A new look at path integral optimization

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An overview of some related papers
and a brief summary of an upcoming paper

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Outline

- Basics of the Heat kernel method/Spectral geometry
- Applications to Entanglement Entropy
- Applications to Quantum complexity

Basics of Heat Kernels

or Spectral Geometry

A good review is that by Vassilevich, hep-th/0306138.

Physics laws are usually described by some elliptic PDEs

$$\hat{L}(x)u(x) = f(x)\,,$$

where $\hat{L}$ is the Laplace differential operator.
One can solve the PDE, by the impulse response, known as the Green's function

$$\hat{L}G(x, x') = \delta(x - x')\,.$$

then

$$u(x) = \int f(x')\,G(x, x')\,dx'\,.$$

In 1937 V. Fock noted that writing the Green's function as an integral over an auxiliary time, we get a kernel which satisfies the heat equation

$$G(x, x') = \int_{-\infty}^{\infty} ds\, K(s; x, x')\,,$$

then

$$(\partial_s + \hat{L})K(s; x, x') = 0\,,$$

with initial condition

$$K(0; x, x') = \delta(x - x')\,.$$

In this way, associated to each Laplace operator, a flow of an impulse can be defined and the spectrum of a heat operator

$$e^{-s\hat{L}}$$

can be analyzed through a flow process.
Note that

$$K(s; x, x') = \langle x | e^{-s\hat{L}} | x' \rangle \, .$$

This (heat) flow certainly depends on the geometry of the base manifold, the geometry that is deduced in this way is called spectral geometry.

A very useful identity gives us the effective action

$$W = \frac{1}{2} \ln \det(\hat{L}) = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \, K(s) \, ,$$

where

$$K(s) = \, \mathrm{Tr} \, e^{-s\hat{L}} = \int d^d x \sqrt{g} \, K(s; x, x) \, .$$

We have a simple answer to the request of regularization: just stay away from impulse

$$W = -\frac{1}{2} \int_{\epsilon}^\infty \frac{ds}{s} \, K(s) \, .$$

The trace of the heat kernel is characterized by its small s expansion,

$$K(s) = \sum_{p=0} a_p(\hat{D}^2)\, s^{\frac{(p-d)}{2}}\,, \quad s \to 0$$

where $a_p(\hat{D}^2)$ are the heat kernel coefficients that are represented by the bulk and boundary integrals,

$$a_p(\hat{D}^2) = \int_M \mathrm{Tr}\, A_p(x) + \int_{\partial M} \mathrm{Tr}\, B_p(x)\,,$$

In QFT, the Laplace operator takes the form of $-(\nabla^2 + E)$ in various representations, for instance

- Scalar theory

$$\hat{L} = -(\nabla^2 - \frac{d-2}{4(d-1)} R)\,,$$

- Dirac theory

$$\hat{L} = (i\gamma^\mu \nabla_\mu)^2 = -(\nabla^2 - \frac{1}{4} R)\,,$$

and so on.

$$A_0(x) = \frac{1}{(4\pi)^{d/2}}\, V_d\,,$$

$$A_2(x) = \frac{1}{6(4\pi)^{d/2}} \int_M (6E + R)\,,$$

$$A_4(x) = \frac{1}{360(4\pi)^{d/2}} \Big(60\Box E + 12\Box R + 2R_{ikj\ell} R^{ikj\ell} - 2R_{ij}R^{ij} + 180E^2 + 60RE + 5R^2 \Big)\,,$$

$\ldots$

In particular, in QFT

• QFT with insertions $\rightarrow$ Heat flow on a manifold with insertions

• QFT in presence of the boundary/defect $\rightarrow$ Heat flow on a manifold with boundary/defect

• QFT with singularities $\rightarrow$ Heat flow on a manifold with singularity
$\ldots$

Boundary value problem

- Scalar with Dirichlet b.c.

$$\phi|_{\partial M} = 0\,.$$

- Scalar with Robin (Generalized Neumann) b.c.

$$(\nabla_n + \frac{d-2}{2(d-1)}K)\phi|_{\partial M} = 0$$

and so on.

Extrinsic geometry

External geometry is about how a boundary is embedded in a manifold. The characteristic measure of this geometry is the extrinsic curvature tensor

$$K_{ij} = h_i^k h_j^\ell \nabla_{(k} n_{l)} \, , \; K = K_i^i \, .$$

$$B_1(x) = \mp \frac{1}{4(4\pi)^{\frac{d-1}{2}}} V_d \,,$$

$$B_2(x) = \frac{1}{6(4\pi)^{d/2}} \int_{\partial M} (2K) \,,$$

$$B_3^{(D)}(x) = -\frac{1}{384(4\pi)^{\frac{d-1}{2}}} \left[96E + 16R - 8R_{nn} - 10 \,\mathrm{Tr}\, K^2 + 7K^2 \right] .$$

$$B_3^{(R)}(x) = \frac{1}{384(4\pi)^{\frac{d-1}{2}}} \Big[96E + 16R - 8R_{nn} + 2 \,\mathrm{Tr}\, K^2 + 13K^2 + 96SK + 192S^2\Phi \Big] \,, S = -\frac{d-2}{2(d-1)} K \,.$$

$\cdots$

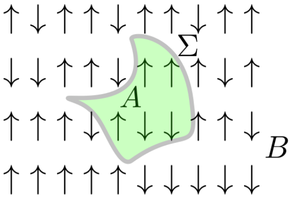
Application 1,

Entanglement Entropy in QFT

A good review can be found in the book by Rangamani & Takayanagi

EE in QM

• Consider a quantum mechanical system in a pure ground state which is described by $|\psi\rangle$ ($\rho = |\psi\rangle\langle\psi|$).



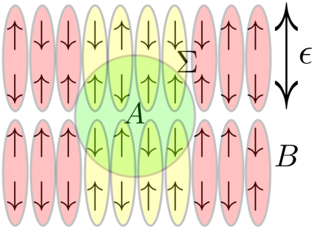
• *Reduced* density operator:

$$\rho_A = \mathrm{Tr}_B\, \rho = \mathrm{Tr}_B |\psi\rangle\langle\psi|\,.$$

Then the EE is

$$S_{EE}(A) = -\,\mathrm{Tr}\, \rho_A \log \rho_A\,.$$

EE in QFT



$$S_{EE}(\Sigma) = \frac{s_{d-2}}{\epsilon^{d-2}} + \frac{s_{d-4}}{\epsilon^{d-4}} + \cdots + s_0 \log \epsilon + f,$$

where

$$s_{d-2} \propto Area(\Sigma).$$

Rényi entropy

In a QFT, we firstly construct the Rényi entropy as

$$S_{RE}(A) = \frac{1}{1-n} \log \mathrm{Tr}\, {\rho_A}^n ,$$

The EE reads then

$$S_{EE}(A) = \lim_{n \to 1} S_{RE}(A).$$

Step 1, Defining the Reduced density operator in QFT

The first step is to define the wave functional of the fields

$$\Psi[\phi_0(x, y^i)] = \int_{\phi(x^\mu)|_{\tau=0}=\phi_0(x,y^i)} \mathcal{D}\phi \, e^{-\int d\tau \mathcal{L}[\phi]} \, .$$

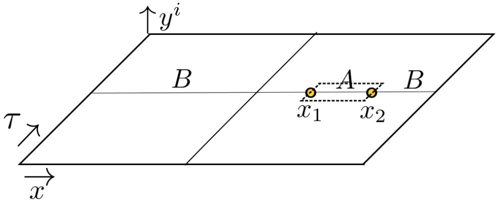
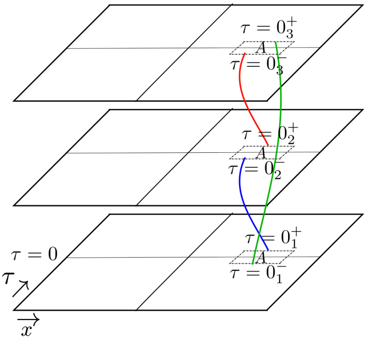


Figure 1: manifold $\mathcal{M}$

Then the reduced density matrix will be found as

$$\rho_A^{+-} = \int \mathcal{D}\phi_B \; \Psi[\phi_A^+, \phi_B] \bar{\Psi}[\phi_A^-, \phi_B] \, ,$$

Step 2, Replica Trick



1. Making n copies

$$\rho_{A,1}^{+-}\ \rho_{A,2}^{+-} \cdots \rho_{A,n}^{+-}.$$

2. Identification

$$(x \in A, \tau = 0_i^+) \sim (x \in A, \tau = 0_{i+1}^-),$$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Partition function on $\mathcal{R}_n$

Finding the RE reduces to computing the partition function on n-sheeted Riemann surface

$$\mathrm{Tr}\,\hat{\rho}_A^n = Z_1^{-n} \int_{\mathcal{R}_n} \mathcal{D}\phi\, e^{-\int_{\mathcal{R}_n} d\tau \mathcal{L}[\phi]} \equiv \frac{Z_n}{Z_1^n}\,,$$

then after an analytical continuation in n we will have

$$S_{EE}(A) = -\,\mathrm{Tr}\,\hat{\rho}_A \log \hat{\rho}_A = -\partial_n \log\, \mathrm{Tr}\,\rho_A^n\Big|_{n=1} = -(n\partial_n - 1)\log Z_n\Big|_{n=1}$$

But the deficit angle $\alpha = 2\pi(1-n)$ introduces a conical singularity such that $\mathcal{R}_n \sim \mathcal{C}_n \times \Sigma$.
The main challenge: **calculation on a manifold with conical singularity?**

Heat flow on a cone.

Applying the Sommerfeld formula

In two dimensions

$$K(x, x', \tau, \tau'; s) = \frac{1}{4\pi s} e^{-\frac{1}{4s}[(\tau-\tau')^2+(x-x')^2]} ,$$

where in polar coordinates

$$(\tau - \tau')^2 + (x - x')^2 = 4r^2 \sin^2(\frac{\phi - \phi'}{2}) .$$

Replication would be possible through changing the periodicity of ϕ to $2\pi n$, then we need to use the Sommerfeld formula, suppressing the r coordinate we have

$$K_n(\phi, \phi'; s) = K(\phi, \phi'; s) + \frac{i}{4\pi n} \int_{\mathcal{C}} d\omega \cot(\frac{\omega}{2n}) K(\phi - \phi' + \omega; s) ,$$

Taking the trace we get

$$\frac{i}{4\pi n}\int_{\mathcal{C}} d\omega \cot(\frac{\omega}{2n})\frac{1}{4\pi s}\int_0^{2\pi n} d\phi \int_0^{\infty} rdre^{-\frac{1}{s}r^2\sin^2(\frac{\omega}{2})}$$

we get

$$\frac{i}{4\pi n}\int_{\mathcal{C}} d\omega \cot(\frac{\omega}{2n})\frac{n}{4\sin^2(\frac{\omega}{2})}\,.$$

By the calculus of residues we find

$$\frac{i}{4\pi n}\int_{\mathcal{C}} d\omega \cot(\frac{\omega}{2n})\sin^{-2}(\frac{\omega}{2}) = \frac{1}{3n^2}(1-n^2)\,.$$

Therefore

$$\mathrm{Tr}\, K_n(s) = \mathrm{Tr}\, K(s) + \frac{1}{3n}(1-n^2)\,.$$

The effective action reads

$$W_n = -\frac{1}{2}\int_{\epsilon}^{\ell} \frac{ds}{s}\,\mathrm{Tr}\,K_n(s) = \frac{1}{6n}(n^2-1)\log\left(\frac{\ell}{\epsilon}\right)\,,$$

We can ultimately find the entropy as follows

$$S_{EE} = (n\partial_n - 1)\,W_n|_{n=1} = \frac{1}{3}\log\left(\frac{\ell}{\epsilon}\right)\,.$$

Conformal Anomaly

It tells us about the universal physical properties in the one-loop level.

$$Z = e^{-W} = \int \mathcal{D}\Phi \;\, e^{-S[\Phi,g]} \,,$$

$$A = \delta_\sigma W(e^{2\sigma} g_{\mu\nu})\Big|_{\sigma=0} = \int_{\mathcal{M}_d} \langle\, T^\mu_\mu \rangle \,.$$

This is the *integrated conformal anomaly*.
log term in the expansion of EE is one of the universal features.

EE and conformal anomaly

Lets remind ourselves the universal logarithmic term in EE, $s_0 \log \epsilon$. Interestingly it is related to the trace anomaly in even dimensions. To see this consider a Weyl scaling as

$$\ell \to e^{-\omega}\ell \qquad \leftrightarrow \qquad g_{\mu\nu} \to e^{-2\omega} g_{\mu\nu}\,,$$

then

$$\ell\frac{\partial}{\partial \ell}(\log\,\mathrm{Tr}\,\hat{\rho}_A^n) = 2\int d^dx\, g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}[\log Z_n - n\log Z_1]\,.$$

but since

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}\,,$$

we will finally arrive at

$$\ell\frac{\partial}{\partial \ell}S_{EE}(A) = \lim_{n\to 1} n\partial_n\langle\, T^\mu_\mu\rangle_{\mathcal{M}_n} \sim \langle\mathcal{R}\rangle_{\mathcal{M}_n}\,.$$

Physics on the cone!

For example in $2D$, $\langle T^{\mu}_{\mu} \rangle = -\frac{c}{12} R$, we get

$$^{(n)}R =^{(reg.)} R + 4\pi(1-n)\delta_{\Sigma} \,,$$

therefore, again

$$\ell \frac{\partial}{\partial \ell} S_{EE}(\Sigma) = \frac{c}{3} \to S_{EE}(\Sigma) = \frac{c}{3} \log \frac{\ell}{\epsilon} \,.$$

Application 2,

Quantum Complexity

Series of papers by Takayanagi et al. for instance arXiv:1706.07056.

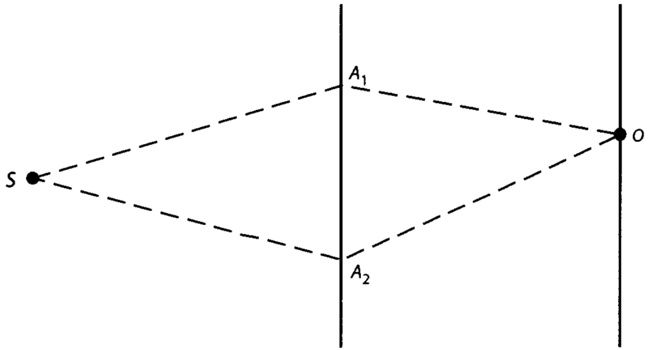
Quantum Complexity

- Complexity: How complicated is it to do a task?
- How advanced should a circuit that implements an algorithm be?
- in QM: How complicated is it to prepare a quantum state?

$$|\uparrow\rangle|\uparrow\rangle \quad \rightarrow \quad |\uparrow\rangle|\downarrow\rangle \quad vs. \quad \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)$$

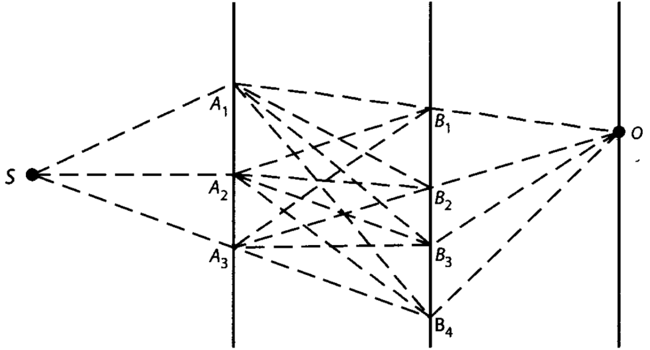
Optimal arrangement of correlated nodes

Consider a grid of correlated node which are distributed in a bulk. The correlation could be of any kind but for illustrative purposes we may consider a multi slits thought experiment.

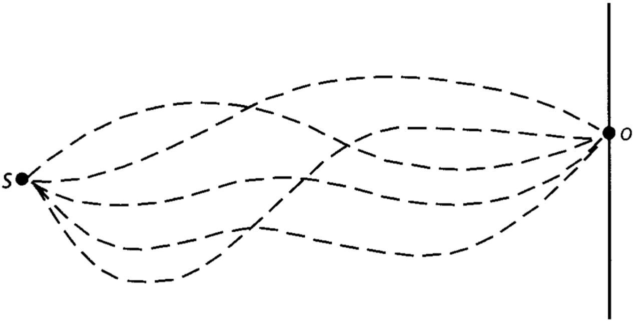


I have borrowed the figures from Zee's book.

Suppose we start in an iterative process to remove the nodes that do not contribute significantly to the signal transferring and thus the final formed pattern. The question is that at the end of the day, what geometry will the distribution of the nodes give?

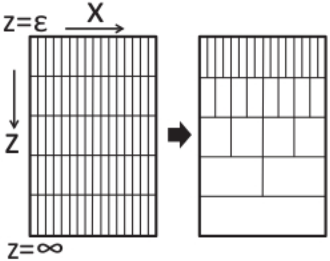


Therefore, our problem is reduced to the optimal heat flow on a manifold and the weight function that measures the optimality of different flows is the effective action.
So the question is: how to minimize effective action. This question is closely related to the question of how to optimally take the path integral and optimally prepare a quantum state.



State in QFT

$$\Psi[\Phi_0] = \int \prod_x \prod_{\epsilon \le z < \infty} \mathcal{D}\Phi \, e^{-S[\Phi]} \times \delta\left(\Phi(z=\epsilon, x) - \Phi_0(x)\right) \, .$$



$$ds^2 = dz^2 + dx^2 \xrightarrow{\text{optimization}} g_{ij}(z, x) \, .$$

What is the form of g_{ij}?

I have borrowed the figure from arXiv:1706.07056 by Takayanagi et al.

complexity in 2 dimensions

In 2d all of the metrics are conformally flat

$$g_{ij}(z,x) = e^{2\phi}(dz^2 + dx^2)\,.$$

The question is now which ϕ optimizes the path integral?

Note:

$$\mathcal{D}\Phi\big|_{e^{2\phi}\eta_{ij}} = e^{S_L}\mathcal{D}\Phi\big|_{\eta_{ij}}\,,$$

where S_L is the Liouville action

$$S_L = \frac{c}{24\pi}\int d^2x\left((\partial\phi)^2 + e^{2\phi}\right)\,.$$

In our approach

$$W[e^{(2\phi)}\eta_{ij}] - W(\eta_{ij}) = S_L\,.$$

complexity in 2 dimensions

Optimizing the path integral, i.e. solving the e.q.m for the Liouville action we will get

$$e^{2\phi} = \frac{1}{z^2} \to g_{ij} = \frac{1}{z^2}(dz^2 + dx^2)\,,$$

This is the time constant slice of AdS_3 !

Optimization of the complexity $\sim$ Solving the Einstein equation

My conclusion is very simple and short:

The most symmetric is the simplest!

Thank You!