Holographic complexity

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Based on

M.A, "Holographic complexity," arXiv:1509.06614

And Works in progress with

A. F. Astameh, M.R.M. Mozaffar, A. Naseh and J. Pedraza, " Reduced Free Energy and Complexity,"

and Motivated by

Susskind, " Computational Complexity and Black Hole Horizons," arXiv:1402.5674

- AdS/CFT correspondence as a concrete realization of holographic principle could provide a framework to study quantum gravity and black hole physics.

- Quantum information theory may also provide a useful tool to study physics of black holes in gravitational theories.

- It would be interesting or might even be crucial to understand quantum information theory holographically.

• Holographic entanglement entropy is an explicit example in this direction.

Indeed quantum entanglement might be used to understand the nature of the space time geometry.

• In quantum information theory there are other quantities, such as n-partite information, which might be of interest from holography point of view.

Actually n-partite information has been also studied holographically.

Entanglement may not be enough!

Calabrese, Cardy, Tonni, arXiv:1011.5482 [hep-th]

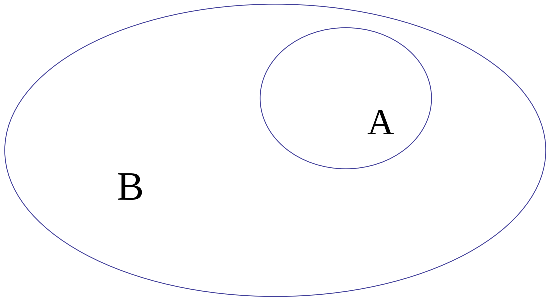
Even if we could compute entanglement entropy or in general n-partite information (directly or holographically), it might not be enough to fully understand the system under consideration quantum mechanically.

It is because no matter which entanglement measure is being computed, we may lose some information of the system simply because the whole system is not a sum of the subsystems.

Motivated by holographic entanglement entropy we would like to define holographic complexity and explore its properties within the context of AdS/CFT correspondence.

Entanglement entropy

Assume that the quantum system has multiple degrees of freedom and so one can decompose the total system into two subsystems A and B



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

The reduced density matrix of the subsystem A

$$\rho_A = \mathrm{Tr}_B(\rho_{\mathrm{total}})$$

Then the entanglement entropy is defined as the von Neumann entropy for A

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

AdS/CFT correspondence

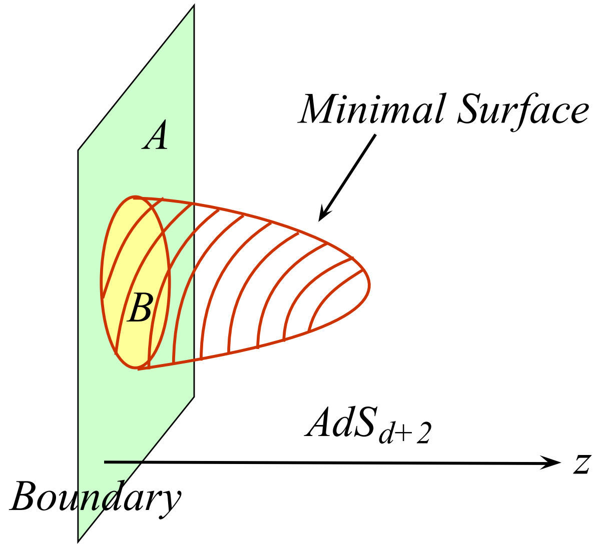
Basically AdS/CFT correspondence is a duality or a relation between two theories one with a gravity and the other without gravity.

The gravitational theory is usually defined in higher dimension.

Well developed case is the one where the gravity is defined on an AdS geometry and the dual theory is a CFT living in the conformal boundary of AdS space.

Holographic Formula for Entanglement Entropy

For static background and fixed time divide the boundary into A and B. Extend this division $A \cup B$ to of the bulk spacetime. Extend ∂A to a surface γ_A in the entire spacetime such that $\partial\gamma_A = \partial A$.

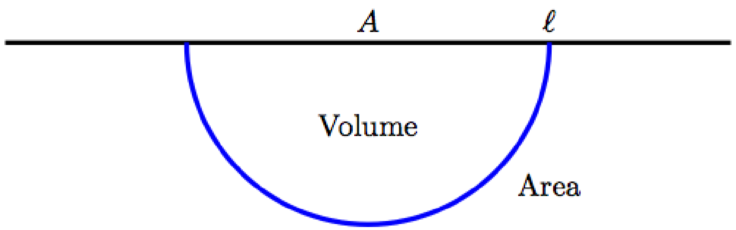


$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}$$

Ryu, Takayanagi, hep-th/0603001.

Holographic complexity

For a static background for a time slice space consider the volume enclosed by the RT curve.



$$\mathcal{C}_A = \frac{\text{Volume}(\gamma_A)}{8\pi G L}$$

where L is the radius of the space time. It is clear that

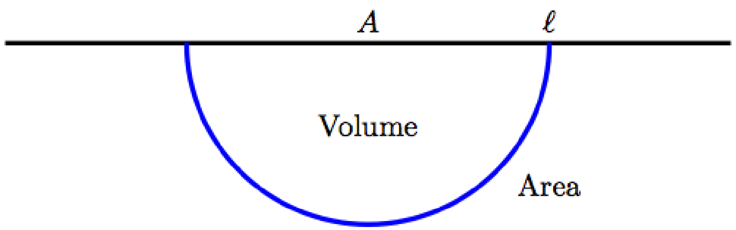
$$\mathcal{C}_A + \mathcal{C}_{\bar{A}} \leq V_{\text{ts}}$$

V_{ts} is the volume of the time slice.

For a time-dependent case, one could consider the volume enclosed by the extremum curve.

Holographic Complexity for a sphere

Consider a gravitational theory on an AdS_{d+2} geometry. Using RT prescription one may compute holographic entanglement entropy for a sphere with radius ℓ.



To do so, it is more convenient to take the following parametrization for the AdS geometry

$$ds^2 = \frac{L^2}{r^2}\left(-dt^2 + dr^2 + d\rho^2 + \rho^2 d\Omega_{d-1}^2\right).$$

Then the entangling region is given by $t =$ fixed, $\rho \leq \ell$.

To compute the holographic entanglement entropy one needs to minimize the area of a co-dimension two hyper-surface in the bulk which may be parametrized by $\rho = f(r)$

$$A = \Omega_{d-1} L^d \int dr \, \frac{f(r)^{d-1}\sqrt{1+f'(r)^2}}{r^d}$$

It is easy to see that the area is minimized for

$$f(r) = \sqrt{\ell^2 - r^2}$$

Ryu, Takayanagi, hep-th/0603001.

Following our proposal one needs to evaluate the volume enclosed by the above minimal area

$$
\begin{aligned}
V &= \Omega_{d-1} L^{d+1} \int_{\rho \leq f(r)} d\rho \, dr \, \frac{\rho^{d-1}}{r^{d+1}} \\
&= \frac{\Omega_{d-1} L^{d+1}}{d} \int_{\varepsilon}^{\ell} dr \, \frac{(\ell^2 - r^2)^{d/2}}{r^{d+1}},
\end{aligned}
$$

whereΩ_{d-1} is the volume of the unit sphere S^{d-1} and ε should be thought of as a UV cut-off.

For even dimensional CFT's one arrives at

$$\mathcal{C}_A = \frac{\Omega_{d-1} L^d}{8d\pi G}\Bigg(\frac{1}{d}\frac{\ell^d}{\varepsilon^d} - \frac{d}{2(d-2)}\frac{\ell^{d-2}}{\varepsilon^{d-2}} + \frac{d(d-2)}{8(d-4)}\frac{\ell^{d-4}}{\varepsilon^{d-4}} + \cdots - (-1)^{[\frac{d}{2}]}\frac{\pi}{2}\Bigg),$$

for $d = 1, 3, 5, \cdots$.

For odd dimensional CFT's one gets

$$\mathcal{C}_A = \frac{\Omega_1 L^2}{16\pi G}\left(\frac{\ell^2}{2\varepsilon^2} - \log\frac{\ell}{\varepsilon} - \frac{1}{2}\right),$$

$$\mathcal{C}_A = \frac{\Omega_3 L^4}{32\pi G}\left(\frac{\ell^4}{4\varepsilon^4} - \frac{\ell^2}{\varepsilon^2} + \log\frac{\ell}{\varepsilon} + \frac{3}{4}\right),$$

$$\mathcal{C}_A = \frac{\Omega_5 L^6}{48\pi G}\left(\frac{\ell^6}{6\varepsilon^6} - \frac{3\ell^4}{4\varepsilon^4} + \frac{3\ell^2}{2\varepsilon^2} - \log\frac{\ell}{\varepsilon} - \frac{11}{12}\right),$$

for $d = 2, 4, 6,$ respectively.

One observes that:

• The most divergent term in the expression of holographic complexity is proportional to the volume of the subsystem $V(A)$

$$\mathcal{C}_A = \frac{L^d}{8d\pi G} \frac{V(A)}{\varepsilon^d} + \cdots ,$$

This should be thought of as an analogous to the celebrated area law of the entanglement entropy.

• For arbitrary d, the holographic complexity contains a universal term - independent of the UV cut off

$$\mathcal{C}_A^{\text{uni}} = (-1)^{[\frac{d}{2}]} \begin{cases} \frac{\Omega_{d-1} L^d}{16 d G} & \text{odd}\, d, \\ \frac{\Omega_{d-1} L^d}{8 d \pi G} & \text{even}\, d\,. \end{cases}$$

Also independent of the size of the subsystem ℓ, indicating that it could reflect certain intrinsic properties of the theory under consideration.

Holographic complexity for strip

Let us redo the same computations for the case where the entangling region is a strip. In this case it is useful to parametrize the metric as follows

$$ds^2 = \frac{L^2}{r^2}(-dt^2 + dr^2 + dx^2 + dY_{d-1}^2)$$

Then the entangling region is given by $-\ell \leq x \leq \ell$ and setting $x = f(r)$ the entanglement entropy is obtained by minimizing the area functional

$$A = V_{d-1} L^d \int dr \frac{\sqrt{1 + f'(r)^2}}{r^d}$$

One finds

$$f'(r) = -\frac{\left(\frac{r}{r_t}\right)^d}{\sqrt{1 - \left(\frac{r}{r_t}\right)^{2d}}}$$

Now let us compute the complexity defined by the volume in the bulk enclosed by the above RT curve

$$\mathcal{C} = \frac{\mathsf{V}(\gamma)}{8\pi G L} = 2 \times \frac{V_{d-1} L^d}{8\pi G} \int dr\, \frac{f(r)}{r^{d+1}},$$

By making use of an integration by part one arrives at

$$\begin{aligned}\mathcal{C} &= \frac{V_{d-1} L^d}{4d\pi G} \frac{f(\varepsilon)}{\varepsilon^d} - \frac{V_{d-1} L^d}{4d\pi G} \int dr \frac{f'(r)}{r^d} \\ &= \frac{V_{d-1} L^d}{4d\pi G} \frac{\ell}{\varepsilon^d} - \frac{V_{d-1} L^d}{4d\pi G r_r^{d-1}} \int_0^1 d\zeta \frac{1}{\sqrt{1-\zeta^{2d}}}\end{aligned}$$

Finally one gets

$$\mathcal{C} = \frac{V_{d-1} L^d}{4 d \pi G} \left(\frac{\ell}{\varepsilon^d} - \frac{C_0}{\ell^{d-1}} \right)$$

Note also that

$$\mathcal{C} = \frac{L^d}{4 d \pi G} \frac{V(A)}{\varepsilon^d} - \cdots$$

which is the volume law.

Thermal state

It is also interesting to compute the holographic complexity for a subsystem in an excited state.

$$ds^2 = \frac{R^2}{r^2}\left(-h(r)dt^2 + \frac{dr^2}{h(r)} + d\rho^2 + \rho^2 d\Omega_{d-1}^2\right),$$

where $h(r) = 1 - mr^{d+1}$ with m being a constant. Taking into account that in the present case the energy of the excited state is proportional to $\mathcal{E} \sim \frac{R^d}{G} ml^d$, up to numerical factors, one arrives at ($m\ell^{d+1} \ll 1$)

$$\begin{aligned}
\Delta S_{\mathsf{EE}} &\sim \frac{R^d}{G}\epsilon \sim \mathcal{E}\ell, \\
\Delta \mathcal{C}_A &\sim (d-1)\frac{R^d}{G}\epsilon^2 \sim (d-1)\frac{G}{R^d}\mathcal{E}^2\ell^2,
\end{aligned}$$

The first relation known as the first law of entanglement entropy.

Bhattacharya, Nozaki, Takayanagi, Ugajin, 1212.1164; Allahbakhshi, M. A., Naseh,1305.2728; Blanco, Casini, Hung, Myers,1305.3182.

Field theory description

In quantum information there are several quantities which could provide measures to compare two states.

- Relative entropy

- Fidelity

The aim of this section is to investigate whether there is any connection between holographic complexity and fidelity.

Fidelity

In quantum information theory fidelity is a measure for closeness of two states: How close or similar two stets are.

Consider a one parameter quantum state $|\Psi(\lambda)\rangle$, where λ is tunable parameter.

Let us also denoting by $\varrho(\lambda_1)$ and $\varrho(\lambda_2)$ the density matrices associated with two states $|\Psi(\lambda_1)\rangle$ and $|\Psi(\lambda_2)\rangle$ respectively. Then the fidelity may be defined as follows (Uhlmann, 1976)

$$F = \mathrm{Tr}\sqrt{\sqrt{\varrho(\lambda_1)}\,\varrho(\lambda_2)\,\sqrt{\varrho(\lambda_1)}},$$

If both states are pure

$$F = |\langle\Psi(\lambda_1)|\Psi(\lambda_2)\rangle|$$

Let us start with a quantum pure state $|\Psi(\lambda_1)\rangle$ in the Hilbert of a quantum system. Now consider a neighboring pure state $|\Psi(\lambda_2)\rangle$ which may be reached by changing, infinitesimally, the parameter λ.

For sufficiently small $\delta\lambda = \lambda_2 - \lambda_1$ the fidelity may be expanded as follows

$$F = |\langle\Psi(\lambda_1)|\Psi(\lambda_2)\rangle| = 1 - (\delta\lambda)^2 G_\lambda$$

G_λ is fidelity susceptibility. This expression, considered as a metric, could measure the distance between two neighboring quantum pure states. (Braunstein, Caves, 1994)

For certain case there is a holographic description for the fidelity susceptibility.

The fidelity susceptibility for a $d+1$ dimensional CFT deformed by an exactly marginal perturbation is holographically estimated by

$$G_\lambda = n_d \frac{\mathrm{Vol}(\Sigma)}{R^{d+1}}$$

where n_d is an order one constant, R is the AdS radius. The $d+1$ dimensional space-like surface Σ is the time slice with the maximal volume in the AdS which ends on the time slice at the AdS boundary(ies).

In particular using three dimensional Janus solution it is found

$$G_\lambda = \frac{c}{12\pi}\frac{V_1}{\varepsilon},$$

where $c = \frac{3R}{2G}$ is central charge, L is the volume of one dimensional spatial direction and ε is a UV cut-off.

Miyaji, Numasawa, Shiba, Takayanagi, Watanabe, arXiv:1507.07555.

In our case and for a three dimensional space-time the holographic complexity reads

$$\mathcal{C}_A = \frac{c}{12\pi}\frac{\ell}{\varepsilon} - \frac{c}{24}$$

For large entangling region setting $\ell = V_1$ one finds

$$\mathcal{C}_A = G_\lambda$$

More generally for a $d+1$ dimensional CFT and for large entangling region one gets

$$\mathcal{C}_A = \frac{R^d V_d}{8 d\pi G \varepsilon^d} = G_\lambda$$

For general finite entangling region one has

$$\mathcal{C}_A + \mathcal{C}_{\bar{A}} \leq G_\lambda$$

• Note that the above comparison works just for the extremely large ℓ limit. In other words there is, a priori, no way to understand the subleading divergences in this picture.

• Moreover the holographic information metric computations were based on the assumption that two states are pure, which could not be case when we restrict ourselves to a subsystem.

• Nonetheless it might be possible to extend the notation of fidelity susceptibility for a subsystem with a finite size.

To explore this point better let us consider the general definition of fidelity

$$F = \mathrm{Tr}\sqrt{\sqrt{\varrho(\lambda_1)}\,\varrho(\lambda_2)\,\sqrt{\varrho(\lambda_1)}},$$

Since we are dealing with a subsystem, it is natural to consider a reduced density matrix and therefore define the fidelity for two reduced density matrices (Zhou:2007)

$$F_A = \mathrm{Tr}_A\sqrt{\sqrt{\varrho_A(\lambda_1)}\,\varrho_A(\lambda_2)\,\sqrt{\varrho_A(\lambda_1)}},$$

where $\varrho_A(\lambda_1)$ and $\varrho_A(\lambda_2)$ are the corresponding reduced density matrices.

Now the aim is to expand the reduced fidelity for a small perturbation to find an expression for reduced fidelity susceptibility.

To proceed, we will take advantage of having a subsystem in the shape of a sphere.

Actually when we have a subsystem with spherical symmetry in the ground state of a CFT, one may conformally map the system to a thermal system whose temperature is $\beta = 2\pi\ell$

Under this conformal map the reduced density matrix maps to a thermal density matrix given by

$$\varrho_{\text{th}}(\lambda) = \frac{e^{-2\pi\ell H_\tau(\lambda)}}{\text{Tr}(e^{-2\pi\ell H_\tau(\lambda)})},$$

H_τ is the standard Hamiltonian of the thermal system which corresponds to the time translation.

Casini, Huerta, Myers, arXiv:1102.0440.

As a result we will have to compute fidelity at finite temperature.

Fidelity for a mixed state at finite temperature has been studied and two possibilities have been considered: (Zanardi, Quan, Wang, 2007.)

Either to change the temperature while keeping the parameter λ fixed, or another way around.

In our model since we are dealing with a system with a fixed temperature (fixed ℓ) the corresponding thermal fidelity should be given as follows

$$F(2\pi\ell, \lambda_1, \lambda_2) = \text{Tr}\sqrt{\sqrt{\varrho_{\text{th}}(\lambda_1)}\; \varrho_{\text{th}}(\lambda_2)\; \sqrt{\varrho_{\text{th}}(\lambda_1)}}.$$

Therefore our problem reduces to evaluating fidelity for a thermal system.

For the case we are interested in the fidelity can be approximated as follows (*e.g.* Quan, Cucchietti, 0806.4633)

$$F(2\pi\ell, \lambda_1, \lambda_2) \approx \frac{Z(2\pi\ell, \frac{\lambda_1+\lambda_2}{2})}{\sqrt{Z(2\pi\ell, \lambda_1)Z(2\pi\ell, \lambda_2)}},$$

where Z is partition function

$$Z(2\pi\ell, \lambda) = \mathrm{Tr}\left(e^{-2\pi\ell H_\tau(\lambda)}\right)$$

It may be also written in terms of free energy $\mathcal{F}_{th}$.

Let us compute (approximately) the susceptibility

$$
\begin{aligned}
\chi_\lambda &= \frac{\partial^2 \mathcal{F}_{th}}{\partial \lambda^2} \\
&\approx \frac{\mathcal{F}_{th}(\lambda_1 + \delta\lambda/2) + \mathcal{F}_{th}(\lambda_1 - \delta\lambda/2) - 2\mathcal{F}_{th}(\lambda_1)}{(\delta\lambda/2)^2} \\
&= -\frac{8}{(\delta\lambda)^2 \beta} \frac{2 \ln Z(\lambda_1) - \ln Z(\lambda_1 + \delta\lambda/2) - \ln Z(\lambda_1 - \delta\lambda/2)}{2} \\
&= -\frac{8}{(\delta\lambda)^2 \beta} \ln \frac{Z(\lambda_1)}{\sqrt{Z(\lambda_1 + \delta\lambda/2) Z(\lambda_1 - \delta\lambda/2)}}
\end{aligned}
$$

Therefore one arrives at

$$
\begin{aligned}
F(2\pi\ell, \lambda_1, \lambda_1 + \delta\lambda) &\approx \frac{Z(\lambda_1)}{\sqrt{Z(\lambda_1 + \delta\lambda/2) Z(\lambda_1 - \delta\lambda/2)}} \\
&\approx 1 - (2\pi\ell)\, \chi_\lambda \, \frac{\delta\lambda^2}{8}
\end{aligned}
$$

χ_λ is fidelity susceptibility.

If one perturbs the system by an operator with dimension Δ, then the fidelity susceptibility scales as $\frac{R^d}{\varepsilon^{2\Delta-2-d}}$, where R is a scale of the model (S J. Gu, arXiv:0811.3127)

Thus for a marginal operator where $\Delta = d+1$ one gets

$$\chi_\lambda \sim \left(\frac{R}{\varepsilon}\right)^d$$

.

The free energy and also susceptibilities receive finite temperature corrections which have an expansion in power of T^2 (M. Laine, " Basics of Thermal Field Theory, " 2013).

Therefore for our mixed thermal state, where the temperature is given by $T = \frac{1}{2\pi\ell}$, one arrives at

$$\chi_\lambda \sim \frac{R^d}{\varepsilon^d}\left(1 + c_2\frac{\varepsilon^2}{\ell^2} + c_4\frac{\varepsilon^4}{\ell^4} + \cdots\right),$$

in qualitative agreement with our results in the previous section.

Using the inverse of the conformal map to return to the original picture of the reduced density matrix, one gets

$$F_A = 1 - \partial_\lambda^2 \mathcal{F} \, \frac{\delta\lambda^2}{8} + \mathcal{O}(\delta\lambda^3),$$

where

$$\mathcal{F}(\lambda) = \mathrm{Tr}\Big(\varrho_A(\lambda) H(\lambda)\Big) - S_{\mathrm{EE}}(\lambda)$$

S_{EE} is the entanglement entropy and, $H(\lambda)$ is the modular Hamiltonian.

$$\varrho_A(\lambda) = \frac{e^{-H(\lambda)}}{\mathrm{Tr}(e^{-H(\lambda)})}.$$

In terms of the modular Hamiltonian the fidelity susceptibility $\chi = \partial_\lambda^2 \mathcal{F}$ is given by

$$\chi = \langle H^2 \rangle - \langle H \rangle^2.$$

The proposal

For a field theory with a gravitational dual one can define a quantity which is proportional to the volume of a co-dimension one time slice in the bulk geometry enclosed by the extremal co-dimension two hyper-surface appearing in the computation of the holographic entanglement entropy.

The most divergent term of the computed quantity may be related to the fidelity susceptibility appearing in quantum information literature.

How essential the volume is?

Essentially we have computed the volume of certain part of a time slice

$$\mathcal{C}_A = \frac{1}{8\pi G L}\int d^{d+1}x\sqrt{\tilde{g}},$$

where $\tilde{g}$ is the determinant of the induced metric on $t=$constant hypersurface. Note that L is inserted by hand which look unnatural!

As far as our considerations are concerned this expression every things remain unchanged up to an overall factor.

A natural numerical factor could naturally come from the Lagrangian density evaluated for the solution

$$\tilde{\mathcal{F}} = \int d^{d+1}\sqrt{\tilde{g}}\,(R-2\Lambda)|_{\text{on-shell}}.$$

which might be compared with the free energy

$$\mathcal{F} = \int d^{d+1}\sqrt{g}\,(R-2\Lambda)|_{\text{on-shell}}.$$

Reduced free energy

For Einstein gravity where the action is given by

$$I = -\frac{1}{16\pi G}\int d^{d+1}\sqrt{g}(R - 2\Lambda),$$

and taking into account that $R - 2\Lambda = -\frac{2(d+1)}{L^2}$ one gets

$$\tilde{\mathcal{F}} = \frac{d+1}{8\pi G L^2}\int d^{d+1}x\,\sqrt{\tilde{g}}.$$

It is then natural to define

$$\frac{d\mathcal{C}}{dt} = \tilde{\mathcal{F}}.$$

Up to an overall factor it has the same expression as the holographic complexity we have been considering.

Its divergent terms could still be dual to fidelity susceptibility. There is however a finite term in any dimension.

When we are dealing with an action it usually comes with certain boundary terms. Adding these one finds a finite term. This can be used to find the finite term of the reduced free energy.

$$\tilde{\mathcal{F}} = \bar{I}_{\rm EH} + \bar{I}_{GH} + \bar{I}_{\rm ct},$$

where

$$\bar{I}_{\rm EH} = -\frac{1}{16\pi G}\int d^{d+1}x\,\sqrt{\bar{g}}\,(R - 2\Lambda), \qquad \bar{I}_{\rm GH} = -\frac{1}{8\pi G}\,\frac{d+1}{d}\int d^d x\sqrt{\bar{h}}\bar{K},$$

and

$$\begin{aligned}\bar{I}_{\rm ct} \;=\; & \frac{1}{16\pi G}\frac{d+1}{d}\int d^d x\sqrt{\bar{h}}\Big[\frac{2(d-1)}{L} + \frac{L}{(d-2)}\bar{\mathcal{R}} \\ & + \frac{L^3}{(d-1)(d-3)(d-4)}\Big(\bar{\mathcal{R}}_{ij}\bar{\mathcal{R}}^{ij} - \frac{d+1}{4d}\bar{\mathcal{R}}^2\Big) + \cdots\Big].\end{aligned}$$

Note that for $d = 2$ and $d = 4$ we have log divergent terms as follows

$$\text{For } d = 2 \quad : \quad -\frac{1}{16\pi G}\int d^2x\sqrt{\bar{h}}\,\frac{3L}{2}\,\bar{\mathcal{R}}\log\frac{\varepsilon}{2L},$$

$$\text{For } d = 4 \quad : \quad -\frac{1}{16\pi G}\int d^dx\,\sqrt{\bar{h}}\frac{15L^3}{36}\left(\bar{\mathcal{R}}_{ij}\bar{\mathcal{R}}^{ij} - \frac{d+1}{4d}\bar{\mathcal{R}}^2\right)\log\frac{\varepsilon}{2L}.$$

One may still need more counter terms depending on the shape of the entangling region.

Strip in black brane solution

The black brane solution is

$$ds^2 = \frac{L^2}{r^2}\left(-F(r)dt^2 + \frac{dr^2}{F(r)} + dx^2 + dY_{d-1}^2\right), \qquad F(r) = 1 - \frac{r^{d+1}}{r_H^{d+1}}.$$

and the entangling region is given by $t = 0$, $-\ell \le x \le \ell$.

$$\frac{d}{dt}\Delta\mathcal{C} \approx \frac{d^2-1}{d^2}\,\frac{\ell}{L}\,T_{\mathrm{E}}\,S_{\mathrm{EE}}^{\mathrm{finite}}$$

where $T_E \sim \frac{1}{\ell}$ and $\Delta\mathcal{C} = \mathcal{C}_A - \mathcal{C}_{\mathrm{total}}$.

The coefficients are not universal and depend on the state and the entangling region.

Reduced free energy for total subspace

One may also compute reduced free energy for the co-dimension one hyper-surface given by $t = 0$.

- For AdS metric in Poincare coordinates the finite part is zero.

- Black brane

$$ds^2 = \frac{L^2}{r^2}\left(-F(r)dt^2 + \frac{dr^2}{F(r)} + dx^2 + dY_{d-1}^2\right), \qquad F(r) = 1 - \frac{r^{d+1}}{r_H^{d+1}}.$$

one gets

$$\frac{d\mathcal{C}}{dt} = \frac{(d-1)c_0}{(d+1)d}\,\frac{r_H}{L}\,TS_{\text{th}}.$$

• Black hole solution

$$ds^2 = \frac{L^2}{r^2}\left(-F(r)dt^2 + \frac{dr^2}{F(r)} + L^2 d\Omega_d^2\right), \qquad F(r) = 1 + \frac{r^2}{L^2}\left(1 - \frac{r^{d-1}}{r_H^{d-1}}\right) - \frac{r^{d+1}}{r_H^{d+1}},$$

In this case the internal space is curved and one gets contributions from intrinsic curvature as well. Taking all terms into account for $\frac{L}{r_H} \ll 1$ one finds

$$\frac{d}{dt}\Delta\mathcal{C} \approx \frac{d+1}{d-1} T S_{th} + \cdots$$

To conclude up to a non-universal factor the reduced free energy is

$$\frac{d\mathcal{C}}{dt} \sim TS$$

and for static case we get linear growth: $\mathcal{C} \sim TSt$.

A possible interpretation may be given by the complexity

Computational complexity is a concept from computer science that has to do with quantifying the difficulty of carrying out a task.

A task is generally to start with a simple state and transform the system to some other state.

Computational complexity: The minimum number of simple operations required to carry out the task.

In the context of eternal black hole it is conjectured that the rate of growth

$$\frac{d\mathcal{C}}{dt} \sim TS$$

Susskind, rXiv:1411.0690

Summary

- Inspired by holographic entanglement entropy one may define a quantity given by on-shell action evaluated on the volume of a co-dimension one time slice enclosed by the co-dimension two hyper surface appearing in the computation of holographic entanglement entropy.

- In all dimensions it has a universal terms which could provide a charge for the underling theory.

- The most divergent part may be related to fidelity in the dual theory for a state perturbed by a marginal operator.

- The finite part of the quantity has a right feature to be identified with the rate of growth of the complexity.