Shallow Water Memory

M.M. Sheikh-Jabbari, **Vahid Taghiloo**, M.H. Vahidinia

Based on: 2302.04912

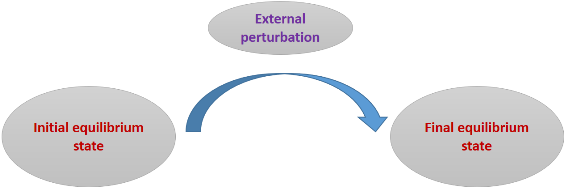
IPM
Institute for Research in Fundamental Sciences

Institute for Advanced Studies in Basic Sciences
Gava Zang, Zanjan, Iran

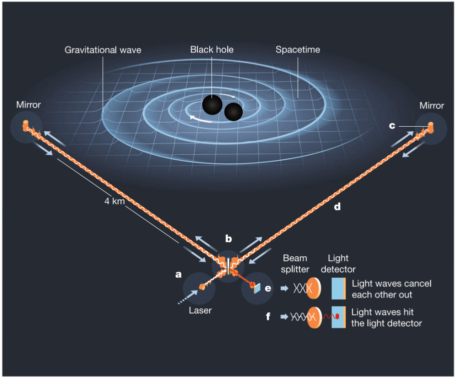
Memory Effect

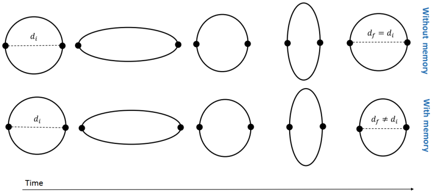
Memory Effect

A stationary physical system is perturbed by external sources. It evolves over time and eventually returns to a stationary state. If the two final states can be distinguished by probes, the system has memory.

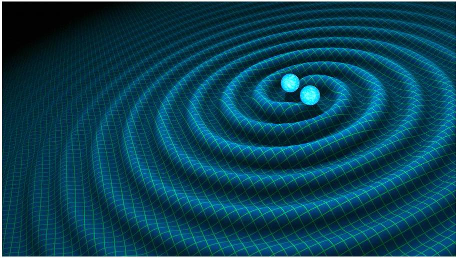


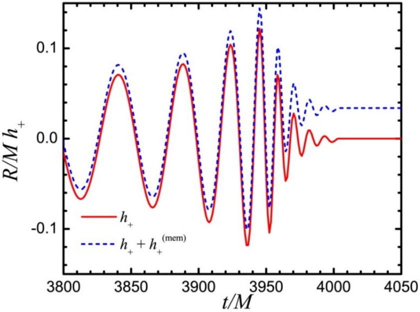
Gravitational memory effect





Gravitational wave





Shallow Water

Shallow water

Shallow water

Shallow water systems are fluid systems with a much larger extent in two dimensions than the third.

The atmosphere and oceans are typical examples of shallow fluid systems.

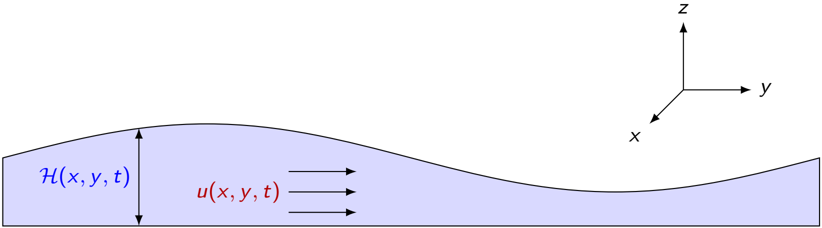
Example 1 : Caspian Sea

Max. length=10^3 km Max. width=0.5×10^3 km Ave. depth=10^2 m

Example 2 : Pacific Ocean

Ave. length/ width $\sim 10^4$ km Max. depth=10 km

Shallow water



Shallow water equations

Shallow water equations

Shallow water equations:

$$\frac{\partial \mathcal{H}}{\partial t} + u \cdot \nabla \mathcal{H} = -\mathcal{H} \nabla \cdot u$$

$$\frac{\partial u^i}{\partial t} + (u \cdot \nabla) u^i = f \epsilon_{ij} u^j - g \partial_i \mathcal{H}$$

g: gravitational constant $\qquad$ f: Coriolis parameter

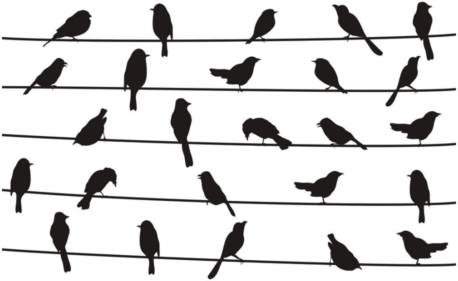
Linear shallow water equations

Linearization procedure: $\mathcal{H}(x^i, t) = H + \eta(x^i, t)$ and $u(x^i, t) = 0 + u(x^i, t)$.

$$\partial_t \eta + H \nabla \cdot u = 0$$

$$\partial_t u_i = f \epsilon_{ij} u^j - g \partial_i \eta$$

Gauge theory description



Kelvin's theorem vs. Gauss law

Kelvin's circulation theorem

The conserved current associated with vorticity:

$$\partial_t(\zeta + f) + \nabla \cdot [(\zeta + f)u] = 0 \qquad\qquad \zeta := \epsilon^{ij}\partial_i u_j$$

Circulation:

$$\Gamma := \int_\Sigma \mathrm{d}^2x\,(\zeta + f) = \oint_{\mathcal{B}} \mathrm{d}l_i\,\left(u^i - \frac{f}{2}\epsilon^{ij}x_j\right)$$

The analogy with the Gauss law:

$$Q_e = \int \mathrm{d}^3x\rho = \int \mathrm{d}^3x\nabla \cdot E = \oint E \cdot \mathrm{d}a$$

This low dimensional property suggests that this system should have a gauge theory description.

Maxwell-Chern-Simon (MCS)

Maxwell-Chern-Simon (MCS)

Action: [Tong: 2209.10574]

$$S[A_\mu] = -\frac{g}{4v}\int d^3x(F^{\mu\nu}F_{\mu\nu} - \frac{f}{v}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho})$$

where $v := \sqrt{gH}$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad F_{0i} = E_i \qquad F_{ij} = \epsilon_{ij}B$$

Gauge symmetry:

$$A_\mu \to A_\mu + \partial_\mu \Lambda(t, x, y)$$

Dictionary:

$$B = \eta \qquad E_i = H\epsilon_{ij}u^j$$

Equations of motion:

$$\nabla^\mu F_{\mu\nu} + \frac{f}{2v}\epsilon_{\nu\alpha\beta}F^{\alpha\beta} = 0$$

Shallow water memory

Memory

Memory

Memory is a “permanent” (long time scales) trace remaining in a system after a perturbation or fluctuation has passed.

1. Early times ($t < -T$): $u \to 0 \quad \Rightarrow \quad E \to 0 \qquad A \neq 0$
2. Middle times ($-T < t < +T$): $u \neq 0$
3. Late times ($t > +T$): $u \to 0 \quad \Rightarrow \quad E \to 0 \qquad A \neq 0$

Memory effect is encoded in the Memory field

$$\Delta A = A(+T) - A(-T)$$

Memory from probes

Pathline equation:

$$\frac{\mathrm{d}x^i(y,t)}{\mathrm{d}t} = u^i(x(y,t),t)$$
$$= -\frac{1}{H}\epsilon^{ij} E_j(x(y,t),t)$$
$$= -\frac{1}{H}\epsilon^{ij} \frac{\partial A_j(x,t)}{\partial t}\Big|_{x=x(y,t)}$$

Pathline memory effect

Pathline equation yields

$$\Delta x^i = -\frac{\epsilon^{ij}}{H}\Delta A_j + \frac{\epsilon^{jk}\epsilon^{il}}{H^2}\int_{-T}^{+T} \mathrm{d}t\, \partial_j E^l(y,t)\Delta A_k(t) + \mathcal{O}(u^3)$$

Lagrangian disp. = Eulerian disp. + Stoksian disp.

Forced linearized shallow water

Forced linearized shallow water

Forced linearized shallow water:

$$\partial_t \eta + H \nabla \cdot u = 0$$

$$\partial_t u_i = f \epsilon_{ij} u^j - g \partial_i \eta + \mathcal{F}_i$$

Kelvin's circulation theorem implies $\mathcal{F}_i = -\partial_i P$.

Action for forced linearized shallow water:

$$S = -\frac{g}{4v} \int \mathrm{d}^3 x \left(F^{\mu\nu} F_{\mu\nu} - \frac{f}{v} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + \frac{2}{g} \epsilon^{ij} F_{ij} P \right)$$

Memory implants in shallow water

Memory implants

Consider an external potential with a shock-wave profile

$$P = \bar{P}(x)\delta(t - t_0)$$

Assume $\Delta u_i = 0$

$$\frac{f}{H}\Delta A_i - \frac{g}{f}\partial_i(\nabla \cdot \Delta A) - \partial_i \bar{P} = 0$$

$\Delta u_i = 0$ also implies $\Delta E_i = 0$, and as a result $\Delta A_i = \partial_i \Lambda$

$$\Lambda(x) = -\frac{f}{2\pi g}\int \mathrm{d}^2 x' K_0\left(\frac{|x - x'|}{R}\right)\bar{P}(x')$$

where $R = \frac{v}{f}$. This is called the memory implant equation.

Summary and Outlook

- Reviewed the gauge theory description of shallow water equation.
- Introduced the concept of the fluid memory effect.
- It would be interesting to experimentally consider the effect of fluid memory.

Thank You!