# Holographic Entanglement Entropy & Field Redefinition

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**Based on:**

M.R. Mohammadi Mozaffar, A. Mollabashi, M.M. Sheikh-Jabbari, M. H .V,[hep-th/1603.05713]

Recent Trends in String Theory and Related Topics
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# Outline


1. Entanglement Entropy
2. HEE in Higher Curvature Gravitates
3. HEE & Field Redefinition
4. Sum up

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1 Entanglement Entropy

2 HEE in Higher Curvature Gravitates

3 HEE & Field Redefinition

4 Sum up

# EE is an useful quantity

- an important measure in quantum information
- a good order parameter for some quantum phase transitions at $T = 0$
- to better understand RG structure of QFT's [Casini-Huerta]
- emergent gravity and space-time ?!
  [Van Raamsdonk'10-Swingle'09, Jacobson '15]
- Von-Newmann entropy with reduced density matrix may be useful to study out of equilibrium systems .
- ...?

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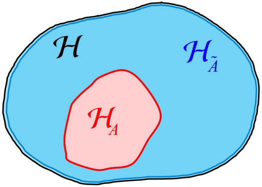
- Von-Newmann entropy with reduced density matrix may be useful to study out of equilibrium systems .
- ...?

It is very difficult to compute in the general case!

# Entanglement Entropy: a measure of entanglement

- Divide a quantum system into two parts $A$ and $\tilde{A}$: $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{H}_{\tilde{A}}$.



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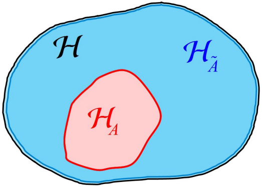

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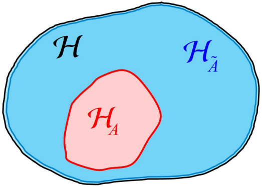

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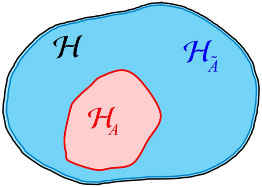

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- Entanglement Entropy (Von-Neumann entropy): $S_A \equiv -\mathrm{Tr}(\rho_A \ln \rho_A)$

**Entanglement entropy**

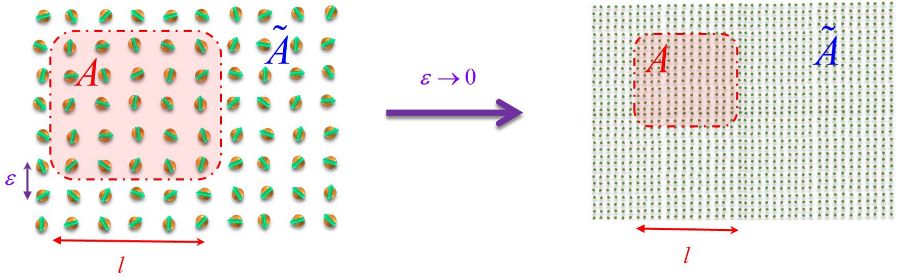
A measure of entanglement of a system with $\rho = |\psi\rangle\langle\psi|$.

$$S(A) = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

# Entanglement Entropy in Field Theory

- In a local QFT, a space-like boundary $\partial A$ divides $\mathcal{H}$ to two parts.
- EE obeys area law [Sorkin '84- Bombelli et.al. '86 - Srednicki '93,...].

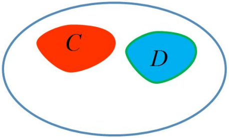
$$
\begin{aligned}
S_{EE}(A) &= c_1 (l/\epsilon)^{d-2} + c_3\,(l/\epsilon)^{d-4} + \cdots \\
&\cdots + \begin{cases} c_{d-2}\,(l/\epsilon) + c_{d-1} + \mathcal{O}(\epsilon/l), & d:\ \text{odd}, \\ c_{d-3}\,(l/\epsilon)^2 + q \ln(l/\epsilon) + \mathcal{O}(1), & d:\ \text{even}, \end{cases}
\end{aligned}
$$

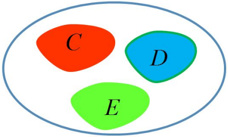


## properties of entanglement entropy

- Area law: $S_{EE} \propto \frac{Area(\partial A)}{\epsilon^{d-1}}$
- Pure state: $S(A) = S(\tilde{A})$
- Thermal state: $\rho_A \to \rho \quad \Rightarrow \quad S_{EE} \to S_{th}$
- Subadditivity: $S(C) + S(D) \geq S(C \cup D)$
- Strong Subadditivity:

$$S(C \cup E) + S(D \cup E) \geq S(E) + S(C \cup D \cup E)$$



# Holographic Entanglement Entropy

- Partition function $\mathcal{Z}$ gives entanglement entropy.

# Holographic Entanglement Entropy

- Partition function $\mathcal{Z}$ gives entanglement entropy.
- AdS/CFT should be useful!

# AdS/CFT (Gauge/Gravity duality)

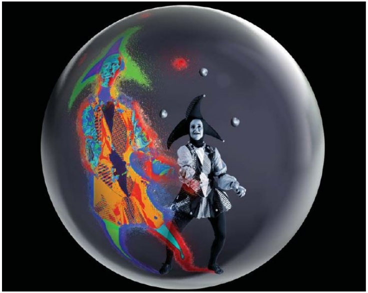
Classical gravity in $\text{AdS}_{d+1}$
$S \sim \int \sqrt{g}(R - 2\Lambda + ...)$

$\xleftrightarrow{\text{AdS/CFT}}$

Strongly Coupled $\text{CFT}_d$
with large # d.o.f

$$\mathcal{Z}_{CFT} = e^{iS|_{on-shell}}$$

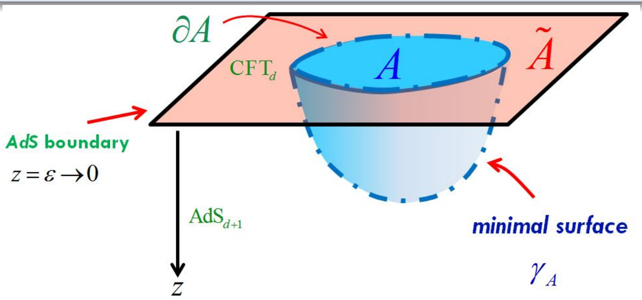
operator on the boundary $\leftrightsquigarrow$ classical field in the bulk.



# Entanglement entropy in the CFT dual to *Einstein gravity*

## Ryu-Takayanagi (RT) prescription

$$S_{EE}(A) \;=\; \frac{\text{minimal Area}(\bar{\gamma}_A)}{4G}\Big|_{\partial\bar{\gamma}_A=\partial A}$$




[Ryu-Takayangi '06]

# Some properties of RT

- RT has passed many consistency checks.
- It can be generalized to time dependent cases.
  [Hubeny-Rangamani-Takayanagi '07]
- It satisfies different entanglement inequalities. [Headrick-Takayanagi '07]
- It is proved in the context of AdS/CFT [Maldacena-Lewkowycz '13]
- It works for Einstein gravity plus minimally coupled matter fields.
- RT gives both the HEE functional and the corresponding surface.

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# Motivations for higher curvature gravities

- They generically arise in the Wilsonian approach.
- As stringy correction to Einstein gravity.
- To better understand AdS/CFT.
- Einstein gravity is limited to CFT's with $c = a$ central charges.

$$\langle T^m_m \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4. \quad (d = 4)$$

- Higher curvatures may extend $AdS_5/CFT_4$ to $CFT_4$ with $c \neq a$. [Buchel-Myers-Sinha '08]

# HEE in Higher Curvature Gravities

**How can we generalize RT to higher curvature gravities?**

$$I \sim \int \sqrt{g} f(R, R_{\mu\nu}, R_{\mu\rho\nu\sigma}) \quad \Rightarrow S_{EE} = ?$$

**It couldn't be area of surface. It could be Wald (In general, No!).**

## Different proposals

- Lovelock Gravity [deBoer-Kulaxizi-Parnachev, Hung-Myers-Smolkin '11]
- Curvature Squared Gravity [Fursaev-Patrushev-Solodukhin '13]
- More General Cases [Dong, Camps '13, Miao-Guo '14]

**They suffer from ambiguity in the determination of surface.**

[Bhattacharyya-Sharma '14]

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# Field Redefinition Invariance

- Physical observables must be invariant under a field reparametrization (change of basis in the Hilbert space).

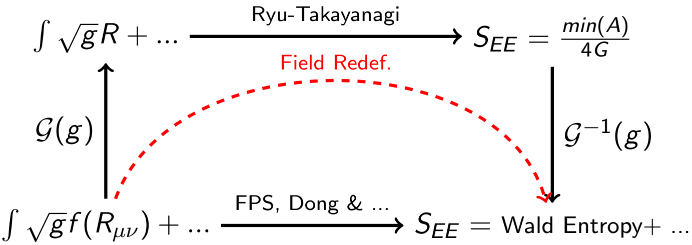
$$\mathcal{Z} = \int \mathcal{D}\Phi e^{iS[\Phi]} = \int \mathcal{D}\bar{\Phi} e^{i\bar{S}[\bar{\Phi}]}$$

- Entanglement entropy should be invariant under an invertible field redefinition.

- AdS/CFT dictionary implies that field redefinition in the one side, should induce a field redefinition in the other side.

# Field Redefinition Invariance of HEE & HCG

- RT has very nice properties!
- We like them and we will save them.
- HEE formula for Einstein gravity gives HEE in $f(R_{\mu\nu})$




[Mohammadi-Mollabashi-Sheikh Jabbari-Vahidinia'16]

# From $f(R_{\mu\nu})$ to Einstein

- Consider the following action for gravity

$$I = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{|g|}\left[f(R_{\mu\nu}) + \mathcal{L}_{matter}(j^1 g_{\mu\nu}, \Phi)\right]$$

- Field equation is given by

$$\frac{\partial f}{\partial g^{\mu\nu}} - \frac{1}{2} f g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\nu}\nabla_\lambda\nabla_\rho - 2g_{\mu\rho}\nabla_\lambda\nabla_\nu + g_{\lambda\nu}g_{\rho\mu}\Box\right)\frac{\partial f}{\partial R_{\lambda\rho}} = T_{\mu\nu},$$

- In general, it is fourth order equation.
- In general, RT formula doesn't work for this theory.

# From $f(R_{\mu\nu})$ to Einstein

$$I = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{|g|} \left[ f(R_{\mu\nu}) + \mathcal{L}_{matter} \right]$$

**Field Redefinition → Einstein gravity + new matter field**

$$I = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{|\bar{g}|} \left[ \bar{R}(\bar{g}) + \bar{\mathcal{L}}_{matter} \right].$$

$$\bar{g}^{\mu\nu} = |g|^{\frac{-1}{d-1}} |\det(\mathcal{F}^{\mu\nu})|^{\frac{-1}{d-1}} \mathcal{F}^{\mu\nu}, \quad \mathcal{F}^{\mu\nu} \equiv \frac{\partial f}{\partial R_{\mu\nu}}$$

[Magnano, Ferraris, Francaviglia'87]

We can apply RT to obtain HEE in CFT dual to Einstein theory.

# General formula

- HEE in CFT dual to Einstein (RT functional)

$$S_{EE} = \frac{1}{4G_{d+1}} \min \int_{\bar{\gamma}_A} d^{d-1}\xi \sqrt{\bar{\gamma}}, \qquad \bar{\gamma}_{ab} = \bar{g}_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^b} \frac{\partial x^\nu}{\partial \xi^b}$$

- Field redefinition of metric $\bar{g}^{\mu\nu} = |g|^{\frac{-1}{d-1}} |\det(\mathcal{F}^{\mu\nu})|^{\frac{-1}{d-1}} \mathcal{F}^{\mu\nu}$

**HEE in CFT dual to $f(R_{\mu\nu})$**

$$S_{EE} = \frac{1}{4G_{d+1}} \min \int_{\gamma_A} d^{d-1}\xi \sqrt{|g| \det\left(\mathcal{F}^{\mu\nu}\right) \det\left(\mathcal{F}^{-1}_{ab}\right)};$$

$$\mathcal{F}_{ab} = \mathcal{F}_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^b} \frac{\partial x^\nu}{\partial \xi^b}$$

[Mohammadi-Mollabashi-Sheikh Jabbari-Vahidinia'16]

# $det(\mathcal{F}^{\mu\nu}) = ?$

Calculation of $det(\mathcal{F}^{\mu\nu})$ is not easy! We restrict ourselves to 3 cases:

1. f(R) theories:
$$f(R_{\mu\nu}) = f(R)$$
2. Perturbations around Einstein-Hilbert theory:
$$f(R_{\mu\nu}) = R - 2\Lambda + \lambda\tilde{f}(R_{\mu\nu}) + \mathcal{O}(\lambda^2)$$
3. Einstein manifold solutions:(It includes AdS space)
$$R_{\mu\nu} = \frac{R}{d+1} g_{\mu\nu}$$

# Special cases

### HEE in $f(R)$

$$S_{EE} = \frac{1}{4G_{d+1}} \min \int d^{d-1}\xi \; f'(R)\sqrt{\gamma},$$

### HEE in $R - 2\Lambda + \lambda \tilde{f}(R_{\mu\nu}) + \mathcal{O}(\lambda^2)$

$$S_{EE} = \frac{1}{4G_{d+1}} \min \int d^{d-1}\xi \; \sqrt{\gamma} \left[ 1 + \frac{\lambda}{2} \tilde{\mathcal{F}}^{\mu\nu} n^i_\mu n^i_\nu \right] + \mathcal{O}(\lambda^2),$$

### HEE in Einstein manifold solutions of $f(R_{\mu\nu})$

$$S_{EE} = \frac{1}{4G_{d+1}\mathcal{X}} \min \int_{\gamma_A} d^{d-1}\xi \sqrt{\gamma}, \qquad \mathcal{X} \equiv \frac{2R}{(d+1)f(R)}.$$

## Cross-check

- Vacuum EE in the even dim. CFT has the following form

$$S_{EE}(A) = c_1 (l/\epsilon)^{d-2} + \cdots + c_{d-3} (l/\epsilon)^2 + q \ln (l/\epsilon) + \mathcal{O}(1)$$

- For *AdS* our formula is RT (up to $\chi$) so it has the correct form.
- To check the overall factor we consider the universal constant $q$.
- $q = q(B_i, A)$ is proportional to central charges of CFT.

$$< T^m_m >= \sum B_i I_i - 2(-1)^{(d/2)} A E_d + \ldots$$

where $I_i$ are the independent Weyl invariants and $E_d$ Euler density [Deser - Schwimmer '93].

- It is known $A \sim \mathcal{L}|_{AdS}$, where $\mathcal{L}$ is Lagrangian of gravity. It is easy to show $A \sim \chi^{-1} \sim f|_{AdS}$. (with the exact coef.) [Sinha & Myers '10]

# A lesson about Riemann

- $q = q(B_i, A)$ depends on shape of region, e.g in 4 dim

$$< T^m_m > \sim c I_4 - a E_4 + \ldots$$

$$S_{EE} = \cdots - 4a \, ln(l/\epsilon) + \cdots , \quad \text{sphere entangling region}$$
$$S_{EE} = \cdots - \# \, c \, ln(l/\epsilon) + \cdots , \quad \text{cylinder entangling region}$$

[Solodukhin '08]

- For Einstein theory $c = a$.
- We learn it is also true for generic $f(R_{\mu\nu})$.
- To go beyond $c = a$, (it seems) one needs Riemann tensor in the gravity action.

# Comparison

- There are several proposals to extend the RT the HEE of higher derivative gravities. [Fursaev-Patrushev-Solodukhin, Dong, Camps'13]

$$S_{EE} = \text{Wald Entropy} + F(\text{Riemann, Extrinsic curveture})$$

- For example, for the curvature squared theory

$$I \sim \int \sqrt{g}\left[R - 2\Lambda + (\alpha_1 R^2 + \alpha_2 R_{\alpha\beta}R^{\alpha\beta} + \alpha_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})\right] ,$$

$$S_{\text{other}} \sim \int d^{d-1}\xi\sqrt{\gamma}[\frac{\partial f}{\partial R_{\mu\nu\rho\sigma}}(n^i_\mu n^i_\rho n^j_\nu n^j_\sigma - n^i_\mu n^i_\sigma n^j_\nu n^j_\rho) - \frac{\alpha_2}{2}\mathcal{K}^i\mathcal{K}_i - 2\alpha_3\mathcal{K}^i_{\mu\nu}\mathcal{K}^{\mu\nu}_i)].$$

# Comparison

- Let $f(R_{\mu\nu}) = R - 2\Lambda + (\alpha_1 R^2 + \alpha_2 R_{\alpha\beta} R^{\alpha\beta})$ perturbatively or on Ein. manifolds
- Field redef. proposal EE

$$S_{\rm FR} \sim \underset{\checkmark}{\min} \int d^{d-1}\xi \sqrt{\gamma} \frac{\partial f}{\partial R_{\mu\nu}} n^i_\mu n^i_\nu$$

- Other proposal for EE e.g. (Dong or FPS)

$$S_{\rm other} \sim \underset{???}{\min} \int d^{d-1}\xi \sqrt{\gamma} \left[ \frac{\partial f}{\partial R_{\mu\nu}} n^i_\mu n^i_\nu - \frac{\alpha_2}{2} \mathcal{K}^i \mathcal{K}_i \right].$$

- Field redef. only cares about Wald!
- Variation of $S_{\rm others}$ functional (at least for Einstein manifolds),

$$\mathcal{A}\mathcal{K}^i + \mathcal{B}_{ij}\mathcal{K}^j = 0,$$

which admits $\mathcal{K}^i = 0$ surface.

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  which admits $\mathcal{K}^i = 0$ surface.
- One may needs a minimal functional to prove subadditivity. Is unitarity of gravity important?
- In the other proposals, when $\alpha_2 > 0$ the minimal functional has $\mathcal{K}^i \neq 0$:
- Only $\mathcal{K}^i = 0$ leads to correct universal term! [Ghodsi-Moghadassi '15]

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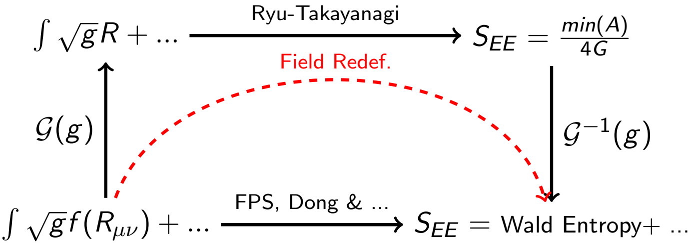
# Sum up

- Most of the generalization of RT suffer from some ambiguities.
- Using field redefinition, we generalise RT to a family of $f(R_{\mu\nu})$.
- It is Wald entropy for a surface which minimize Wald entropy.



# Sum up

- It enjoys all nice properties of RT (e.g. inequalities).
- $(\text{Ricci})^n$ is not enough to have a gravity dual for $c \neq a$ $\text{CFT}_4$.
- Our method, does not cover gravities with Riemann such as Lovelock theories.
- Riemann tensor can be written in terms of Ricci tensor in 3 dim cases, our method captures the most general parity-conserving 3-dim, case.

# Lookout

- In principle, the field redefinition method should also work for more general theories. It would be desirable to explore this direction.
- Using field redef. one may map a family of higher curvature theories with Riemann to Lovelock theory.
- In this case it should be possible to use Jacobson-Myers functional.
- Considering EE in non-minimal coupling gravities is interesting.
- AdS/CFT is less studied in the presence of non-minimal coupling.
- Lots to explore!

**Thank you!**