0^{L_\pm} dx^\pm\, \bar{\pi}^\pm_\pm(x^\pm) = \int_0^{L_\mp} dx^\mp\, \bar{\pi}^\pm_\mp(x^\mp)$

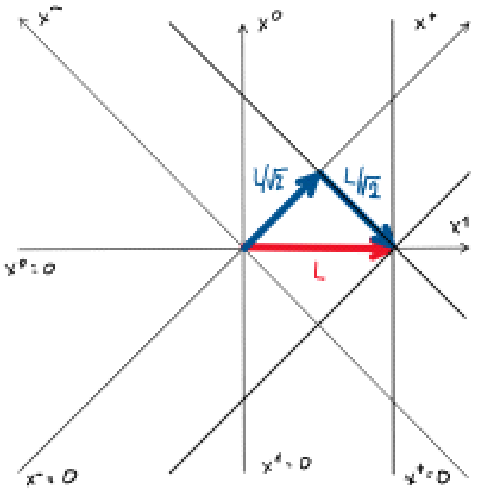


Figure 3: Stokes' theorem

Conserved currents & Stoke's theorem

$\partial_\mu j^\mu \approx 0$

$$j = d^{n-1}x_\mu \, j^\mu , \qquad dj = \partial_\mu j^\mu d^n x \qquad \int_{\partial V} j = \int_V dj \approx 0$$

$$j = dx^1 j^0 - dx^0 j^1 = dx^- j^+ - dx^+ j^-$$

$$j^\pm = -\frac{1}{\sqrt{2}}\left(j^0 \pm j^1\right) \qquad j'^\mu(x') = \left(\det \frac{\partial x}{\partial x'}\right)\frac{\partial x'^\mu}{\partial x^\nu} j^\nu(x)$$

$$Q = \int_0^L dx^1 j^0 \Big|_{x^0=0} \approx -\int_0^{L_+} dx^+ j^- \Big|_{x^-=0} - \int_{-L_-}^{0} dx^- j^+ \Big|_{x^+=L_+}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$ periodicity in x^-

$$-\int_0^{L_-} dx^- j^+ \Big|_{x^+=L_+}$$

NB: if $\partial_+ j^+ = 0 = \partial_- j^-$ separately, the intersection point does not matter
integrals may be evaluated at any $x^\pm = c^\pm$

Conserved symplectic $(2, n-1)$ form

first variational formula $\quad d^n x \; \delta_V \mathcal{L} = d^n x \; \delta_V \phi^i \frac{\delta \mathcal{L}}{\delta \phi^i} + d_H\, a \qquad a = d^{n-1}x_\mu \; a^\mu$

second variational formula $\quad 0 = -d^n x \; \delta_V \phi^i \, \delta_V \frac{\delta \mathcal{L}}{\delta \phi^i} + d_H\, \sigma \qquad \sigma = (-)^{n-1} \delta_V a = d^{n-1}x_\mu \underbrace{\delta_V a^\mu}_{\sigma^\mu}$

$\partial_\mu \sigma^\mu \approx 0$ linearized field equations

$$a = dx^- \pi^+ \delta_V \phi - dx^+ \pi^- \delta_V \phi \qquad \sigma = dx^- \underbrace{\delta_V \pi^+ \delta_V \phi}_{\sigma^+} - dx^+ \underbrace{\delta_V \pi^- \delta_V \phi}_{\sigma^-}$$

$$\partial_+ \sigma^+ \not\approx 0 \not\approx \partial_- \sigma^-$$

non-vanishing Poisson brackets $\quad \{\phi(L_+, x^-), \pi^+(L_+, y^-)\}_+ = \delta(x^-, y^-) \qquad \{\phi(x^+, 0), \pi^-(y^+, 0)\}_- = \delta(x^+, y^+)$

Lesson 3: the Lagrange multipliers are the canonical momenta along rather than off the front

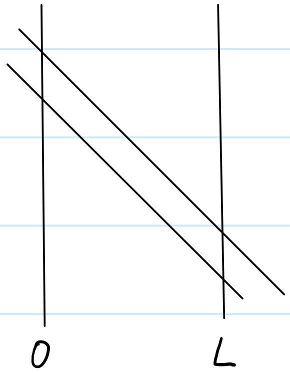
$\lambda^+ = \pi^-, \; \lambda^- = \pi^+$

missing bracket for quantization

more rigorous inversion: Peierls bracket

Matching & equivalent off-shell descriptions of a theory ?

$$S = \frac{1}{2}\int_{-\infty}^{+\infty} dx^0 \int_0^L dx^1\, \partial_\mu \phi\, \partial^\mu \phi \iff \begin{cases} S_H^+ = \int_{-\infty}^{+\infty} dx^+ \left[2\left(\bar{\pi}_+^+ \partial_+ \bar{\Phi}_+ - \frac{1}{L_-} \bar{\pi}_+^- \bar{\pi}_+^+\right) + \int_{x^+-2L_-}^{x^+} dx^- \left(\tilde{\pi}_+^+ \partial_+ \tilde{\Phi}_+ - \tilde{\pi}_+^- \tilde{g}_+^+\right)\right] \\ S_H^- = \int_{-\infty}^{+\infty} dx^- \left[2\left(\bar{\pi}_-^- \partial_- \bar{\Phi}_- - \frac{1}{L_+} \bar{\pi}_-^+ \bar{\pi}_-^-\right) + \int_{x^-}^{x^-+2L_+} dx^+ \left(\tilde{\pi}_-^- \partial_- \tilde{\Phi}_- - \tilde{\pi}_-^+ \tilde{g}_-^-\right)\right] \end{cases}$$



$$S \iff \frac{1}{2} S_H^+ + \frac{1}{2} S_H^- = S^P + \tilde{S}^R[\tilde{\Phi}_+, \tilde{\pi}_+^\pm] + \tilde{S}^L[\tilde{\Phi}_-, \tilde{\pi}_-^\mp]$$

$$\tilde{S}^R = \int_{-\infty}^{+\infty} dx^+ \int_0^{L_-} dx^- \left[\tilde{\pi}_+^+ \partial_+ \tilde{\Phi}_+ - \tilde{\pi}_+^- \tilde{g}_+^+\right]$$ right movers / chiral bosons

$$\tilde{S}^L = \int_{-\infty}^{+\infty} dx^- \int_0^{L_+} dx^+ \left[\tilde{\pi}_-^- \partial_- \tilde{\Phi}_- - \tilde{\pi}_-^+ \tilde{g}_-^-\right]$$ left movers / chiral bosons

$$S^P = \int_{-\infty}^{+\infty} dx^+ \left[\bar{\pi}_+^+ \partial_+ \bar{\Phi}_+ - \frac{1}{L_-} \bar{\pi}_+^- \bar{\pi}_+^+\right] + \int_{-\infty}^{+\infty} dx^- \left[\bar{\pi}_-^- \partial_- \bar{\Phi}_- - \frac{1}{L_+} \bar{\pi}_-^+ \bar{\pi}_-^-\right]$$

old problem : massless KG $\iff$ left + right chiral bosons up to zero mode on-shell field $\phi(x^+,x^-) = \phi(0,0) + \frac{x^- \bar{\pi}^+_+(L_+)}{L_-} + \frac{x^+ \bar{\pi}^-_-(0)}{L_+} + \int_0^{x^-} dy^- \, \tilde{\pi}^+_+(L_+,y^-) + \int_0^{x^+} dy^+ \, \tilde{\pi}^-_-(y^+,0)$

$\partial_\mu \pi^\mu \approx 0$ conserved current but $\pi'^\mu(x') = \left|\det \frac{\partial x}{\partial x'}\right| \frac{\partial x'^\mu}{\partial x^\nu} \pi^\nu(x)$

$\partial_+ \pi^+ \approx 0$ & $\partial_- \pi^- \approx 0$ $\qquad \pi^\pm = \pm \frac{\pi^0 \pm \pi^1}{\sqrt{2}}$

Stokes' theorem

$$\bar{\pi}^0_0 = \int_0^L dx^1 \, \pi^0 = + \int_0^{L_+} dx^+ \, \pi^- \big|_{x^-=0} + \int_0^{L_-} dx^- \, \pi^+ \big|_{x^+=L_+} = \bar{\pi}^-_-(0) + \bar{\pi}^+_+(L_+) = \boxed{\bar{\pi}^-_-(0) + \bar{\pi}^+_+(0)}$$

$$\phi(x^+,x^-) = \phi(0,0) - \phi^R(0) - \phi^L(0) + \frac{x^- \bar{\pi}^+_+(0)}{L_-} + \frac{x^+ \bar{\pi}^-_-(0)}{L_+} + \phi_R(x^-) + \phi_L(x^+)$$

entangled periodicity iff matching condition $\bar{\pi}^+_+(0) = \bar{\pi}^-_-(0) = \frac{1}{2} \bar{\pi}^0_0$

to be completed: (i) prove that zero mode sector corresponds to single free particle

$\Rightarrow$ correct partition function $Z(\tau,\bar{\tau}) = \frac{1}{\sqrt{\tau_2}} \frac{1}{\eta(q(\tau))} \frac{1}{\eta(q(\bar{\tau}))}$ particle zero mode

(ii) show that conformal trsf are generated by Noether charge in Peierls bracket and that the Peierls bracket of charges form a realization of the algebra.

take inspiration from instant form:

$$S_H = \int_{x_i^0}^{x_f^0} dx^0 \int_0^L dx^1 \, \mathcal{L}^0_H \,, \quad \mathcal{L}^0_H = \pi^0 \partial_0 \phi - \mathcal{H} \qquad \mathcal{H} = \frac{1}{2}\left((\pi^0)^2 + (\partial_1\phi)^2\right)$$

(formulation with auxiliary field π^1: $\quad S'_H = \int dx^0 dx^1 \, \mathcal{L}'_H \,, \quad \mathcal{L}'_H = \pi^\mu \partial_\mu \phi - \frac{1}{2}\pi^\mu \pi_\mu \qquad \partial_\rho\phi - \pi_\rho = 0 \qquad \partial_\mu \pi^\mu = 0$)

$\pi^1 \approx -\partial_1 \phi$

periodic boundary conditions $x^1 \sim x^1 + L$ $\qquad \{\phi(x^1), \pi^0(y^1)\}_0 = \delta^P(x^1, y^1) = \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{ik(x^1-y^1)} \qquad k = \frac{2\pi}{L} n$

decomposition $\quad \phi(x^0, x^1) = \bar{\phi}_0(x^0) + \tilde{\phi}_0(x^0, x^1) \,, \quad \pi^0(x^0, x^1) = \frac{\bar{\pi}^0_0}{L}(x^0) + \tilde{\pi}^0_0(x^0, x^1)$

zero mode: free particle $\quad S_P = \int dx^0 \left[\bar{\pi}^0_0 \partial_0 \bar{\phi}_0 - \frac{1}{2L}(\bar{\pi}^0_0)^2\right] \qquad \{\bar{\phi}_0, \bar{\pi}^0_0\}_0 = 1$

$\{\tilde{\phi}_0(x^1), \tilde{\pi}^0_0(y^1)\} = \delta^P(x^1, y^1) - \frac{1}{L} \quad (*)$

split off-shell $\tilde{\phi}_0$, $\tilde{\pi}^0_0$ into left & right chiral fields without zero modes HT, Chiral forms

$$\begin{cases} \phi^L = \frac{1}{2}\left[\tilde{\Phi}_0 + \int_0^{x^1} dy^1\, \tilde{\pi}^0_0 - \frac{1}{L}\int_0^L dy^1 \int_0^{y^1} dz^1\, \tilde{\pi}^0_0\right] \\ \phi^R = \frac{1}{2}\left[\tilde{\Phi}_0 - \int_0^{x^1} dy^1\, \tilde{\pi}^0_0 + \frac{1}{L}\int_0^L dy^1 \int_0^{y^1} dz^1\, \tilde{\pi}^0_0\right] \end{cases} \Rightarrow \begin{cases} \partial_1 \phi^L = \frac{1}{2}(\partial_1 \tilde{\Phi}_0 + \tilde{\pi}^0_0) \\ \partial_1 \phi^R = \frac{1}{2}(\partial_1 \tilde{\Phi}_0 - \tilde{\pi}^0_0) \end{cases}$$

$$S_H = S^P + \underbrace{\int dx^0 \int_0^L dx^1 \left[\partial_1\phi^L \partial_0 \phi^L - (\partial_1 \phi^L)^2\right]}_{\text{left chiral bosons } S^L} + \underbrace{\int dx^0 \int_0^L \left[-\partial_1\phi^R \partial_0\phi^R - (\partial_1 \phi^R)^2\right]}_{\text{right chiral bosons } S^R} + \underbrace{\frac{1}{2}\left[\int_0^L dx^1 \left[\partial_1\phi^L\phi^R - \phi^L\partial_1\phi^R\right]\right]_{x^0_i}^{x^0_f}}_{\text{boundary term } S^B}$$

$$\partial_1^x \upsilon^P(x^1,y^1) = -\partial_1^y \upsilon^P(x^1,y^1) = \delta^P(x^1,y^1) - \frac{1}{L} \qquad \upsilon^P(x^1,y^1) = \sum_{n\geq 0} \frac{1}{\pi n} \sin k(x^1-y^1) = \frac{1}{2}\epsilon(x^1-y^1) - \frac{x^1-y^1}{L}$$

$$\epsilon(x) = \sum_{n\geq 0} \frac{2(1-\cos \pi n)}{\pi n} \sin k x \qquad \frac{x}{L} = -\sum_{n\geq 0} \frac{\cos \pi n}{\pi n} \sin k x$$

(x) after change of variables $\begin{cases} \{\phi^R(x^1), \phi^R(y^1)\}_0 = \frac{1}{4}\epsilon(x^1-y^1) - \frac{x^1-y^1}{2L} = -\{\phi^L(x^1), \phi^L(y^1)\}_0 \\ \{\phi^L(x^1), \phi^R(y^1)\}_0 = 0 \end{cases}$

mode expansion of on-shell fields

$$\phi = \Phi_0(0) + \frac{x^0}{L}\,\bar{\pi}^0_0(0) + \underbrace{\sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left(a_k\, e^{-ik_- x^-} + c.c. \right)}_{\phi_R(x^+)} + \underbrace{\sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left(\tilde{a}_k\, e^{-ik_+ x^+} + c.c. \right)}_{\phi_L(x^-)}$$

same as before

Conformal transformations $\quad \partial_0 \xi^0 = \partial_1 \xi^1 , \quad \partial_1 \xi^0 = \partial_0 \xi^1$

$$\delta^H_\xi \phi = \xi^0 \pi^0 + \xi^1 \partial_1 \phi , \quad \delta^H_\xi \pi^0 = \partial_1 \left(\xi^0 \partial_1 \phi + \xi^1 \pi^0 \right) \qquad \frac{\delta \mathcal{L}_H}{\delta \phi} \delta^H_\xi \phi + \frac{\delta \mathcal{L}_H}{\delta \pi^0} \delta^H_\xi \pi^0 + \partial_\mu j^\mu_\xi = 0$$

$$j^0_\xi = \xi^0 \mathcal{H} + \xi^1 \mathcal{P} \qquad j^1_\xi = -\xi^0 \mathcal{P} - \xi^1 \mathcal{H} - \left(\xi^0 \partial_1 \phi + \xi^1 \pi^0 \right)\left(\partial_0 \phi - \pi^0 \right) \qquad \mathcal{P} = \pi^0 \partial_1 \phi$$

$$Q_\xi = \int_0^L dx^1 \, j^0_\xi \qquad \left\{ Q_{\xi_1}, Q_{\xi_2} \right\} = Q_{[\xi_1, \xi_2]}$$

canonical realization of conformal algebra

Decomposition $\quad \delta_\xi \bar\Phi_0 = \frac{\bar\Pi_0^0}{L^2}\int_0^L dy^1\, \xi^0 + \frac{1}{L}\int_0^L dy^1 [\xi^0 \tilde\Pi_0^0 + \xi^1 \partial_1 \tilde\Phi_0]$, $\quad \delta_\xi \bar\Pi_0^0 = 0$

$$\delta_\xi \tilde\Phi_0 = \xi^0 \tilde\Pi_0^0 + \xi^1 \partial_1 \tilde\Phi_0 - \frac{1}{L}\int_0^L dy^1 (\xi^0 \tilde\Pi_0^0 + \xi^1 \partial_1 \tilde\Phi_0) + \left(\xi^0 - \frac{1}{L}\int_0^L dx^1 \xi^0\right)\frac{\bar\Pi_0^0}{L}$$

$$\delta_\xi \tilde\Pi_0^0 = \partial_1 (\xi^0 \partial_1 \tilde\Phi_0 + \xi^1 \tilde\Pi_0^0) + \frac{1}{L}\partial_1 \xi^1\, \bar\Pi_0^0$$

canonical generator $\quad Q_\xi = \int_0^L dx^1 [\xi^0 \tilde{\mathcal{H}} + \xi^1 \tilde{\mathcal{P}}] + \frac{\bar\Pi_0^0}{L}\int_0^L dx^1\, \xi^0 [\tilde\Pi_0^0 + \xi^1 \partial_1 \tilde\Phi_0] + \frac{(\bar\Pi_0^0)^2}{2L}\int_0^L dx^1 \xi^0$

$$= \frac{1}{\sqrt2}\int_0^L dx^1 [\xi^+ (\partial_1 \phi^L)^2 - \xi^- (\partial_1 \phi^R)^2] + \frac{\bar\Pi_0^0}{\sqrt2 L}\int_0^L dx^1 [\xi^+ \partial_1 \phi^L - \xi^- \partial_1 \phi^R] + \frac{(\bar\Pi_0^0)^2}{2L}\int_0^L dx^1 \xi^0$$

no mixing $\quad$ (i) if there is no zero mode sector

but then no modular invariant partition function

(ii) only for $\xi^0 = c^0$, $\xi^1 = c^1$ constants (spacetime translations)

$$\int_0^L dx^1 \tilde\Pi_0^0 = 0 = \int_0^L dx^1 \partial_1 \tilde\Phi_0$$

$H = \int_0^L dx^1 (\partial_1 \phi^L)^2 + \int_0^L dx^1 (\partial_1 \phi^R)^2 + \frac{1}{2L}(\bar\Pi_0^0)^2$, $\quad P = -\int_0^L dx^1 (\partial_1 \phi^L)^2 + \int_0^L dx^1 (\partial_1 \phi^R)^2$

Massive case :

- no shift symmetries
- conformal $\rightarrow$ Poincaré

$$S = \int dx^+ dx^- \left(\partial_+\phi\, \partial_- \phi - \tfrac{1}{2} m^2 \phi^2 \right)$$

$$\partial_+ \xi^- = \partial_- \xi^+ = 0 \,, \quad \partial_+ \xi^+ + \partial_- \xi^- = 0 \,, \qquad T_{\pm\pm} = (\partial_\pm \phi)^2$$

$$\xi^+ = a^+ + \omega x^+ \,, \quad \xi^- = a^- + \omega x^- \,, \qquad T_{\pm\mp} = \frac{m^2}{2} \phi^2$$

Dirac algorithm

$$S_H = \int dx^+ dx^- \left[\pi^+ \partial_+ \phi - \frac{m^2}{2} \phi^2 - \lambda^+ (\pi^+ - \partial_- \phi) \right]$$

$$\left\{ \pi^+ - \partial_- \phi \,, \int_0^L dx^- \left[\frac{m^2}{2} \phi^2 + \lambda^+ (\pi^+ - \partial_- \phi) \right] \right\}_+ \approx 0 \iff \partial_- \lambda^+ = -\frac{m^2}{2} \phi \Rightarrow \quad \Phi_+ = 0$$

secondary constraint

$$\left\{ \Phi_+ , \int_0^L dx^- \left[\frac{m^2}{2} \phi^2 + \lambda^+ (\pi^+ - \partial_- \phi) \right] \right\} \approx 0 \iff \quad \lambda^+_+ = 0$$

all constraints $\tilde{\Phi}_+$, $\tilde{\pi}^+_+$, $\tilde{\pi}^+_+ - \partial_- \tilde{\Phi}_+$ are second class reduced theory : free data $\tilde{\phi}_+(0, x^-)$, $H^R = \int_0^{L_-} dx^- \frac{m^2}{2} (\tilde{\phi}_+)^2$

$$\{ \tilde{\phi}_+(x^-), \tilde{\phi}_-(y^-) \}^*_+ = -\frac{1}{4} \epsilon(x^- - y^-) + \frac{x^- - y^-}{2L_-}$$

$$\tilde{\phi}_+(x^+, x^-) = e^{-x^+ \{\cdot, H^R\}^*_+} \tilde{\phi}_+(0, x^-)$$

only data on $x^+ = 0$ is needed

Mode expansion $\tilde{\phi}_+(x^-) = \sum_{n>0} \frac{1}{\sqrt{2 k_- L_-}} \left(a_{k_-} e^{-i k_- x^-} + c.c. \right) = \phi_R(x^-)$, $k_- = \frac{2\pi n}{L_-}$

$$\{ a_{k_-}, a^+_{k'_-} \}^*_+ = -i \delta_{n,n'}$$

Instant form $\phi(x^0, x^1) = \Phi_0 + \frac{x^0}{L} \Pi_0 + \sum_{n \in \mathbb{Z}^*} \frac{1}{\sqrt{2|k|L}} \left(a_k e^{-i k_\mu x^\mu} + c.c. \right)$

Bogoliubov test.

$k = \frac{2\pi}{L} n =: k_1$ $k_0 = \sqrt{k^2 + m^2}$

hard to compare

$\boxed{\text{Lesson 4: Massless \& massive cases are very different}}$

Peierls bracket

$$\phi(x^+,x^-) = \bar{\phi}_0(0) + (\frac{x^+ + x^-}{\sqrt{2}L})\bar{\pi}_0^0(0) + \phi^R(x^-) + \phi^L(x^+),$$

General solution, entangled periodicity, but not separate periodicities !

$$\tilde{G}(x^0,x^1) = G^+(x^0,x^1) - G^-(x^0,x^1),$$

Difference of advanced and retarded propagator, - Pauli-Jordan commutation function

$$(\partial_0^2 - \partial_1^2)\tilde{G}(x^0,x^1) = 0, \quad \tilde{G}(0,x^1) = 0, \quad \partial_0\tilde{G}(x^0,x^1)|_{x^0=0} = -\delta(x^1),$$

Solution to the homogeneous equations, initial conditions determined by canonical equal time commutation relations

$$\begin{aligned}\tilde{G}(x^0,x^1) &= -\int_{-\infty}^{+\infty} dk^1 \frac{1}{4\pi k^1}[\sin k_1(x^0+x^1) + \sin k^1(x^0-x^1)] \\ &= -\frac{1}{4}[\varepsilon(x^0+x^1) + \varepsilon(x^0-x^1)] = -\frac{1}{2}\varepsilon(x^0)\theta(x_\mu x^\mu), \\ &= \int_{-\infty}^{+\infty} dk^0 \int_{-\infty}^{+\infty} dk^1 \frac{1}{2\pi i} e^{-ik_\mu x^\mu}\delta(k_\mu k^\mu)\varepsilon(k^0).\end{aligned}$$

$$\{\phi(x^+,x^-), \phi(y^+,y^-)\} = -\frac{1}{4}[\varepsilon(x^+-y^+) + \varepsilon(x^- - y^-)]$$

Reproduces correctly all equal-time brackets on the two different fronts

$$\{\phi(x^+,x^-),\phi(x^+,y^-)\} = -\frac{1}{4}\varepsilon(x^- - y^-),$$
$$\{\phi(x^+,x^-),\phi(y^+,x^-)\} = -\frac{1}{4}\varepsilon(x^+ - y^+),$$
$$\{\phi(x^+,x^-),\pi^+(x^+,y^-)\} = \frac{1}{2}\delta(x^-,y^-),$$
$$\{\phi(x^+,x^-),\pi^-(y^+,x^-)\} = \frac{1}{2}\delta(x^+,y^+),$$
$$\{\phi(x^+,x^-),\pi^-(x^+,y^-)\} = 0 = \{\phi(x^+,x^-),\pi^+(y^+,x^-)\},$$
$$\{\pi^+(x^+,x^-),\pi^+(x^+,y^-)\} = \frac{1}{2}\delta'(x^-,y^-),$$
$$\{\pi^-(x^+,x^-),\pi^-(y^+,x^-)\} = \frac{1}{2}\delta'(x^+,y^+),$$
$$\{\pi^-(x^+,x^-),\pi^+(x^+,y^-)\} = 0 = \{\pi^+(x^+,x^-),\pi^-(y^+,x^-)\}.$$

Shift and conformal symmetries: On-shell non-vanishing charges on one of the fronts

$$Q^+_{\epsilon^+} = \int_{-L_-/2}^{L_-/2} dx^-(\pi^+ - \partial_-\phi)\epsilon^+ \approx 0,$$
$$Q^-_{\epsilon^+} = \int_{-L_+/2}^{L_+/2} dx^+(\pi^- + \partial_+\phi)\epsilon^+ \approx \int_{-L_+/2}^{L_+/2} dx^+ 2\partial_+\phi\epsilon^+,$$
$$Q^+_{\epsilon^-} = \int_{-L_-/2}^{L_-/2} dx^-(\pi^+ + \partial_-\phi)\epsilon^- \approx \int_{-L_-/2}^{L_-/2} dx^- 2\partial_-\phi\epsilon^-,$$
$$Q^-_{\epsilon^-} = \int_{-L_+/2}^{L_+/2} dx^+(\pi^- - \partial_+\phi)\epsilon^- \approx 0.$$

$$Q'^+_\xi \approx \int_{-L_-/2}^{L_-/2} dx^-\xi^-(\partial_-\phi)^2 = Q'^+_{\xi^-}, \quad Q'^-_\xi \approx \int_{-L_+/2}^{L_+/2} dx^+\xi^+(\partial_+\phi)^2 = Q'_{\xi^+}.$$

Correct representation of the conformal algebra, separately on the two fronts

To be done: adapt to torus topology !

$$\{Q'^+_{\xi_1^-},Q'^+_{\xi_2^-}\} = Q'^+_{[\xi_1^-,\xi_2^-]}, \quad \{Q'^-_{\xi_1^+},Q'^-_{\xi_2^+}\} = Q'^-_{[\xi_1^+,\xi_2^+]}, \quad \{Q'^+_{\xi_1^-},Q'^-_{\xi_2^+}\} \approx 0,$$