Introduction to Holographic Entanglement

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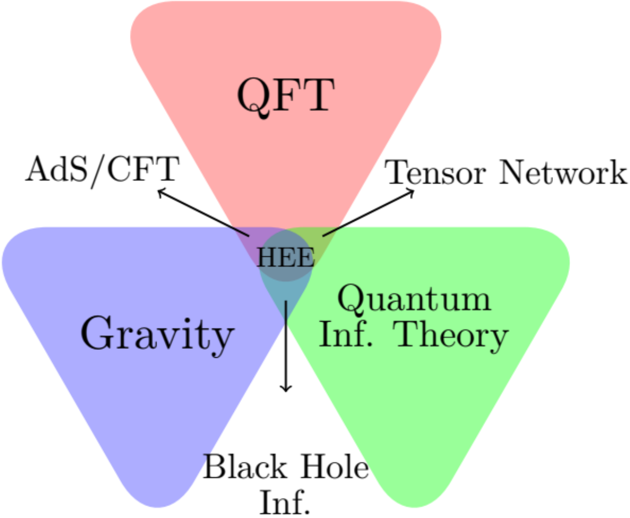
Some References

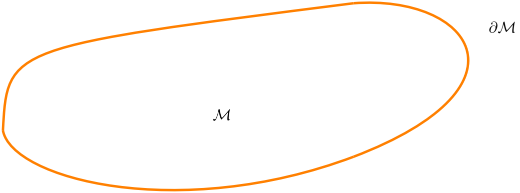
- J. McGreevy, Holographic duality with a view toward many-body physics, arXiv:0909.0518.
- M. Rangamani and T. Takayanagi, Holographic Entanglement Entropy, arXiv:1609.01287.
- T. Nishioka, Entanglement entropy: holography and renormalization group, arXiv:1801.10352.
- A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, The entropy of Hawking radiation, arXiv:2006.06872.

Motivation


QFT
AdS/CFT
Tensor Network
HEE
Gravity
Quantum
Inf. Theory
Black Hole
Inf.

Holographic Principle and AdS/CFT Duality

Holographic Principle



Naive View

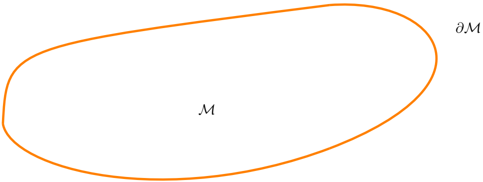
- Information content of a quantum gravity theory in $\mathcal{M}$ can be encoded in an effective theory at $\partial\mathcal{M}$

- An equivalence between boundary and bulk DOFs!

Bekenstein–Hawking Entropy

$$S_{BH} = \frac{A}{4G_N}$$

Holographic Principle



Gauss's Law

$$\oint_{\partial\mathcal{M}} \vec{E}.\vec{ds} = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\int_{\mathcal{M}} \rho \; dV$$

Cauchy's Integral Theorem

$$\oint_{\partial\mathcal{M}} \frac{f(z)}{z - z_0} = 2\pi i \, f(z_0) \qquad z_0 \in \mathcal{M}$$

- The crucial rule of gravity?!

Gauge/Gravity Correspondence

- In QFTs observables correspond to n-point functions

$$\langle \phi(x)\phi(y)\cdots\rangle$$

- $\langle \phi\phi\cdots\rangle$ can be derived from partition function

$$\langle \phi(x)\phi(y)\cdots\rangle = \frac{\delta^n \mathcal{Z}(J)}{\delta J^n}\Big|_{J=0}$$

- $\mathcal{Z}(J)$ depends on the action (Lagrangian)

$$\mathcal{Z}(J) = \int D\phi \; e^{i\int d^dx(\mathcal{L}+J(x)\phi(x))}$$

Challenge

Can we find $\mathcal{Z}$ for any QFT?!

Gauge/Gravity Correspondence

Free QFTs: $\mathcal{Z}$ can be computed exactly

Interacting QFTs: $\mathcal{Z}$ can be computed perturbatively

Example: QED

- $\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{Dirac}}(\psi) + \mathcal{L}_{\text{Maxwell}}(A_\mu) + \alpha\, \mathcal{L}_{\text{int.}}$
- $\mathcal{Z} = \mathcal{Z}_0 + \alpha \mathcal{Z}_1 + \alpha^2 \mathcal{Z}_2 + \cdots$

  In low energy $\alpha < 1$, so perturbation is meaningful.

QED is a weakly coupled theory at low energy

Gauge/Gravity Correspondence

Challenge

- $\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{Dirac}}(\psi) + \mathcal{L}_{\text{gauge}}(A_\mu) + \alpha_s \, \mathcal{L}_{\text{int.}}$

  $\alpha_s > 1$ in low energy!

  QCD is a strongly coupled theory at low energy

  The perturbation method breaks down

- In CMT context: non-Fermi liquids!

Gauge/Gravity duality may help us to overcome this problem!

Gauge/Gravity Correspondence

In GR the maximal entropy in any region scales with area

$$S_{\text{GR}} \sim A$$

In QFT the maximal entropy in any region scales with volume

$$S_{\text{QFT}} \sim V$$

$$\# \text{ of DOFs in a QFT}_d \sim \# \text{ of DOFs in a Grav.}_{d+1}$$

Extra dimension in GR $\sim$ Energy scale of QFT

Gauge/Gravity Correspondence

✓ $SU(N)$ Gauge (YM) Theories at Large N

- N : number of colors g_{YM} : coupling constant

$$\lambda \equiv g_{YM}^2 N \qquad \text{'tHooft coupling}$$

- 't Hooft limit

$$g_{YM} \ll 1 \xrightarrow{N\to\infty} \lambda \sim 1$$

Strong coupling regime may take place in the large N limit!

- Partition Function

$$\log Z_{YM} = \sum_{\chi=2} N^{\chi} f_{\chi}(\lambda)$$

Gauge/Gravity Correspondence

✓ String theory

- ℓ_s : string length $\qquad$ g_s : (closed) string coupling
- Partition Function

$$\log Z_s = \sum_{\chi=2} \left(\frac{1}{g_s}\right)^{\chi} \widetilde{f}_\chi(\ell_s)$$

$$Z_{YM} \sim Z_s$$

$$N \leftrightarrow \frac{1}{g_s} \qquad\qquad \lambda \leftrightarrow \ell_s$$

AdS/CFT correspondence: the best understood example of holographic principle (gauge/gravity duality)

CFT

- Conformal transformations

$$x \to x' \qquad\qquad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$$

- $d-$dimensional conformal symmetry $\qquad SO(1, d+1)$

  Translation + Rotation + Boost + Dilatation + Special

- Conformal field theory (CFT)

  A QFT that is invariant under conformal transformations

- Invariance under conformal symmetry severely constrain the spacetime dependence of correlation functions

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}$$

AdS geometry

- Einstein–Hilbert action with cosmological constant

$$I = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\,(R-2\Lambda) \qquad \Lambda < 0$$

- Pure AdS geometry in Poincare coordinate

$$ds^2 = \frac{L^2}{r^2}\left(-dt^2 + d\vec{x}^2 + dr^2\right) \qquad \Lambda = -\frac{d(d-1)}{2L^2}$$

- center: $r \to \infty$ boundary: $r \to 0$
- Exercise Show that radial null rays sent from the bulk reach the boundary at finite proper time.

AdS/CFT duality

- Classical gravity in $\text{AAdS}_{d+1} \sim$ Strongly coupled CFT_d

$$\log \mathcal{Z}_{\text{CFT}_d}(N \gg \lambda \gg 1) \sim \log \mathcal{Z}_{\text{AdS}_{d+1}} \sim -I^{\text{on shell}}_{\text{AdS}_{d+1}}$$

- Operator/Field correspondence

$$CFT_d: \quad \mathcal{O}, \mathcal{J}_\mu, \mathcal{T}_{\mu\nu}, \cdots \qquad AdS_{d+1}: \quad \phi, A_\mu, g_{\mu\nu}, \cdots$$

- Boundary behavior of the bulk field acts as a source for the operator!

$$\mathcal{L}_{QFT} \sim \phi_{(0)} O + A_{\mu(0)} J^\mu + g_{\mu\nu(0)} T^{\mu\nu} + \ldots$$

- $n-$point function $\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle \sim \frac{\delta}{\delta\phi_0(x_1)} \cdots \frac{\delta \mathcal{Z}_{\text{CFT}_d}}{\delta\phi_0(x_n)}$

AdS/CFT duality

✓ Specific relation between parameters in $AdS_5 \times S^5$

- String theory has two expansion parameters: g_s, ℓ_s
- CFT has two expansion parameters: λ, N
- AdS background has a length scale L (AdS radius)

$$N = \frac{\lambda}{4\pi g_s} \qquad\qquad \lambda = \left(\frac{L}{\ell_s}\right)^4$$

- For fixed λ, $N \to \infty \implies g_s \to 0$

  $g_s \sim 0$: String theory at tree level!

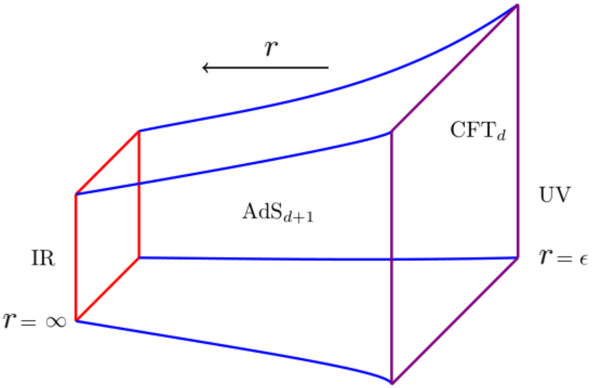
- Now $\lambda \to \infty \implies L \gg \ell_s$

  $\ell_s \sim 0$: Classical string theory becomes classical gravity!

- quantum corrections: $\frac{1}{N}$ stringy corrections: $\frac{1}{\lambda}$

Role of extra dimension!

- Geometrization of the RG flow!



Dictionary

States of the field theory ↔ Asymptotically AdS geometries

CFT_d	AdS_{d+1}
$\mathcal{O}, \mathcal{J}_\mu, \mathcal{T}_{\mu\nu}$	$\phi, A_\mu, g_{\mu\nu}$
Scaling dimension	mass
Energy scale	Radial coordinate
$T \neq 0, \; S \neq 0$	Black-hole
$T = 0, \; S \neq 0$	Extremal black-hole
(Global) Quantum quench	Black-hole formation
...	...

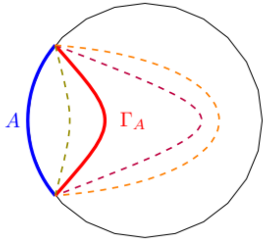
Challenge

Is there any way to compute EE using AdS/CFT?

Holographic Entanglement Entropy

Ryu-Takayanagi Proposal

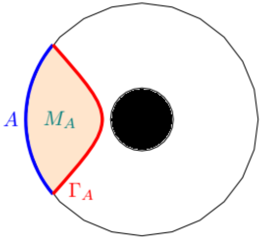
$$S_A = \min\left(\frac{\text{Area}(\Gamma_A)}{4G_N}\right)$$

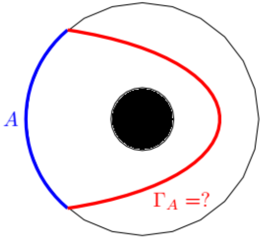


- Properties of the RT surface:
  1. $\partial\Gamma_A = \partial A$
  2. Γ_A is a codimension-2 spacelike hypersurface
  3. Γ_A is homologous to A (homology constraint)

Ryu-Takayanagi Proposal

- Homology constraint



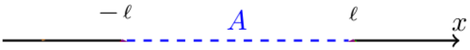


$$\exists\, M_A :\ \partial M_A = \Gamma_A \cup A$$

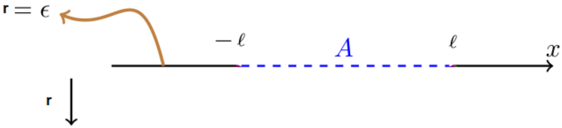
- M_A is a constant time slice of the entanglement wedge.

Ryu-Takayanagi Proposal

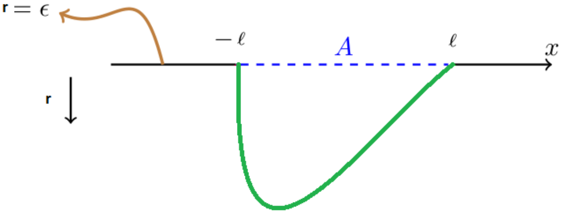
- Example Finite interval in CFT_2



- Assume that the region located on AdS boundary at $r = \epsilon$



- Choose a profile for the RT surface, e.g., $x = x(r)$



Ryu-Takayanagi Proposal

- Find the induced metric on $x = x(r)$ and $t = \text{const.}$

$$ds^2 = \frac{L^2}{r^2}\left(-dt^2 + dx^2 + dr^2\right) \;\rightarrow\; ds^2_{\text{ind}} = \frac{L^2}{r^2}\left(x'(r)^2 + 1\right) dr^2$$

- Compute the area functional

$$\mathcal{A} = \int \sqrt{g_{\text{ind}}}\, dr = \int \frac{L}{r}\sqrt{x'(r)^2 + 1}\; dr$$

- Employ the Euler-Lagrange equation to minimize the area

$$\mathcal{A} = L \int \mathcal{L}\left(x'(r), r\right) dr, \qquad \mathcal{L}(x'(r), r) = \frac{\sqrt{x'(r)^2 + 1}}{r}$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial}{\partial r}\frac{\partial \mathcal{L}}{\partial x'} = 0$$

Ryu-Takayanagi Proposal

$$\mathcal{L}(x'(r), r) = \frac{\sqrt{x'(r)^2+1}}{r} \qquad \cancelto{0}{\frac{\partial \mathcal{L}}{\partial x}} - \frac{\partial}{\partial r}\frac{\partial \mathcal{L}}{\partial x'} = 0$$

- A constant of motion

$$\frac{\partial \mathcal{L}}{\partial x'} = \text{const.} \longrightarrow \frac{1}{r}\frac{x'(r)}{\sqrt{x'(r)^2+1}} = \text{const.} \equiv \frac{1}{r_t}$$

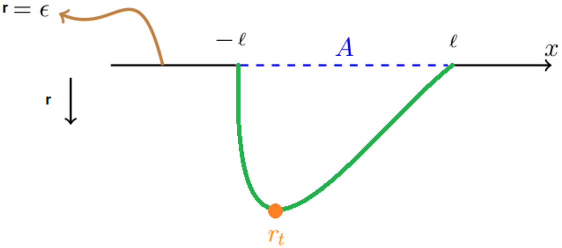
- Solve the differential equation to find the profile

$$x'(r) = \mp\frac{r}{\sqrt{r_t^2 - r^2}} \longrightarrow x_\pm(r) = \pm\sqrt{r_t^2 - r^2} + c_\pm$$

Ryu-Takayanagi Proposal

- At $r = r_t$ we have $x'(r_t) \to \infty$ or $r'(x)\big|_{r=r_t} \to 0$

r_t denotes the turning point of the RT surface.



- Boundary inversion symmetry $x \leftrightarrow -x$ implies that $c_\pm = 0$
- Using the boundary condition $x_\pm(\ell) = 0$ we find $r_t = \ell$

$$r^2 + x^2 = \ell^2$$

Ryu-Takayanagi Proposal

- A circle in $r - x$ plane: $r^2 + x^2 = \ell^2$
- Compute the on-shell area functional

$$\mathcal{A}_{\text{on-shell}} = \int \frac{L}{r}\sqrt{x'(r)^2 + 1}\, dr = 2L\ell \int_\epsilon^\ell \frac{dr}{r\sqrt{\ell^2 - r^2}}$$

- HEE

$$S_A = \frac{\mathcal{A}_{\text{on-shell}}}{4G_N} = \frac{L}{2G_N}\log\frac{2\ell}{\epsilon} \xrightarrow{c=\frac{3L}{2G_N}} \frac{c}{3}\log\frac{2\ell}{\epsilon}$$

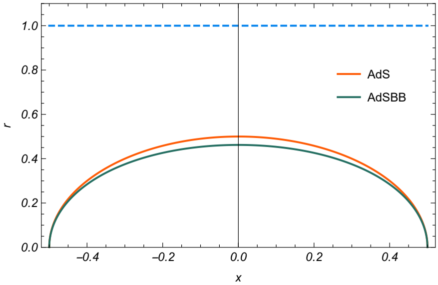
- Exactly matches with previous CFT results!
- Leading large N behavior is N^2 $(G_N \sim N^{-2})$

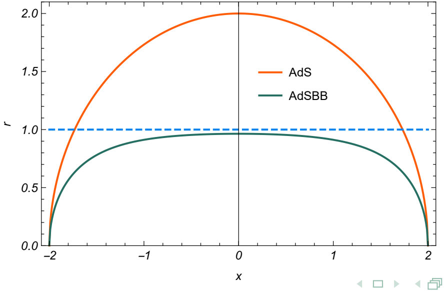
Ryu-Takayanagi Proposal

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)\,dt^2 + dx^2 + \frac{dr^2}{f(r)}\right) \qquad f(r) = 1 - \frac{r^2}{r_0^2}$$

- Exercise In an AdS_3 black brane geometry show that
  1. the length of the entangling region is given by
  $$\ell = r_0 \log \frac{r_0 + r_t}{r_0 - r_t}$$
  2. the HEE is
  $$S = \frac{c}{3} \log \frac{\sinh \pi \ell T}{\pi \epsilon T} \qquad T = \frac{1}{2\pi r_0}$$
  3. the low and high temperature behavior of HEE is given by
  $$S \sim S_{\mathrm{AdS}} + \frac{c}{3}\begin{cases} \frac{1}{6}(\pi \ell T)^2 + \cdots & \ell T \ll 1 \\ \pi \ell T + \cdots & \ell T \gg 1 \end{cases}$$

Ryu-Takayanagi Proposal



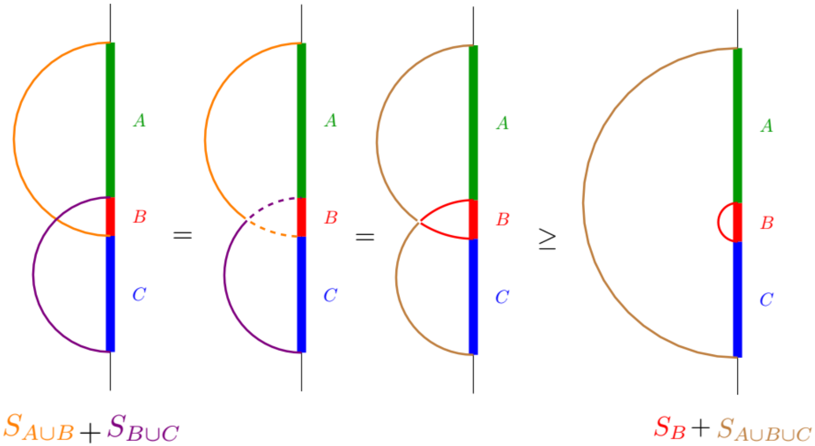


Ryu-Takayanagi Proposal

- Further generalizations: higher dimensions, mixed and excited states, entanglement inequalities, information and entanglement measures, multipartite entanglement, $\cdots$

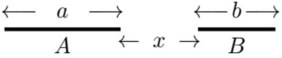
Ryu-Takayanagi Proposal

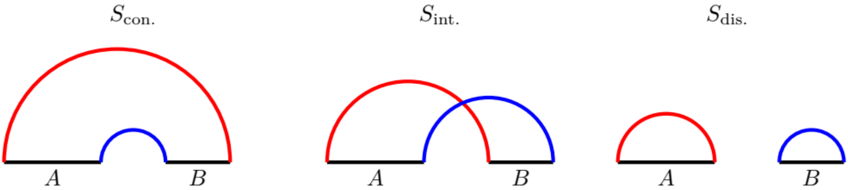
- Proof of SSA $S_{A\cup B} + S_{B\cup C} \geq S_B + S_{A\cup B\cup C}$



Ryu-Takayanagi Proposal

- Multipartite entanglement $\quad S_{A\cup B}, S_{A\cup B\cup C}, \cdots$

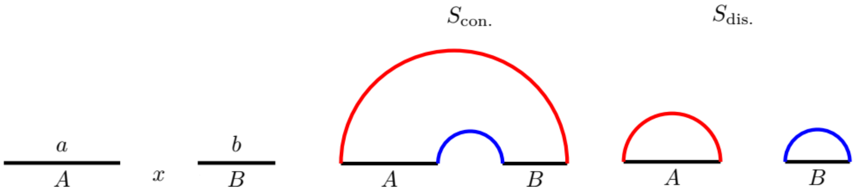




$$S_{A\cup B} = \begin{cases} S_{\text{dis.}} = S(a) + S(b) & x > x_{\text{crit}} \\ S_{\text{con.}} = S(a+b+x) + S(x) & x < x_{\text{crit}} \end{cases}$$

Ryu-Takayanagi Proposal

- Mutual information $I(A, B) = S_A + S_B - S_{A\cup B}$



$$I(A,B) = \begin{cases} 0 & x > x_{\text{crit}} \\ S(a) + S(b) - S(a+b+x) - S(x) & x < x_{\text{crit}} \end{cases}$$

Ryu-Takayanagi Proposal

- Exercise

  Show that holographic mutual information is monogamous!

$$I(A, B \cup C) \geq I(A \cup B) + I(A \cup C)$$

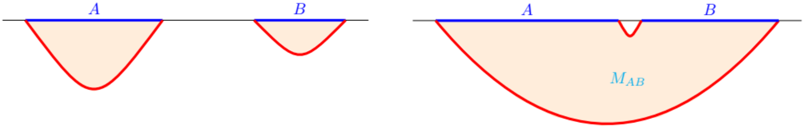
- Holographic tripartite information is negative!

$$I_3(A, B, C) = I(A \cup B) + I(A \cup C) - I(A, B \cup C) \leq 0$$

- In a typical QFT I_3 can be either $+, -$ or 0!
- $I_3 \leq 0$ is not satisfied by generic states in QFTs!

Ryu-Takayanagi Proposal

- The entanglement wedge in disconnected phase is trivial!



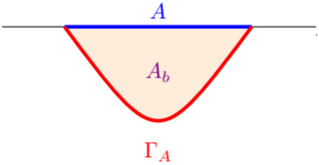
- Quantum corrections gives nonzero I for $x > x_{\text{crit}}$ $(G_N \sim N^{-2})$

$$I = \frac{(\cdots)}{G_N} \xrightarrow{} 0 + \# G_N^0 + \cdots$$

Holographic entanglement entropy beyond classical gravity!

Ryu-Takayanagi Proposal

- Quantum correction at 1 loop!



$$S(A) = S_{\text{cl}}(A) + S_{\text{q}}(A) + \mathcal{O}(G_N)$$

$$S_{\text{cl}}(A) = \min\left(\frac{\text{Area}(\Gamma_A)}{4G_N}\right) \qquad S_{\text{q}}(A) = S_{\text{bulk-ent}}(A_b)$$

- Minimize the area and then add $S_{\text{bulk-ent}}$.
- Ignore the variation of Γ_A at leading order!

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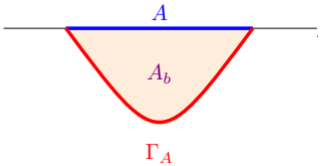
- All order quantum corrections!



- Generalized entropy

$$S_{\text{gen}}(\Gamma_A) = \frac{\text{Area}(\Gamma_A)}{4G_N} + S_{\text{ent}}(\Gamma_A)$$

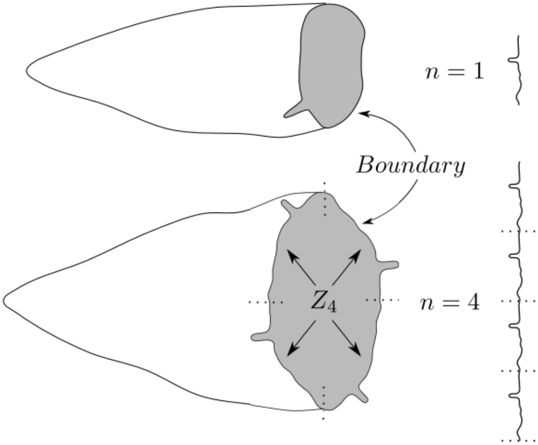
- Minimize the total generalized entropy functional!

Quantum minimal surface!

Ryu-Takayanagi Proposal

- Derivation of the RT prescription
- Earlier attempts to prove the formula: Fursaev 2006.
- Proof of the formula for spherical regions: CHM 2011.
- Generalized gravitational entropy: LM 2013.
- Basic idea: extending the replica trick to the bulk!
- For $n = 1$: boundary $\mathcal{M}$, bulk $\mathcal{B}$, metric $g_{\mu\nu}$
- For $n > 1$: boundary $\mathcal{M}_n$, bulk $\mathcal{B}_n$, metric $g_{\mu\nu(n)}$
- $\mathcal{M}_n$ has a Z_n replica symmetry which extends to the $\mathcal{B}_n$.

Ryu-Takayanagi Proposal



Adapted from arXiv:1304.4926.

Ryu-Takayanagi Proposal

- Boundary replica trick:

$$S_A = -n\,\partial_n \left(\log Z(n) - n \log Z(1)\right)\Big|_{n=1} \qquad Z(n) = Tr\left(\rho^n\right)$$

- Remember that due to AdS/CFT $\log Z(n) = -I^{on}(n)$

$$S_A = n\,\partial_n \left(I^{on}(n) - n I^{on}(1)\right)\Big|_{n=1}$$

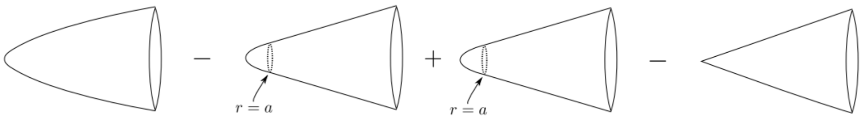
- $I^{off}(n)$ is a $\mathcal{O}(n-1)$ off-shell configuration

$$S_A = n\,\partial_n \Big(I^{on}(n) - I^{off}(n) + I^{off}(n) - n I^{on}(1)\Big)\Big|_{n=1}$$

- All configurations have the same b.c. with $\tau \sim \tau + 2\pi n$

Ryu-Takayanagi Proposal

$$S_A = n\,\partial_n \left(I^{on}(n) - I^{off}(n) + I^{off}(n) - n I^{on}(1) \right) \Big|_{n=1}$$



Adapted from arXiv:1304.4926.

$$I^{on}(n) - I^{off}(n) \propto \frac{\delta I}{\delta g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial n} \sim 0$$

$$I^{off}(n) - n I^{on}(1) \propto (\text{Area}) \int_{vertex} d^2x \sqrt{g} R$$

Ryu-Takayanagi Proposal

$$S_A = \frac{\text{Area}}{16\pi G_N} n\,\partial_n \left(4\pi(n-1)\right) = \frac{\text{Area}}{4G_N}$$

- what is the origin of minimal area condition?

  Demand that $g_{\mu\nu(n)}$ obeys EoMs up to $\mathcal{O}(n-1)$.

- Expand the EoMs near the tip of the cone $(r \to 0)$

$$G_i^i \propto \frac{K^1 + K^2}{r} + \text{regular terms} \qquad i = 1, \cdots, d-1$$

- Vanishing trace of the extrinsic curvature in the normal directions! minimal surface condition

HRT Proposal

- Static case: Euclidean geometries!

✓ boundary Cauchy slice Σ uniquely determines the bulk Cauchy slice $\tilde{\Sigma}$

- Considering a spacelike Σ the minimality is meaningful

$$ds^2 = dx^2 + dy^2 + \cdots$$

- Dynamical case: Lorentzian geometries!
- The minimality condition is not applicable!

$$ds^2 = -dt^2 + dx^2 + dy^2 + \cdots$$

HRT Proposal

- Classical prescription

$$S_A = \min_{\Gamma_A}\left[\mathrm{ext}_{\Gamma_A}\left(\frac{\mathrm{Area}(\Gamma_A)}{4G_N}\right)\right]$$

- Covariant generalized entropy

$$S_{\mathrm{gen}}(\Gamma_A) = \min_{\Gamma_A}\left[\mathrm{ext}_{\Gamma_A}\left(\frac{\mathrm{Area}(\Gamma_A)}{4G_N} + S_{\mathrm{ent}}(\Gamma_A)\right)\right]$$

A formula for the entropy of quantum systems coupled to gravity

- Extremize the total generalized entropy functional!

Quantum extremal surface!

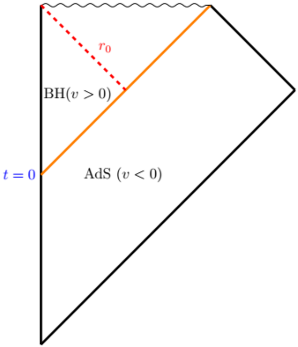
HRT Proposal

- Properties of the co-dimension two HRT surface:
  1. $\partial\Gamma_A = \partial A$
  2. A surface with vanishing expansions of null geodesics
  3. A saddle point of the proper area functional
- Some properties of HRT prescription:
  1. Obeys strong subadditivity of HEE
  2. Obeys monogamy of mutual information

Time evolution of HEE in quenched holographic systems!

HRT Proposal

- Quench process corresponds to black hole formation
- Vaidya geometry describes the collapse of a shell of matter



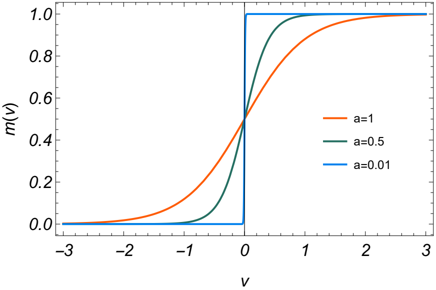
- Early time: Pure AdS   Late time: AdS Black-brane

HRT Proposal

- A holographic set-up dual to a global quench

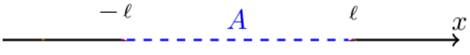
$$ds^2 = \frac{L^2}{r^2}\Big(-f(r,v)\,dv^2 - 2\,dr dv + d\vec{x}^2\Big), \qquad dv = dt - \frac{dr}{f}$$

$$f(r,v) = 1 - m(v)\,r^d \qquad\qquad m(v) = \frac{m}{2}\left(1 + \tanh\frac{v}{a}\right)$$



HRT Proposal

- Example Quantum quench in CFT_2



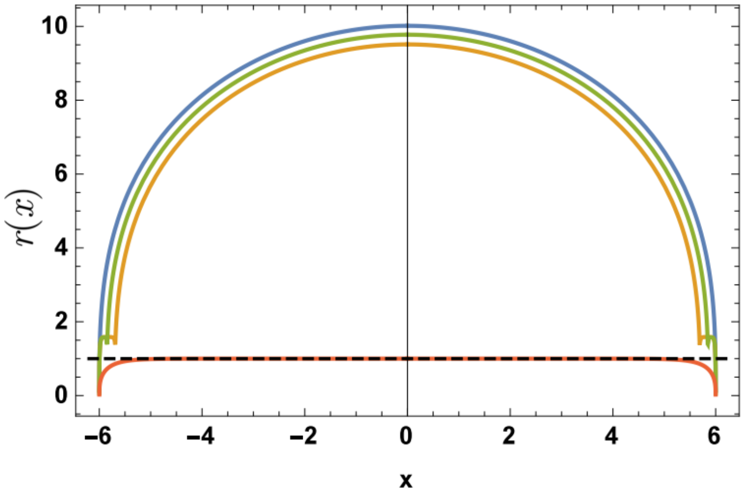
- profile of the HRT surface: $r = r(x)$ and $v = v(x)$
- Area functional

$$\mathcal{A} = \int \frac{dx}{r} \sqrt{1 - 2v'r' - v'^2 f(r,v)}\; dr$$

- Boundary conditions $\quad r\left(\pm\frac{\ell}{2}\right) = 0 \quad v\left(\pm\frac{\ell}{2}\right) = t$
- Exercise

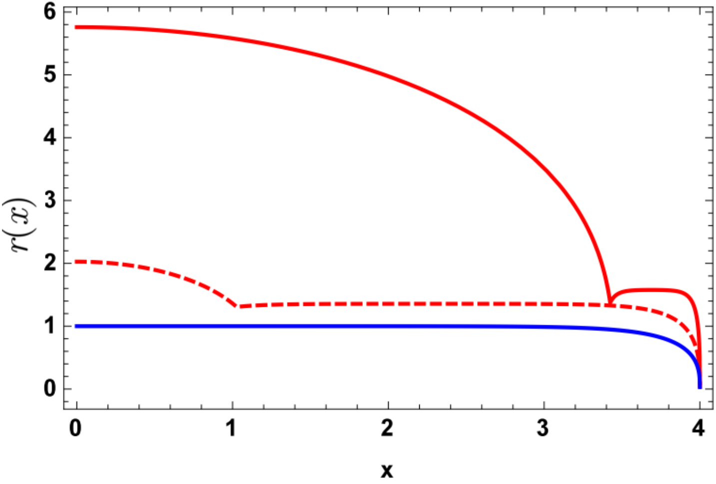
  Find the profile of the HRT surface and compute $S_A(t)$

HRT Proposal



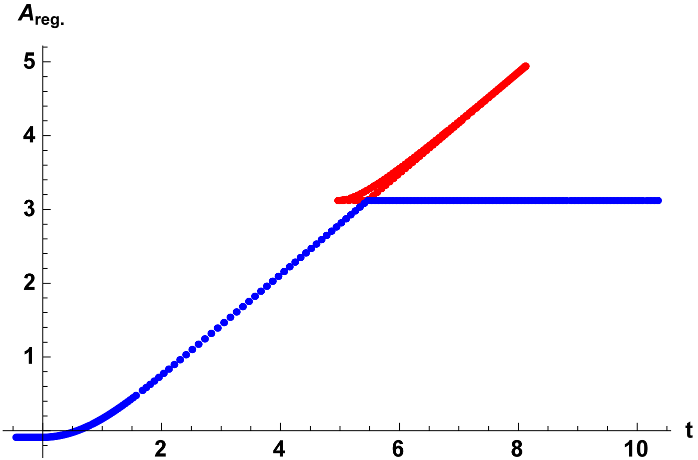
- An evolution from AdS profile to AdS Black-brane profile

HRT Proposal



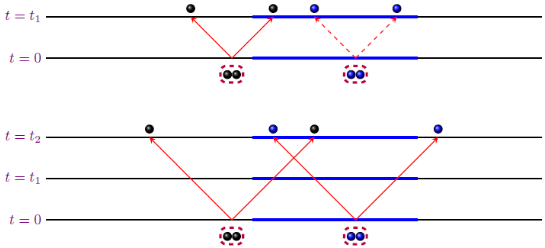
- Presence of multiple solutions to EoMs for a given time!

HRT Proposal



HEE

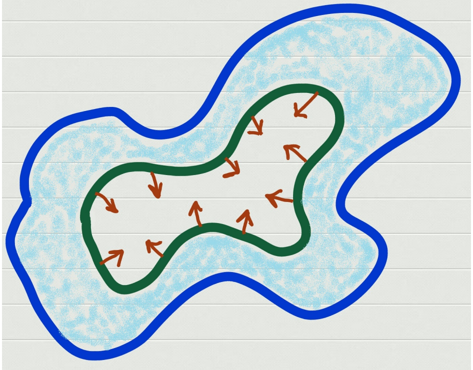
- According to HRT prescription (after a quench) HEE has:
  1. an early time quadratic growth
  2. an intermediate linear growth
  3. a late time saturation
- Quasi particle picture!



HEE

- Entanglement Tsunami

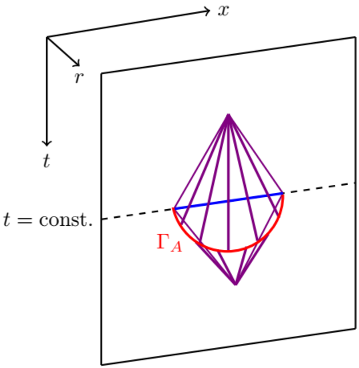
A picture for the growth of EE in a strongly coupled QFT!

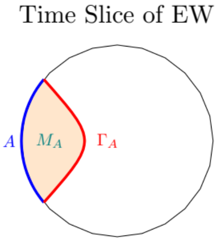


✓ Holographic Entanglement Measures

✓ Geometry From Entanglement

Entanglement Wedge

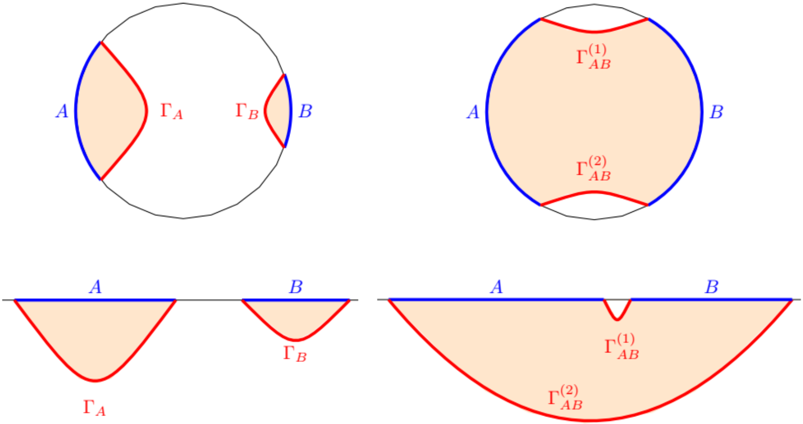




- $\partial M_A = A \cup \Gamma_A$

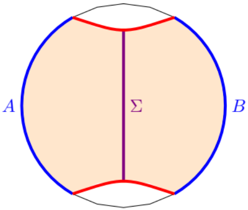
Entanglement Wedge

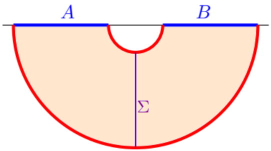
- Two boundary subsystems: $\partial M_{AB} = A \cup B \cup \Gamma_{AB}$



Entanglement Wedge Cross-section E_W

$$E_W = \frac{\text{area}\,(\Sigma)}{4G_N}$$





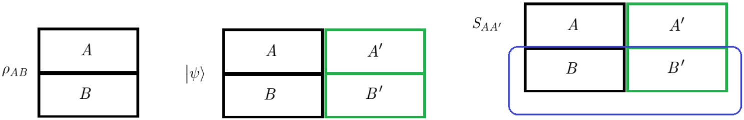
- A Holographic Dual for
  1. E_P [Takayanagi, Umemoto '17, Nguyen et. al. '17]
  2. $\mathcal{E}$ [Kudler-Flam, Ryu'18]
  3. S_O [Tamaoka'18]
  4. S_R [Dutta, Faulkner, '19]

Entanglement of Purification

- Consider a mixed state in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ describe by ρ_{AB}
- Enlarge $\mathcal{H}$ to $\mathcal{H}_{AA'} \otimes \mathcal{H}_{BB'}$ by adding some auxiliary degrees of freedom $\rho_{AB} = \mathrm{Tr}_{A'B'} |\psi\rangle\langle\psi|$

  Purification is not unique!

$$E_P = \min_{\rho_{AB} = \mathrm{Tr}_{A'B'} |\psi\rangle\langle\psi|} S_{AA'}$$



Entanglement of Purification

E_P:

- reduces to S_A for pure states
- enjoys several inequalities
  1. $E_P(A, B) \leq E_P(A, B \cup C)$
  2. $\frac{I(A,B)}{2} \leq E_P(A, B) \leq \min\{S_A, S_B\}$
  3. $\frac{I(A,B)+I(A,C)}{2} \leq E_P(A, B \cup C)$
- is a UV finite quantity
- $\cdots$

First Law of EE

✓ Some basic questions:

- What is the gravity dual of ρ_A?
- What part of the bulk can be fully reconstructed from ρ_A?
- Given ρ_A, in what region of the bulk can we uniquely reconstruct the geometric data ($g_{\mu\nu}$)?
- Given the EEs of a collection of regions in the boundary QFT, what is the corresponding geometry?
- Can we find gravitational dynamics from entanglement pattern?

The bulk reconstruction program!

✓ Modular Hamiltonian (K_A)

- ρ_A is both hermitian and positive semidefinite

$$\rho_A = e^{-K_A} \qquad\qquad K_A : \text{ Modular Hamiltonian}$$

(Similar to $\rho = e^{-\beta H}$!)

- Generically K_A is not a local operator!
- For a spherical region in the vacuum of a CFT

$$K_A = 2\pi \int_{|x|<R} d^{d-1}x \frac{R^2 - r^2}{2R} T_{00}(x)$$

✓ Relative entropy $S(\rho|\sigma)$

- A measure of distinguishability between density matrices

$$S(\rho|\sigma) = \mathrm{tr}\,(\rho \log \rho) - \mathrm{tr}\,(\rho \log \sigma)$$

- Positivity Condition $S(\rho|\sigma) \geq 0$
- Exercise:
  1. Show that $S(\rho|\sigma) = \Delta\langle K_\sigma\rangle - \Delta S$ where

$$\Delta\langle K_\sigma\rangle = \mathrm{tr}\,(\rho K_\sigma) - \mathrm{tr}\,(\sigma K_\sigma) \qquad \Delta S = S(\rho) - S(\sigma)$$

  2. Assume $\rho = \sigma + \epsilon\sigma_1 + \cdots$, show that at $\mathcal{O}(\epsilon)$ we have

$$\delta S = \delta\langle K_\sigma\rangle,$$

First Law of Entanglement! (Similar to $dS = \beta\, d\langle H\rangle$!)

Linearized gravitational dynamics

1. Consider perturbations around the AdS geometry

$$g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} + h_{\mu\nu}, \qquad \mathcal{O}(h) \ll \mathcal{O}(g)$$

2. Choose a ball-shaped entangling region

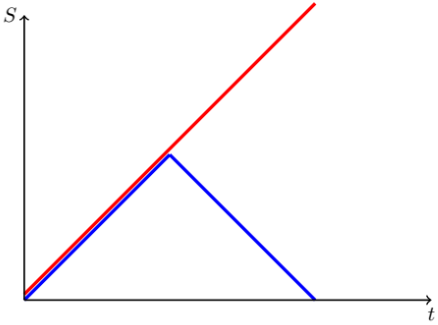
- Linearized gravity from the 1st law

$$\delta\langle S\rangle = \delta\langle K_\sigma\rangle \Leftrightarrow \delta\mathbf{E} = 0$$

Islands and Replica Wormholes

Information Paradox!

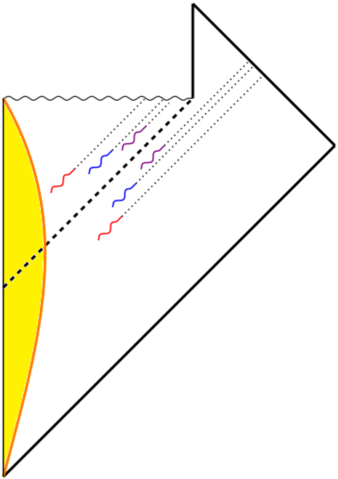
- Hawking vs Page!



- Hawking used the wrong formula for computing S!

Information Paradox!

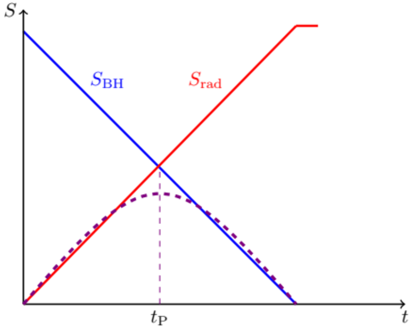
- Formation and evaporation of a black-hole:



Information Paradox!

- Assumption: radiation ∪ black-hole = pure state

$$S_{\rm rad} = S_{\rm bh} \leq S_{\rm BH}$$



Generalized Entropy

- A gravitational formula for the von Neumann entropy

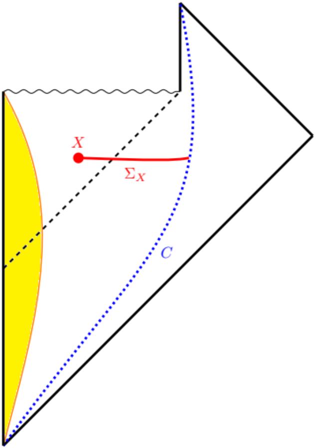
$$S_{\text{gen}}(X) = \min_X \left[\text{ext}_X \left(\frac{\text{Area}(X)}{4G_N} + S_{\text{ent}}(\Sigma_X) \right) \right]$$

- X: quantum extremal surface
- Σ_X: region bounded by X and a cutoff surface C
- C: separates interior region where gravity is dynamical

  from the exterior region

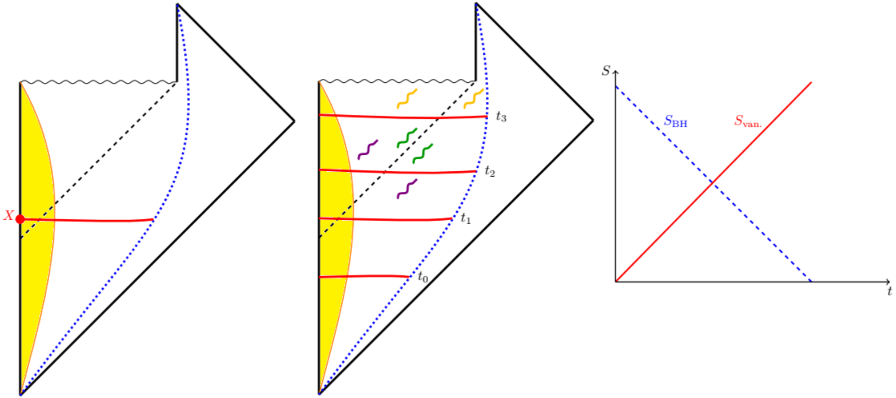
Quantum Extremal Surface

$$S_{\text{gen}}(X) = min_X \left[ext_X \left(\frac{\text{Area}(X)}{4G_N} + S_{\text{ent}}(\Sigma_X) \right) \right]$$



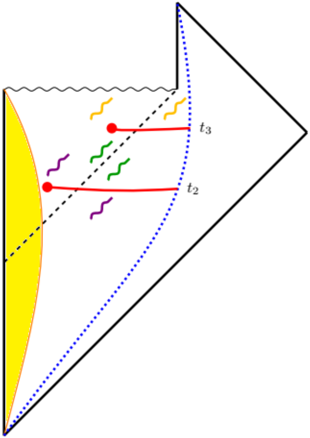
Vanishing Surface

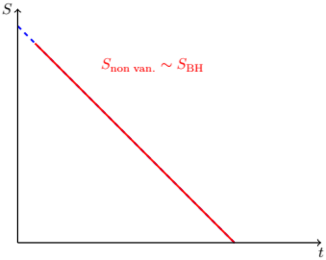
- At early times X shrinks down to zero size!



Non-vanishing Surface

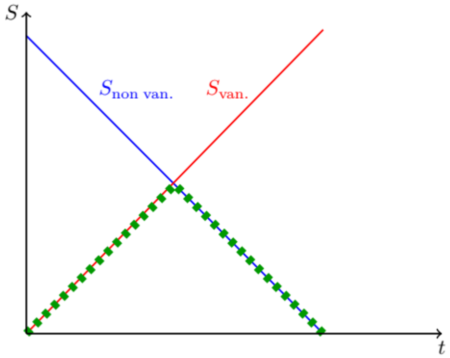
- A new saddle appears shortly after the radiation starts!





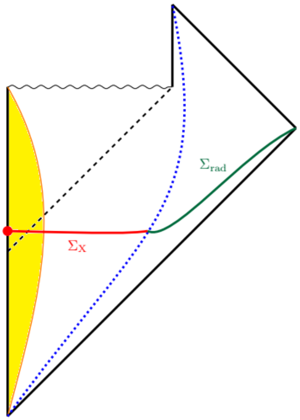
Page Curve

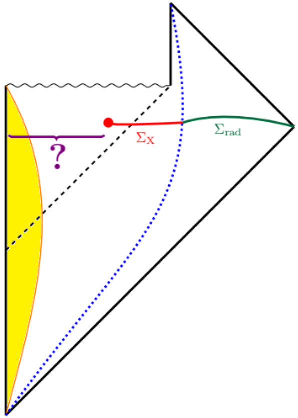
- Choose the minimal area surface!



Island

- The state is pure on the whole Cauchy slice at any time



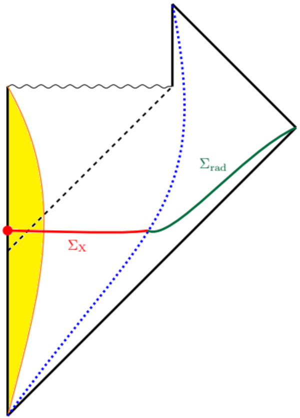


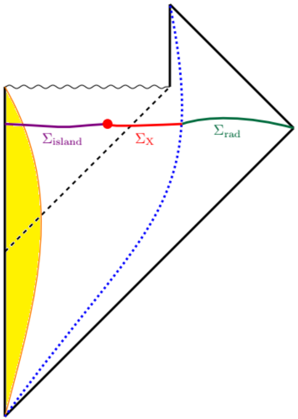
$\Sigma_{\rm early} = \Sigma_X \cup \Sigma_{\rm rad}$

$\Sigma_{\rm late} = \Sigma_X \cup \Sigma_{\rm rad} \cup ?$

Island

- Island: A disconnected region inside the black-hole!



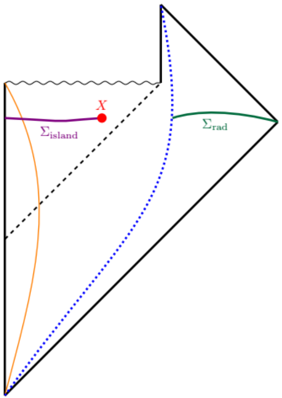


$$\Sigma_{early} = \Sigma_X \cup \Sigma_{rad}$$

$$\Sigma_{late} = \Sigma_X \cup \Sigma_{rad} \cup \Sigma_{island}$$

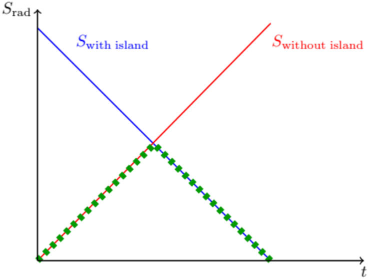
Entropy of Radiation

$$S_{\text{rad}} = \min_X \left[\text{ext}_X \left(\frac{\text{Area}(X)}{4G_N} + S_{\text{ent}}\left(\Sigma_{\text{rad}} \cup \Sigma_{\text{island}}\right)\right)\right]$$



Entropy of Radiation

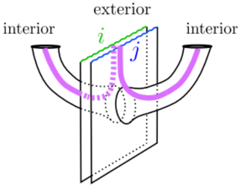
- The entropy of the black hole and the entropy of the radiation region are equal



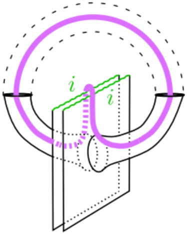
Replica Wormholes

- Compute S_{rad}, assuming that the black hole formed from a pure state.
- Blackhole initial state: $|\psi\rangle$ Final radiation state: $|i\rangle$
- $\rho_{ij} = \langle i|\psi\rangle\langle\rangle\psi|j\rangle$

$\rho_{ij} \approx$



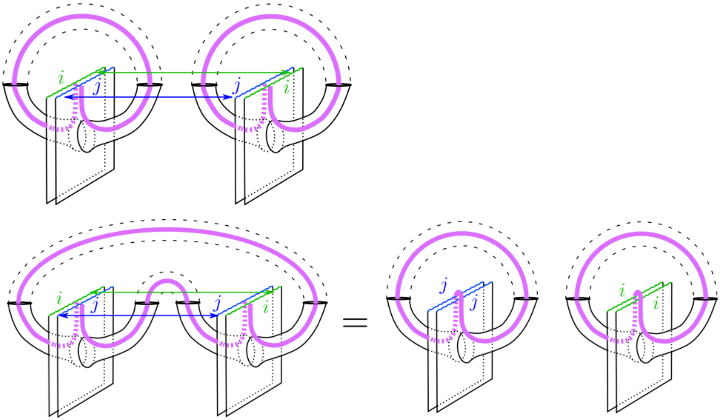
$\mathrm{Tr}\rho \approx$



Adapted from arXiv:2006.06872.

Replica Wormholes

- Hawking saddle: $\mathrm{Tr}(\rho^2) \ll (\mathrm{Tr}\rho)^2$
- Wormhole saddle: $\mathrm{Tr}(\rho^2) = (\mathrm{Tr}\rho)^2 \to S = 0$



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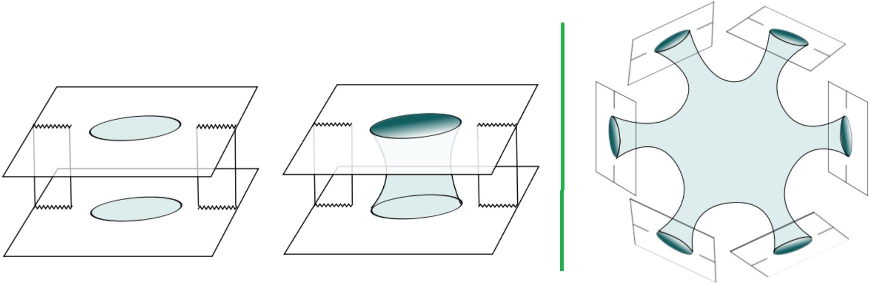
Replica Wormholes



Adapted from arXiv:2006.06872.

- The replica wormholes give rise to the island contributions to the entropy at $n \to 1$!

Some achievements/extensions related to HEE

- Holographic entanglement measures
- Entanglement and renormalization
- Entropic c-functions
- Surface/State correspondence, AdS/cMERA
- HEE & causality in CFT and gravity
- Entanglement inequalities holographic entropy Cone
- Higher dimensional twist operators
- HEE as a probe for QPT
- Holographic tensor network and quantum error correction
- Holographic computational complexity
- ...