Holographic Mutual Information for Singular Surfaces

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25 May 2016

Acknowledgements:

This talk is based on:
arXiv:1511.00244, M. R. Mohammadi Mozaffar, A. Mollabashi, F. Omidi, JHEP 1512 (2015) 082.

I would like to thank my collaborators:

Ali Mollabashi, School of Physics of the IPM
Mohammad Reza Mohammadi Mozaffar, School of Physics of the IPM

I am also very grateful to the School of Physics of IPM for its warm hospitality during this project

Outline

- Introduction:
  1. Some Measures of Quantum Entanglement:
     1.1 Entanglement Entropy
     1.2 Mutual Information
     1.3 n-Partite Information
  2. Holographic approach: Ryu-Takayanagi formula
  3. How can geometrical singularities affect Entanglement Entropy?

- Effects of geometrical singularities on
  1. Holographic Mutual Information (HMI)
  2. Holographic n-Partite Information

  For a 3d CFT with a gravity dual.

- What would happen to HMI when singular entangling regions have/not have common boundaries?
- Generalization to Higher dimensions $CFT_{d\geq 4}$

- Summary

How to Quantify Quantum Entanglement:

Consider a quantum mechanical system whose Hilbert space is $\mathcal{H}$. Devide it into two subsystems

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

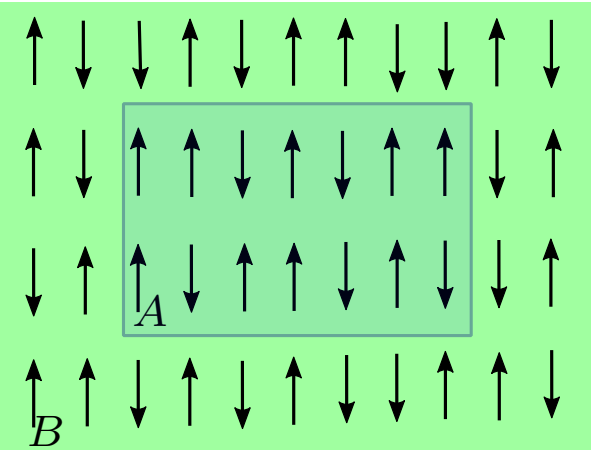


Figure: A 2d Ising Model which is divided into two subsystems A and B

How can one measure the amount of quantum entanglement?

Answer: by Entanglement Entropy!

Entanglement Entropy in Quantum Mechanics:

One can define a reduced density matrix ρ_A for the subsystem A, by tracing out of the states of the subsystem B.

$$\rho_A = Tr_B\left(\rho_{total}\right) = \sum_{i \in \mathcal{H}_B} \langle i|\rho_{total}|i\rangle$$

Entanglement Entropy of A is defined as the Von-Neumann entropy for ρ_A

$$S_A = -Tr\left(\rho_A \log \rho_A\right)$$

If $S_A = 0$, there is no quantum entanglement between the degrees of freedom of A and B. For example for a two-electron system we have:

$$\text{Entangled:} \quad |\Psi_1\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right) \qquad S_A = \log 2$$

$$\text{Unentangled:} \quad |\Psi_2\rangle = \frac{1}{2}\left(|\uparrow\rangle_A + |\downarrow\rangle_A\right) \otimes \left(\uparrow\rangle_B + |\downarrow\rangle_B\right) \qquad S_A = 0$$

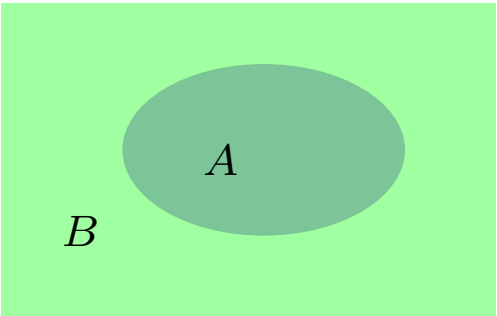
Hence, E.E. is a good quantity to measure Quantum Entanglement.

Entanglement Entropy in QFT:

If one has a quantum field theory on a manifold, one can consider a constant time slice of the manifold, and divide it into two spatial parts A and B. If one can decompose the whole Hilbert space as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

E.E. can be defined in the same manner as in quantum mechanics.



- Consider a smooth (i.e. without any geometrical singularities) entangling surface like a sphere, strip,... in a relativistic QFT, generally EE behaves as :

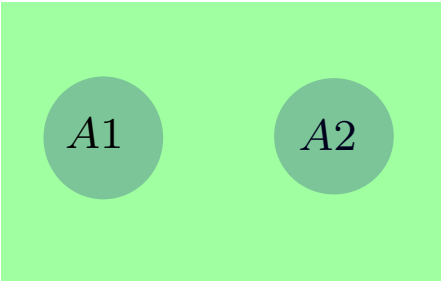
$$S_A = \frac{g_{d-2}}{\epsilon^{d-2}} + \cdots + \frac{g_1}{\epsilon} + g_0 \log \epsilon + S_0$$

- ϵ is the short distance cut off. g_i's depends on the geometry of the boundary of the entangling region A.
- g_0 is universal, in the sense that it is independent of the short distance cut off ϵ.
- EE suffers from UV and IR divergences.

Mutual Information

It is better to use quantities which are independent of the UV cut off such as Mutual and n-partite information. For two disjoint subsystems A_1 and A_2, Mutual Information (MI) is defined by:

$$I(A_1, A_2) = S_{A_1} + S_{A_2} - S_{A_1 \cup A_2},$$



It has several interesting properties:

1. It is written in terms of four-point functions of "Twist Operators", so it can give some information about the field content of the CFT.
2. Its behavior depends on both the geometry of the entangling region and dimension of the CFT. Usually, it is independent of the UV cut off.

one can generalize this definition to n-sectors: and define n-partite information. For $n = 3$:

$$I^{[3]}(A_1, A_2, A_3) = S_{A_1} + S_{A_2} + S_{A_3} - S_{A_1 \cup A_2} - S_{A_1 \cup A_3} - S_{A_2 \cup A_3} + S_{A_1 \cup A_2 \cup A_3},$$

Holographic Entanglement Entropy (HEE):

Computation of EE in Quantum Field Theories is very difficult!
AdS/CFT duality: For a strongly coupled CFT_d with a large central charge, the gravity dual is a classical Einstein gravity on an asymptotically AdS_{d+1} space time.
If the QFT has a dual gravity, there is a spacelike, codimension two, minimal surface in the bulk whose area gives the EE (Ryu-Takayanagi 2006)

$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+1}}$$

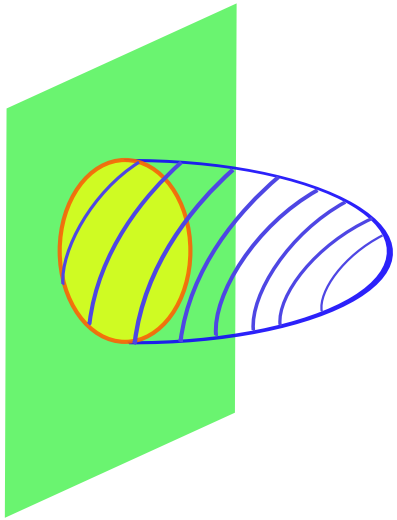


Figure: The minimal surface γ_A in the bulk $\mathcal{M}$ is shown in blue, the boundary of the bulk is shown by the green plane.
This works in Einstein Gravity, otherwise one should use FPS's or Dong's proposal (see the talk by Mohammad Hassan).

Holographic Mutual Information (HMI):

To compute MI one can use holography: For simplicity, just consider a 2d CFT, then A_1 and A_2 can be two separated intervals.
One can use the RT prescription to read MI. S_{A_1} and S_{A_2} are simply given by:

$$S_{A_i} = \frac{Area(m_i)}{4G_N} \qquad i = 1, 2$$

There are always two configurations which contribute to $S_{A_1 \cup A_2}$: Connected and disconnected.

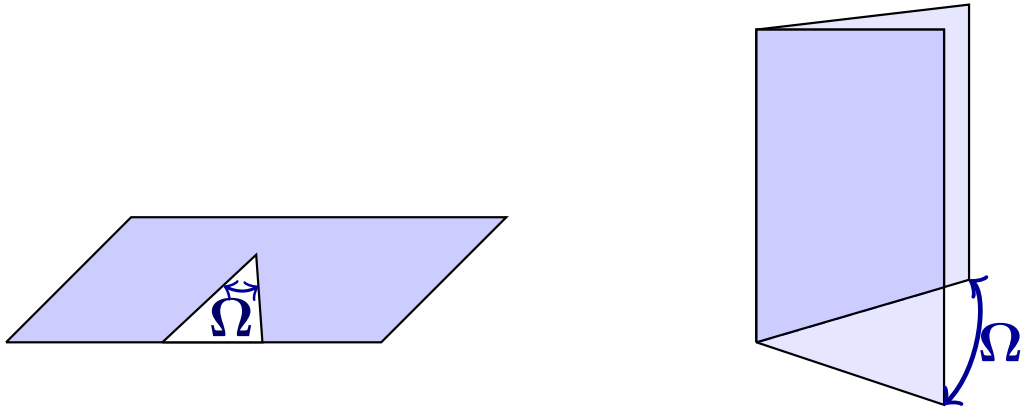
One should choose the minimum of S_{dis} and S_{con}.

$$S_{A_1 \cup A_2} = min\{S_{dis}, S_{con}\}$$

The minimum depends on the length of $A_{1,2}$ as well as the distance between them. HMI shows a first order phase transition (Headrick '10).

Singular Entangling Regions:

Question: what happens to EE, MI, and n-partite information when there are some geometrical singularities in the boundary of the entangling regions.



Geometrical singularities leads to new terms like $log(\frac{l}{\epsilon})$ and $\left(log(\frac{l}{\epsilon})\right)^2$, in EE. For example in a 3d CFT in its vacuum state, one has (Casini, Huerta '06 - Hirata, Takayanagi '06 - Myers, Singh '12)

$$S(\Omega) = \frac{L^2}{2G_N}\frac{H}{\epsilon} - a(\Omega)\log\frac{H}{\epsilon} + \mathcal{O}\left(\frac{\epsilon}{H}\right),$$

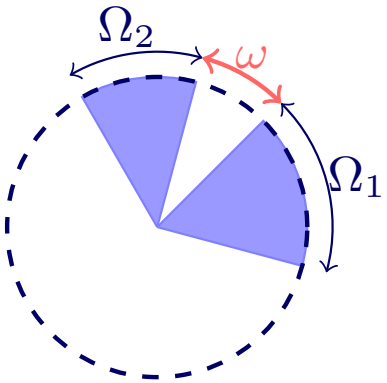
in the Smooth Limit ($\Omega \to \pi$):

$$a(\Omega)\mid_{\Omega\to\pi} = \sigma(\pi-\Omega)^2 + \cdots,$$

σ is related to the central charge of the CFT. (Bueno, Myers, Witczack-Krempa '15)

HMI for two Slices of a Cake in a 3d CFT:

For two smooth and disjoint entangling regions in a 3d CFT, HMI is independent of the UV-cutoff. Can the presence of corners change this behavior? To answer the question we have considered a configuration in the shape of cake slices. We restrict ourselves to a CFT_d which is in its vacuum state and has a gravity dual on AdS_{d+1}. For the moment we set $d = 3$.



$$I(\Omega_1, \Omega_2) = S_{\Omega_1} + S_{\Omega_2} - S_{\Omega_1 \cup \Omega_2}$$

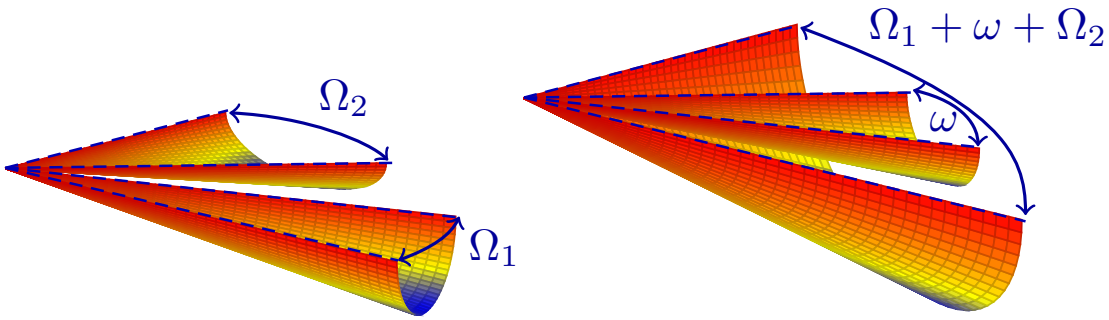
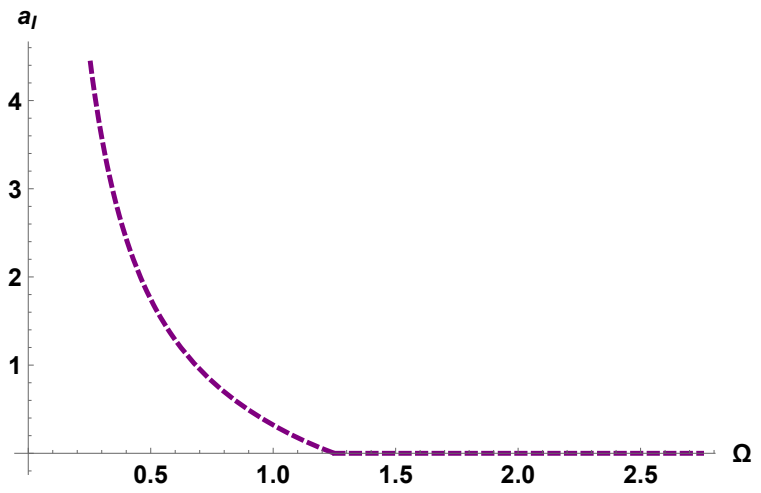


Figure: RT surfaces corresponding to $S_{\Omega_1 \cup \Omega_2}$ for disconnected (left) and connected (right) configurations.

The area law terms cancel each others, but the new log-terms remains.
Hence, HMI depends on the UV- cut off ϵ logarithmically.

$$I = a_I \log \frac{H}{\epsilon} + \mathcal{O}(\epsilon^0)$$

This behavior is in contrast to that of HMI for smooth entangling regions.



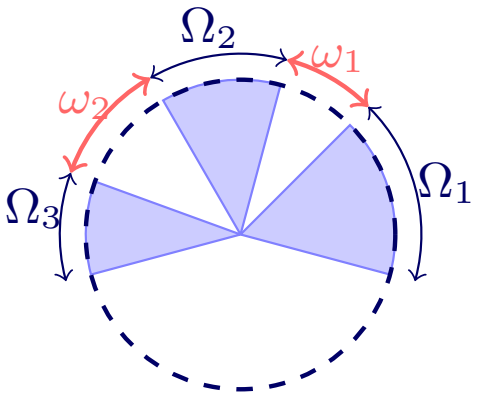
If the separation angle ω becomes large enough, there is a first order phase transition in MI:

- Disconnected configurations dominates the connected one, and MI vanishes.
- Although MI is a continuous function of ω, its first order derivative w.r.t to ω becomes discontinuous.

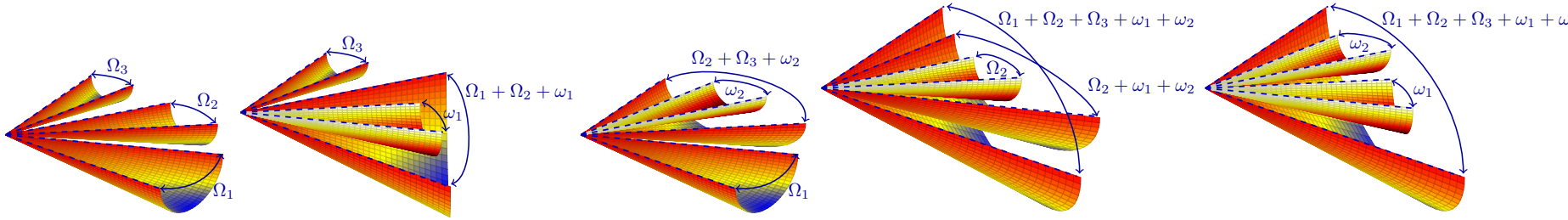
The same behavior happens for smooth entangling regions.

Holographic 3-partite

We have generalized the cake-slice configuration to $n = 3, 4$ and calculated the n-partite information. For $n = 3$



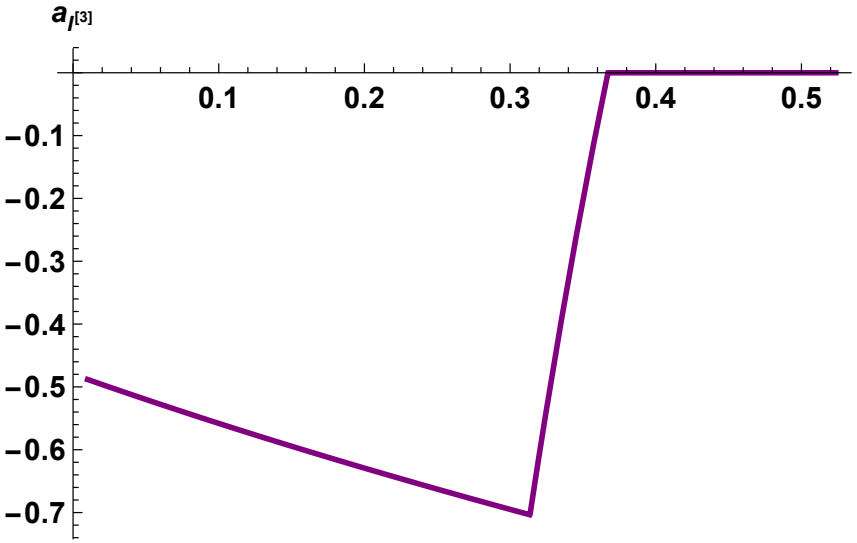
There are more RT surfaces. For example for $S_{\Omega_1 \cup \Omega_2 \cup \Omega_3}$ there are five configurations:

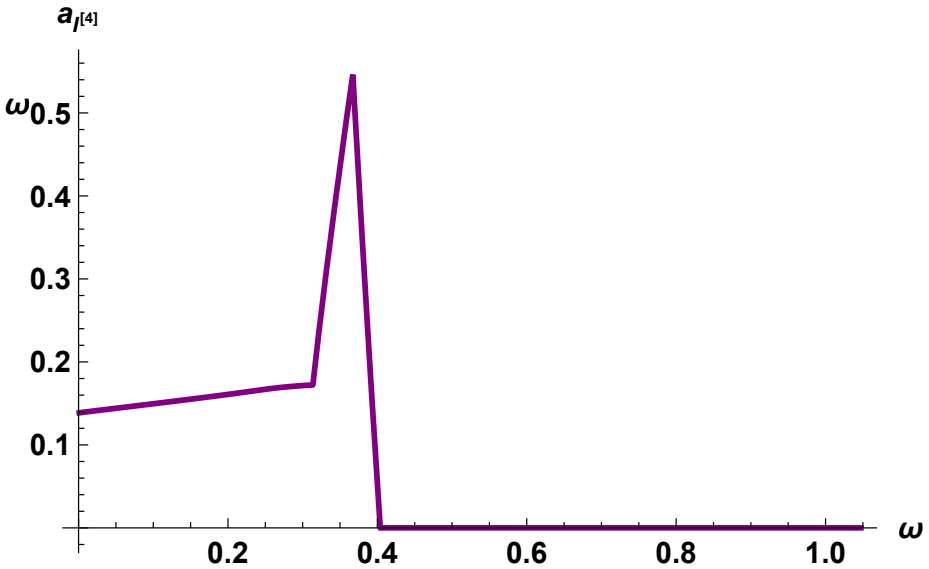


Holographic n-partite information is UV divergent:

$$I^{[n]}(\Omega \sim 0, \omega \sim 0) = \begin{cases} (-1)^n \frac{2\kappa}{n(n-1)(n-2)\Omega} \log \frac{H}{\epsilon} & \omega \ll \Omega \\ 0 & \omega \gg \Omega \end{cases}.$$

This behavior is in contrast to the one for smooth entangling regions (which are independent of the UV cut off).

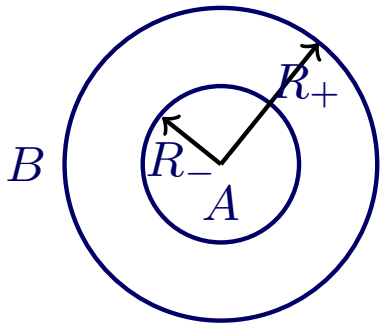


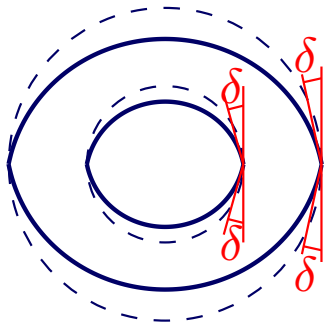


Moreover, In the presence of corners, n-partite information is positive (negative) for even (odd) n.
For smooth entangling surfaces (in some limiting cases) the same behavior had been observed. (Hayden, Headrick, Maloney '11 - Alishahiha, Mozaffar, Tanhayi '14 - Mirabi, Tanhayi- Vazirian '16)

Common Boundary Between Entangling Regions:

Question: What happens to HMI when there is no common boundary between the two entangling regions?





The bulk metric is

$$ds^2 = \frac{1}{z^2}\left(dz^2 - dt^2 + d\rho^2 + \rho^2 d\phi^2\right),$$

the boundary of the entangling regions are parametrized by:

$$t = \text{const.} \quad , \quad \rho = \rho_\pm(\phi) \quad , \quad 0 \leq \phi < 2\pi.$$

such that

$$\frac{\rho_\pm(\phi)}{R_\pm} = \begin{cases} 1 - \delta\,\sin(\phi) & 0 \leq \phi < \pi \\ 1 + \delta\,\sin(\phi) & \pi \leq \phi < 2\pi \end{cases}$$

On the minimal surfaces $\rho = \rho(z, \phi)$, and the area functional is given by

$$\mathcal{A} = \int dz d\phi \frac{1}{z^2}\sqrt{\rho^2\left(1 + \rho'^2\right) + \dot{\rho}^2}$$

Profile of RT surfaces and HMI:

One has to apply perturbation in δ to derive the eom's of the RT surfaces.

$$\rho(z,\phi) = \rho_0(z) + \delta\ \rho_1(z)\sin\phi + \delta^2\ \left[\rho_{20}(z) + \rho_{22}(z)\cos(2\phi)\right],$$

HMI can be computed numerically. It is independent of the UV cut-off to 95%. It is in contrast to the cake-slice configuration in which the regions had a contact point in the center of the cake.

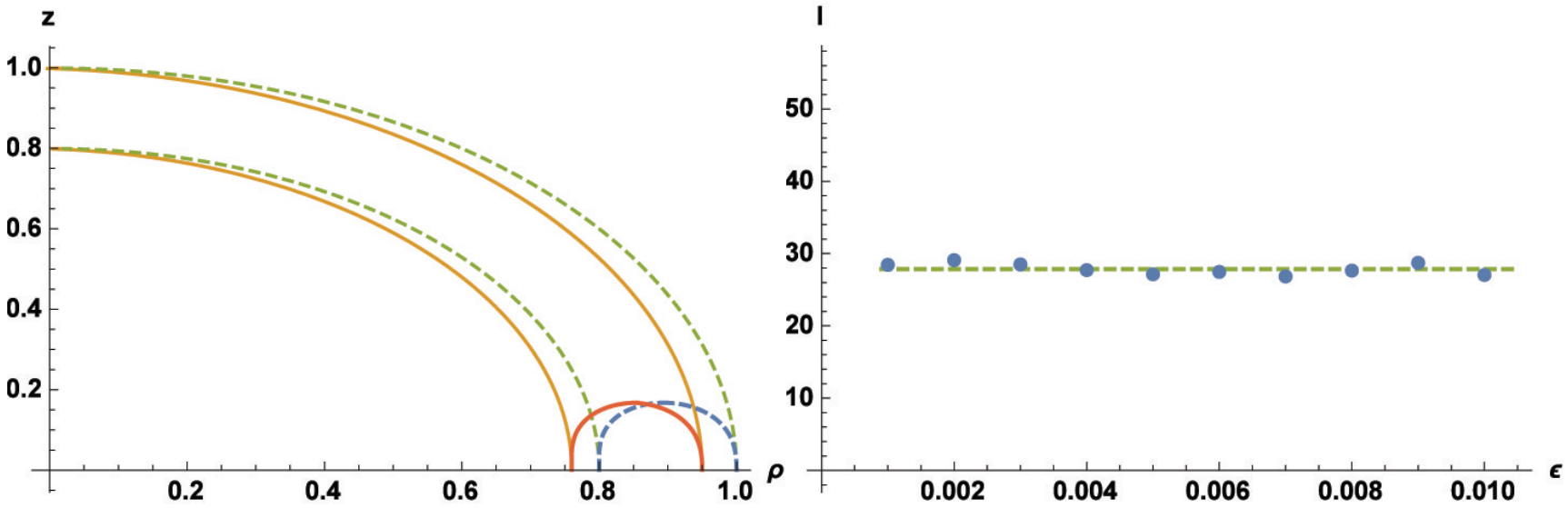


Figure: *Left*: Perturbed minimal surfaces versus unperturbed surfaces. The blue dashed curve represents the connected minimal surface and the dashed green ones represent the disconnected minimal surface for $\delta = 0$. The red and orange curves represent the connected and disconnected minimal surfaces for $\delta = 0.05$. *Right*: The mutual information as a function of ϵ. The dashed green line represents the mean value for mutual information. We have set $R_- = 0.8$, $R_+ = 1$, $\phi = \pi/2$ and $\delta = 0.05$.

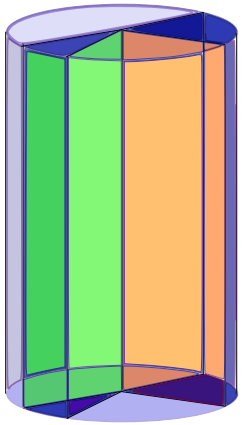
Higher Dimensions

One can generalize the cake-slice geometry to higher dimensional ($d > 4$) CFT's. The bulk metric is an AdS_{d+1}

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\rho^2 + \rho^2 d\theta^2 + \sum_{i=1}^{d-3} dx_i^2\right)$$

For an entangling region of the form

$$t = \text{const.} \quad , \quad 0 < \rho < H \quad , \quad -\frac{\Omega}{2} \leq \theta \leq \frac{\Omega}{2}, \quad 0 < x_i < \tilde{H},$$



Using the RT formula, for each of the entangling regions the HEE is given by

$$S = \frac{L^{d-1}\tilde{H}^{d-3}}{2G_N}\left[\frac{H}{(d-2)\epsilon^{d-2}} + \frac{j(\Omega)}{(d-3)\epsilon^{d-3}}\right] + \mathcal{O}(\epsilon),$$

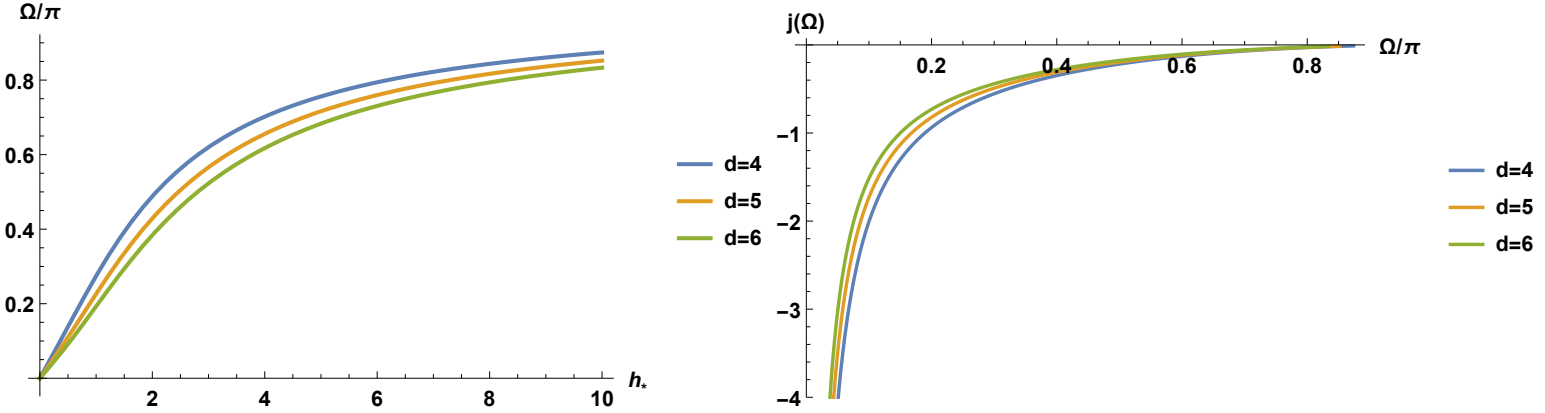


Figure: *Left*: Ω/π as a function of the turning point h_* in different dimensions. *Right*: j as a function of the opening angle Ω in different dimensions.

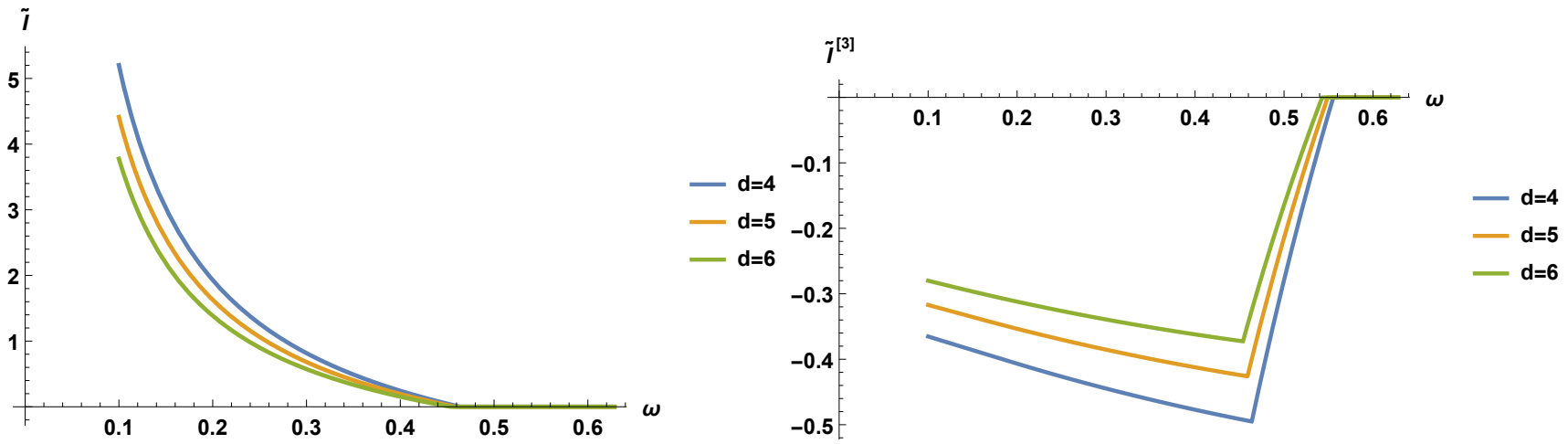


Figure: Holographic mutual (left) and tripartite (right) information in various spatial dimensions for $\Omega = \frac{\pi}{4}$.

Again the 3-partite information is a negative quantity.

Conclusion

Geometrical singularities on the boundary of the entangling regions (i.e. entangling surfaces) lead to some interesting properties:

- HMI shows a first order phase transition, the same as smooth entangling surfaces.
- HMI depends on the UV cut-off, when the singular entangling regions share some boundaries together.
- when there is no common boundary, HMI is an almost independent of UV-cut off.

For Holographic n-partite information one has:

- n-partite information have definite sign $(-1)^n$ for $n = 3, 4$.
- In contrast to the case of smooth entangling surfaces, 3- and 4-partite information depend on the UV cut off of the CFT and are not finite quantities.

Thank you very much