B^0 meson $(J^P = 0^-)$

Interpolating current $\quad J^{B^0}_{(x)} = \bar{b}^a_{(x)} \gamma_5 d^b(x)$

In order to use QCD sum-rules we need to compute the following

$$T = i\int d^4_x \; e^{i p.x} \; \langle 0| \underbrace{T}_{\text{time ordering operator}} \{ \underbrace{J(x)}_{\text{interpolating current}} J^{+}(0)\} |0\rangle \quad (1)$$

Step 1: Express $\langle 0|T\{J(x)J^+(0)\}|0\rangle$ in terms of propagators

$$J = \bar{b}^a(x) \gamma_5 d^b(x)$$

$$J^+ = (b^{a+} \gamma^0 \gamma_5 d^b)^+ = d^{b+} \gamma_5^+ \gamma_0^+ b^a = \bar{d}^b \gamma^0 \gamma_5 \gamma^0 b^a = -\bar{d}^b \gamma_5 b^a$$

In above calculations, we have used these relations

$$(\gamma^\mu)^+ = \gamma^\mu, \quad \gamma^{5+} = \gamma^5, \quad \{\gamma^5, \gamma^\mu\} = 0$$

Repeatation of two indices in high energy physics means summation on that index, therefore for now we rename indices in J^+

$$J = \bar{b}^a \gamma_5 d^b \qquad J^+ = -\bar{d}^{b'} \gamma_5 b^{a'}$$

$$\boxed{J(x) J^+(0) = -\bar{b}^a_{(x)} \gamma_5 d^b_{(x)} \bar{d}^{b'}_{(0)} \gamma_5 b^{a'}_{(0)}} \quad (2)$$

From Wick theorem in QFT, we have

time ordering = Normal ordering + all possible contractions

$$\Rightarrow \langle 0| \text{time ordering} |0\rangle = \underbrace{\langle 0| \text{Normal ordering} |0\rangle}_{=0} +$$
$$+ \langle 0| \text{all possible contractions} |0\rangle$$

Thus, we need to find all contractions of JJ^+

$$\langle 0| T\{J(x)J^+(0)\} |0\rangle = -\langle 0| \bar{b}^a(x) \gamma_5 \underbrace{d^b(x) \bar{d}^{b'}(0)}_{S_d^{bb'}(x-0)} \gamma_5 b^{a'}(0) |0\rangle$$

propagator of d quark

$$\langle 0|T\{J(x)J^\dagger(0)\}|0\rangle = -\langle 0|\bar b^a(x)\,\gamma^5 S_d^{bb'}(x)\gamma^5\, b^{a'}(0)|0\rangle$$

$$= -\langle 0|\,\bar b^a(x)_\alpha \left(\gamma^5 S_d^{bb'}(x)\gamma^5\right)_{\alpha\beta} b^{a'}(0)_\beta\,|0\rangle$$

Grassman valued number anticommute $\bar b^a_\alpha b^{a'}_\beta = -b^{a'}_\beta \bar b^a_\alpha$

$$= +\langle 0|\left(\gamma^5 S_d^{bb'}(x)\gamma^5\right)_{\alpha\beta} b^{a'}(0)_\beta\, \bar b^a(x)_\alpha|0\rangle$$

$$= \left(\gamma^5 S_d^{bb'}(x)\gamma^5\right)_{\alpha\beta}\; S^{a'a}_{b\,\beta\alpha}(0-x)$$

$$= \left(\gamma^5 S_d^{bb'}(x)\gamma^5 S_b^{a'a}(-x)\right)_{\alpha\alpha}$$

$$\boxed{\langle 0|T\{J(x)J^\dagger(0)\}|0\rangle = \mathrm{Tr}\left(\gamma^5 S_d^{bb'}(x)\gamma^5 S_b^{a'a}(-x)\right)} \quad 3$$

From 1 & 3 we have:

$$T = i\int d^4x\; e^{ip\cdot x}\;\mathrm{Tr}\left(\gamma^5 S_d^{bb'}(x)\gamma^5 S_b^{a'a}(-x)\right)$$

Step 2: Substitute propagators

$$S_d^{bb'}(x) = i\,\delta^{bb'}\frac{\not x}{2\pi^2 x^4} - i\,\delta^{bb'}\frac{m_d}{4\pi^2 x^2}$$ Perturbative part of light quark

$$S_b^{a'a}(-x) = i\int\frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\,\delta^{a'a}\,\frac{\not k + m_b}{k^2-m_b^2}$$ Perturbative part of b-quark

For light quark $m_d \ll m_b$, therefore we set $m_d \to 0$ for simplicity.

$$T = i\int d^4x\, e^{ip\cdot x}\,\mathrm{Tr}\left(\gamma^5\; i\,\delta^{bb'}\frac{\not x}{2\pi^2 x^4}\gamma^5\, i\int\frac{d^4k}{(2\pi)^4}\,e^{ik\cdot x}\,\delta^{a'a}\,\frac{\not k + m_b}{k^2-m_b^2}\right)$$

$$= i^3\,\frac{\delta^{a'a}\delta^{bb'}}{32\pi^6}\iint d^4x\, d^4k\; e^{i(p+k)\cdot x}\,\frac{1}{x^4(k^2-m_b^2)}\;\underbrace{\mathrm{Tr}\left(\gamma^5\not x\gamma^5(\not k + m_b)\right)}_{=-4\,k\cdot x}$$

$$= -i^3\,\frac{\delta^{a'a}\delta^{bb'}}{8\pi^6}\iint d^4x\, d^4k\; e^{i(p+k)\cdot x}\,\frac{k\cdot x}{x^4(k^2-m_b^2)}$$

What should we do about $\delta^{a'a}\delta^{bb'}$?

Remember that hadrons are colorless, and as a result $a=b$ and $a'=b'$ $\Rightarrow$ $\delta^{a'a}\delta^{bb'} \to \delta^{a'a}\delta^{aa'} = \delta^{a'a'} = N_c = 3$.

$$\Rightarrow \boxed{T = -i^3\,\frac{3}{8\pi^6}\iint d^4x\, d^4k\; e^{i(p+k)\cdot x}\,\frac{k\cdot x}{x^4(k^2-m_b^2)}} \quad 4$$

Here is a useful identity

$$\frac{1}{(x^2)^n} = \int \frac{d^D t}{(2\pi)^D}\, e^{-ix.t}\; i\,(-1)^{n+1}\, 2^{D-2n}\, \pi^{D/2}\, \frac{\Gamma(D/2-n)}{\Gamma(n)} \left(-\frac{1}{t^2}\right)^{D/2-n}$$

Using this identity in 4 we have

$$T = -i^3 \frac{3}{8\pi^6} \frac{i(-1)^{n+1} 2^{D-2n} \pi^{D/2}}{(2\pi)^D} \frac{\Gamma(D/2-n)}{\Gamma(n)} \iiint d^D t\, d^D x\, d^D k\; e^{i(p+k-t).x} \frac{k.x}{k^2-m_b^2} \left(-\frac{1}{t^2}\right)^{D/2-n}$$

where at the end of the day, we will set $n=2$ and $D\to 4.$

$$T = \underbrace{\boxed{\frac{(-1)^n\, 3}{2^{3+2n}\, \pi^{D/2+6}} \frac{\Gamma(D/2-n)}{\Gamma(n)}}}_{A} \iiint d^D t\, d^D x\, d^D k\; e^{i(p+k-t).x} \frac{x_\mu k^\mu}{k^2-m_b^2} \left(-\frac{1}{t^2}\right)^{D/2-n}$$

We have written $x.k$ as $x_\mu k^\mu$, then we replace $x_\mu \to -i\frac{\partial}{\partial p_\mu}$

$$T = -i A \frac{\partial}{\partial p_\mu} \iiint d^D t\, d^D x\, d^D k\; e^{i(p+k-t).x} \frac{k^\mu}{k^2-m_b^2} \left(-\frac{1}{t^2}\right)^{D/2-n}$$

$$= -i A \frac{\partial}{\partial p_\mu} \iint d^D t\, d^D k\, \frac{k^\mu}{(k^2-m_b^2)} \left(-\frac{1}{t^2}\right)^{D/2-n} \underbrace{\int d^D x\; e^{i(p+k-t).x}}_{=(2\pi)^D \delta(p+k-t)}$$

$$= -i(2\pi)^D A \frac{\partial}{\partial p_\mu} \iint d^D t\, d^D k\, \frac{k^\mu}{k^2-m_b^2} \left(-\frac{1}{t^2}\right)^{D/2-n} \delta(p+k-t)$$

$$= -i(2\pi)^D A \frac{\partial}{\partial p_\mu} \int d^D t\, \frac{(t-p)^\mu}{(t-p)^2-m_b^2} \left(-\frac{1}{t^2}\right)^{D/2-n}$$

$$= \underbrace{\boxed{-i(-1)^{D/2-n} (2\pi)^D A \frac{\partial}{\partial p_\mu}}}_{B} \int d^D t\, \frac{(t-p)^\mu}{((t-p)^2-m_b^2)\,(t^2)^{D/2-n}}$$

Feynman Parametrization

$$\frac{1}{A^n B^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 du\, \frac{u^{n-1}\,\bar{u}^{m-1}}{[Au+B\bar{u}]^{m+n}} \quad ; \quad \bar{u} \equiv 1-u$$

$$= B \int d^D t \int_0^1 du\; (t-p)^\mu \frac{\bar{u}^{D/2-n-1}}{[\underbrace{((t-p)^2-m_b^2)u + t^2\bar{u}}]^{D/2-n+1}} \frac{\Gamma(D/2-n+1)}{\Gamma(1)\Gamma(D/2-n)}$$

$$((t-p)^2-m_b^2)u + t^2(1-u) = (t^2+p^2-2t.p-m_b^2-t^2)u + t^2$$
$$= t^2 - 2t.pu + p^2u^2 - p^2u^2 + p^2u - m_b^2 u$$
$$= (t-pu)^2 + p^2 u\bar{u} - m_b^2 u$$

$$= B \frac{\Gamma(D/2-n+1)}{\Gamma(1)\Gamma(D/2-n)} \int_0^1 du \int d^D t\, \frac{(t-p)^\mu\, \bar{u}^{D/2-n-1}}{[(t-pu)^2 + p^2u\bar{u} - m_b^2 u]^{D/2-n+1}}$$

Change of variables : $t \to t + pu$

$$= B \, \frac{\Gamma(D/2 - n + 1)}{\Gamma(1)\,\Gamma(D/2 - n)} \int_0^1 du \int d^D t \; \frac{(t - p\bar{u})^\mu \, \bar{u}^{D/2 - n - 1}}{[t^2 + \Delta]^{D/2 - n + 1}} \quad , \; \Delta \equiv p^2 u \bar{u} - m_b^2 u$$

$$= \underbrace{-i(-1)^{D/2 - n} (2\pi)^D A \, \frac{\Gamma(D/2 - n + 1)}{\Gamma(D/2 - n)} \frac{\partial}{\partial p_\mu}}_{B} \int_0^1 du \; \bar{u}^{D/2 - n - 1} \int d^D t \; \frac{(t - p\bar{u})^\mu}{[t^2 + \Delta]^{D/2 - n + 1}}$$

$$= -i(-1)^{D/2 - n} (2\pi)^D \, \frac{\Gamma(D/2 - n + 1)}{\Gamma(D/2 - n)} \underbrace{\frac{(-1)^n \, 3}{2^{3+2n} \pi^{D/2 + 6}} \frac{\Gamma(D/2 - n)}{\Gamma(n)}}_{A} \int_0^1 du \; \bar{u}^{D/2 - n - 1} \frac{\partial}{\partial p_\mu} \int d^D t \; \frac{(t - p\bar{u})^\mu}{[t^2 + \Delta]^{D/2 - n + 1}}$$

$$= -i(-1)^{D/2} \, \frac{3\pi^{D/2 - 6}}{2^{3 + 2n - D}} \, \frac{\Gamma(D/2 - n + 1)}{\Gamma(n)} \int_0^1 du \; \bar{u}^{D/2 - n - 1} \frac{\partial}{\partial p_\mu} \int d^D t \; \frac{-4\bar{u} p^\mu}{[t^2 + \Delta]^{D/2 - n + 1}}$$

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}}$$

$$\int d^D t \; \frac{-4\bar{u} p^\mu}{[t^2 + \Delta]^{D/2 - n + 1}} = (2\pi)^D (-4\bar{u}) p^\mu \frac{(-1)^{D/2 - n + 1} i}{(4\pi)^{D/2}} \frac{\Gamma(D/2 - n + 1 - D/2)}{\Gamma(D/2 - n + 1)} \left(\frac{1}{-\Delta}\right)^{D/2 - n + 1 - D/2}$$

$$= \frac{-4 i \bar{u} (-1)^{D/2 - n + 1} \pi^{D/2}}{1} \frac{\Gamma(-n + 1)}{\Gamma(D/2 - n + 1)} \left(\frac{1}{-\Delta}\right)^{-n + 1} p^\mu$$

$$= \frac{-4(-1)^{D - n + 1} \, 3\pi^{D - 6}}{2^{3 + 2n - D}} \frac{\Gamma(-n + 1)}{\Gamma(n)} \frac{\partial}{\partial p^\mu} \int_0^1 du \; \bar{u}^{D/2 - n} \left(\frac{1}{-\Delta}\right)^{-n + 1} p^\mu$$

$$= \frac{(-1)^{D - n + 2} \, 3\pi^{D - 6}}{2^{3 + 2n - D}} \frac{(-n + 1)\Gamma(-n + 1)}{(-n + 1)\Gamma(n)} \frac{\partial}{\partial p^\mu} \int_0^1 du \; \bar{u}^{D/2 - n} \left(\frac{1}{-\Delta}\right)^{-n + 2} (-\Delta) p^\mu$$

$$= \frac{-3}{8\pi^2} \frac{(-1)^{D - n}}{2^{2n - D}} \pi^{D - 4} \frac{\Gamma(-n + 2)}{(-n + 1)\Gamma(n)} \frac{\partial}{\partial p^\mu} \int_0^1 du \; \bar{u}^{D/2 - n} \Delta \left(\frac{1}{-\Delta}\right)^{-n + 2} p^\mu$$

$\epsilon = 4 - 2n$

Peskin (A.52) :

$$\frac{\Gamma(2 - n)}{(4\pi)^n} \left(\frac{1}{\Delta}\right)^{2 - n} = \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \ln \Delta - \gamma + \log(4\pi) + O(\epsilon)\right)$$

$$= -\frac{3}{8\pi^2} \frac{(-1)^{D - n}}{2^{2n - D}} \pi^{D - 4} \frac{(4\pi)^n}{(4\pi)^2} \frac{1}{(-n + 1)\Gamma(n)} \frac{\partial}{\partial p_\mu} \int_0^1 du \; \bar{u}^{D/2 - n} \Delta \left(\frac{2}{\epsilon} - \ln(-\Delta) - \gamma + \log 4\pi + O(\epsilon)\right) p^\mu$$

limit $n \to 2$, $D \to 4$

$$= + \frac{3}{8\pi^2} \frac{\partial}{\partial p^\mu} \int_0^1 du \; \Delta \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \ln(-\Delta)\right) p^\mu$$

Thus, till now we obtain

$$\boxed{T = \frac{3}{8\pi^2}\frac{\partial}{\partial p^\mu}\int_0^1 du\ \Delta\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \ln(-\Delta)\right)p^\mu}$$

Now, notice that in spectral representation we have

$$T = \int \frac{\rho(s)\,ds}{s-p^2} \Rightarrow \rho(s) = \frac{1}{\pi}\,\mathrm{Im}(T)$$

If we chose to work in $\overline{MS}$ scheme, then renormalized T is

$$T = -\frac{3}{8\pi^2}\frac{\partial}{\partial p_\mu}\int_0^1 du\ \Delta \ln(-\Delta)\,p^\mu$$

$$= -\frac{3}{8\pi^2}\frac{\partial}{\partial p_\mu}\int_0^1 du\ \Delta \ln\left(-|\Delta|\,\mathrm{sgn}\,\Delta\right)p^\mu$$

Sign of Δ

$$= -\frac{3}{8\pi^2}\frac{\partial}{\partial p_\mu}\int_0^1 du\ \underbrace{\Delta \ln|\Delta|\,p^\mu}_{\text{real}} + \underbrace{\Delta \ln(-\mathrm{sgn}\,\Delta)\,p^\mu}_{\substack{\mathrm{sgn}\,\Delta<0 \to \text{real} \\ \mathrm{sgn}\,\Delta>0 \to \text{imaginary}}}$$

$$\Rightarrow \mathrm{Im}(T) = -\frac{3}{8\pi^2}\frac{\partial}{\partial p^\mu}\int_0^1 du\ \Delta\ \mathrm{Im}\left(\ln(-\mathrm{sgn}\,\Delta)\right)p^\mu$$

$$= -\frac{3}{8\pi^2}\frac{\partial}{\partial p^\mu}\int_0^1 du\ \Delta\ \mathrm{Im}\left(\ln e^{i\pi}\right)p^\mu$$

$\&\ \Delta>0$

$$\boxed{\mathrm{Im}(T) = -\frac{3\pi}{8\pi^2}\frac{\partial}{\partial p^\mu}\int_\Omega du\ \Delta\, p^\mu}$$

Ω: region for integration; $0<u<1$ & $\Delta>0$

$$\Delta>0 \Rightarrow p^2 u\bar{u} - m_b^2 u > 0$$

$$p^2 \to s \Rightarrow s\bar{u} - m_b^2 > 0 \Rightarrow 1 - u - \frac{m_b^2}{s} > 0 \Rightarrow u < 1 - \frac{m_b^2}{s}$$

$$\Rightarrow \rho(s) = \frac{1}{\pi}\mathrm{Im}(T) = \frac{-3}{8\pi^2}\frac{\partial}{\partial p^\mu}\int_0^{1-\frac{m_b^2}{s}} du\ \left(p^2 u\bar{u} - m_b^2 u\right)p^\mu$$

$$\Rightarrow \boxed{\rho(s) = -\frac{3}{8\pi^2}\int_0^{1-\frac{m_b^2}{s}} du\ \left(6 s u\bar{u} - 4 m_b^2 u\right)}$$