Higher-rank Dirac currents as sources for p-brane stress-energy-momentum tensors in background gravitational fields

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Recent Trends in String Theory and Related Topics
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Contents

1. Introduction and Motivation
2. KY forms and Gravitational Currents
3. KS and Higher-degree Dirac currents
4. Smooth p-fluids and motion
5. Complementary Issues

- We define the **Killing-Yano forms** and relate them to the **symmetries** of the ambient spacetime.

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- Use KY-forms and the inherent **curvature characteristics** for building up **conserved gravitational currents** in spacetime.
- Prefer to understand them through the same lines as interpreted by D. Kastor and J. Traschen.
- We assume that the spacetime admits **Killing spinor(s)** and define the **higher-degree Dirac currents** induced from them.

- Some interesting properties of HDDCs will be reported and will be **classified** into two categories: **kinematical** and **dynamical**.

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- Some interesting properties of HDDCs will be reported and will be **classified** into two categories: **kinematical** and **dynamical**.
- We then focus on the **motion of a fluid particle** in the context of General Relativity, define its **equation of motion** via the **vanishing divergence** of the stress-energy-momentum tensor.
- Applying the definition of EOM for p-brane fluids, for (again) the simple (collisionless, inviscid) case, one can determine the **density** of the fluid by additionally using Einstein's equations.

- In the absence of a **dynamical gravitational field**, if the ambient spacetime contains at least one KS, we propose that the **density function can be determined from the gravitational charges induced by the existence of a KS**.

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- This is a reasonable assumption, since the only definitive quantity admitted by such a smooth p-fluid particle is its density function and should necessarily be in $1-1$ correspondence with such induced gravitational charges.
- Concluding remarks are made in the light of questions that have to be answered for consistency, and possible future directions are considered.

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- A KY-form of degree p satisfies the first order differential equation

$$\nabla_X \omega_{(p)} = \frac{1}{p+1} i_X d\omega_{(p)},$$

and is manifestly co-closed $\delta\omega_{(p)} = 0$ (Yano 1952).

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- Equivalently another definition can be given by

$$ker(d - \nabla) = \{\omega_{(p)}\} \subseteq \Gamma\Lambda^p(M),$$

in terms of certain differential geometric operators.

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- Since KY-forms are their anti-symmetric generalizations, they should have geometrical meanings. Because their meanings are not as clear as KVs, their corresponding symmetries are termed hidden.
- To understand their importance, we consider a spinor field ψ in (M, g) satisfying the dynamical equations of Dirac,

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$$\not{D}\psi = m\psi \, , \; \psi \in \Gamma\mathcal{S}^c(M, g),$$

- where the Dirac operator is defined as

$$\not{D} = e^a . \nabla_{X_a}^{(s=\frac{1}{2})} .$$

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- KY-forms (if they exist) can be used to generate new solutions from a known one:

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- The explicit form of this first-order symmetry operator is (Benn and Kress 2004)

$$L_{\omega_{(p)}}\psi := (i_{X^a}\omega_{(p)}).\nabla_{X_a}\psi + \frac{p}{2(p+1)} d\omega_{(p)}.\psi$$

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- This expression reduces to the ordinary (Lichnerowicz 1963, Kosmann 1971) Lie derivative on spinor fields for $p = 1$

$$L_{\omega_{(p)}} \longrightarrow \mathcal{L}_K^{(s)} = \nabla_K^{(s)} + \frac{1}{4}d\widetilde{K}.$$

here $\omega_{(1)} = \widetilde{K}$.

Conserved gravitational currents

- Fusing KY-forms with inherent curvature characteristics of 'any' spacetime (M, g), it is possible to construct conserved gravitational currents and their associated charges.

$$\mathcal{J}_1(\omega_{(p)}) = i_{X_a}(i_{X_b}\omega_{(p)} \wedge R^{ab}), \quad \mathcal{J}_2(\omega_{(p)}) = (-1)^p i_{X_a}(\omega_{(p)} \wedge P^a).$$

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- These currents are called **basic**! (Açık, Ertem, Önder and Verçin 2010) For example their difference is also a conserved (Y-AD) current,

$$\mathcal{J}(\omega_{(p)}) = -i_{X_a} i_{X_b}\omega_{(p)} \wedge R^{ab} + 2(-1)^p i_{X_a}\omega_{(p)} \wedge P^a + \mathcal{R}\omega_{(p)}$$

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- which gives for $p = 1$ the well known AD current (Abbott and Deser 1982)

$$\mathcal{J}(\tilde{K}) = -2K_a P^a + \mathcal{R}\tilde{K} = K_a *^{-1} G^a.$$

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- Their higher-degree generalizations can be interpreted as defining gravitational charges localized on p-branes (Kastor and Traschen 2004) and we will stick to this interpretation *especially when there is no dynamical gravitational field*!

Conserved gravitational charges

- If $\omega_{(p+1)}$ is a KY $(p+1)$-form then the charge is calculated as

$$Q_i\left[\omega_{(p+1)}, [\Sigma]\right] = \int_\Sigma *\mathcal{J}_i(\omega_{(p+1)});\ i = 1, 2.$$

where Σ is a $(n-p-1)$-chain contained in a constant time hyperplane.

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- If $J_i = *\mathcal{J}_i$ were a source for a dynamical field $F_i \in \Gamma\Lambda^{p+2}(M)$ (e.g. Maxwell's eqs.)then the charge(s) were to be defined by

$$Q_i\left[[\Sigma]\right] = \int_{\partial\Sigma} *F_i;\ i = 1, 2.$$

without any reference to the sources (Benn and Tucker 1987).

Spinors and Clifford algebras

- We adapt the view that spinors correspond to the elements of minimal left ideals of a Clifford algebra

$$\mathcal{S}^*(M,g) \text{ a minimal left (right) ideal of } C(M,g)$$
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- If $\psi \in \Gamma\mathcal{S}$ then $\overline{\psi} := J^{-1}\psi^{\mathcal{J}} \in \Gamma\mathcal{S}^*$:

Spinor Bilinear: $\psi\overline{\psi} = \sum_0^n (\psi, e^{\xi}_{I(p)}\psi) e^{I(p)}$

p-Dirac Current: $(i\psi\overline{\psi})_p$ ($\mathbb{C}^*$ − *symmetric product*)

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- For a massive Dirac spinor, $p = 1$ corresponds to the streamlines for the Dirac electrons and hence the name.

Killing Spinors and Geometry

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- From the compatibility of spinorial covariant derivative with spinor inner products $\overline{\psi}(\phi) = (\psi, \phi)$, one deduces:

$$\overline{\nabla_{X_a}\psi} = \lambda^j \, \overline{\psi}(e_a)^{\mathcal{J}}.$$

Killing Spinors and Geometry

- Another calculation yields:

$$\nabla_{X_a}(\psi\overline{\psi})_p = (\hat{\lambda}^-_{(p)}e_a) \wedge (\psi\overline{\psi})_{p-1} + i_{\widetilde{(\hat{\lambda}^+_{(p)}e_a)}}(\psi\overline{\psi})_{p+1}$$

where the Clifford algebraic operators are $\hat{\lambda}^{\pm}_{(p)} = (\lambda 1 \pm (-1)^p \lambda^j \mathcal{J})$.

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- For any co-frame field the below equations hold,

$$\hat{\lambda}^{\pm}_{(p)}e_a = \langle\hat{\lambda}^{\pm}_{(p)}\rangle e_a \text{ and } \langle\hat{\lambda}^{\pm}_{(p)}\rangle = \frac{e^a\hat{\lambda}^{\pm}_{(p)}e_a}{e^a e_a} \in \{0, 2\lambda\}$$

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- It is clear that the results will depend on the below set:

Data set: $\Delta = \{\lambda, \{j, \mathcal{J}\}, p\}$

Exterior derivative and co-derivative

- It is of use to note the following (pseudo-)Riemannian identities:

$$d = e^a \wedge \nabla_{X_a}$$

and

$$\delta = -i_{X^a} \nabla_{X_a}.$$

also

$$(e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_p})^\eta = (-1)^p e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_p},$$

$$(e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_p})^\xi = e^{a_p} \wedge \ldots \wedge e^{a_2} \wedge e^{a_1}.$$

Principal equations for KS Dirac p-Currents

- By considering possible (12/8) data sets that are compatible with reality conditions (for the case of Hermitian products (12)), one then finds two principal couple of equations corresponding to the $\langle \hat{\lambda}^{+}_{(p)} \rangle = 0$ and $\langle \hat{\lambda}^{-}_{(p*)} \rangle = 0$ cases respectively:

(i) $d(\psi\overline{\psi})_p = 0$ and $\delta(\psi\overline{\psi})_p = -2\lambda(n-p+1)(\psi\overline{\psi})_{p-1}$ (6/4)

(ii) $d(\psi\overline{\psi})_{p*} = 2\lambda(p*+1)(\psi\overline{\psi})_{p^*+1}$ and $\delta(\psi\overline{\psi})_{p*} = 0$ (6/4)

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- For example while $\Delta_1 = \{re, \{Id/*, \xi\}, odd\}$ belongs to (i), $\Delta_1^* = \{re, \{Id/*, \xi\}, even\}$ belongs to (ii).

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$$(\text{K1})\ \nabla_X * (\psi\overline{\psi})_p = \frac{1}{n-p+1} i_X d * (\psi\overline{\psi})_p$$
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\end{aligned}
$$

- Maxwell-like equations

$$(\mathrm{D1})\ d * (\psi\overline{\psi})_p = -2(-1)^p \lambda (n-p+1) * (\psi\overline{\psi})_{p-1} \text{ and } d(\psi\overline{\psi})_p = 0$$

$$(\mathrm{D2})\ d(\psi\overline{\psi})_{p*} = 2\lambda(p*+1)(\psi\overline{\psi})_{p*+1} \text{ and } d * (\psi\overline{\psi})_{p*} = 0$$

Induced Kinematical and Dynamical equations

- Last but not the least the simultaneous usage of both sets give rise to: Generalized Duffin-Kemmer-Petiau equations

$$\text{(D3)} \qquad \begin{array}{ll} d(\psi\overline{\psi})_{p-1} = 2\lambda p(\psi\overline{\psi})_p \quad , & \delta(\psi\overline{\psi})_{p-1} = 0 \\ \delta(\psi\overline{\psi})_p = -2\lambda(n-p+1)(\psi\overline{\psi})_{p-1} \quad , & d(\psi\overline{\psi})_p = 0 \end{array}$$

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- For the special case of spacetime dimension $n = 2p - 1$ they reduce to Duffin-Kemmer-Petiau equations:

$$d\Phi_\pm = \mu\Phi_\mp \; , \; \delta(\Phi_\pm) = 0$$
$$\delta\Phi_\mp = -\mu\Phi_\pm \; , \; d\Phi_\mp = 0$$

where $\Phi_\pm = (\psi\overline{\psi})_{p-1}, \Phi_\mp = (\psi\overline{\psi})_p$ and $\mu = 2|\lambda|p$

Simple Fluids in GR and EOM

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- If $g(V, V) = -1$ then this vanishing divergence condition simultaneously gives ($j = \rho * \widetilde{V}$)

$$(\nabla . T)_{\perp} = 0 \Longleftrightarrow \nabla_V V = 0,$$
$$(\nabla . T)_{\parallel} = 0 \Longleftrightarrow dj = 0.$$

Simple p-Fluids in GR

- The GR collisionless, inviscid p-brane fluid can likely be described by a field $\phi_{\mathbf{c}}$ of $p+1$-dimensional immersions with a SEM tensor:

$$\widetilde{T} = \mathfrak{B}\hat{g}^{\alpha\beta} V_{\alpha} \otimes V_{\beta}\,; \alpha, \beta = 1, ..., p+1$$

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- The vanishing divergence condition results in $(j_\alpha = \frac{\mathfrak{B}}{\sqrt{|det(\hat{\mathbf{g}})|}} * \widetilde{V}_\alpha)$

$$(\nabla . T)_\perp = 0 \Longleftrightarrow h(V^\alpha, V_\alpha) = 0,$$
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- The unknown function $\mathfrak{B}$ together with the metric and immersion functions are determined by the above equations and Einstein's field equations (Mukherjee and Tucker 1987).

Motion in background geometries

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- For the case of our p-brane fluid, there should be additional differential equations in order to determine the unknown function $\mathfrak{B}$.
- MOTIVATION: **Here we turn our attention to background (Lorentzian) spacetimes that contain at least one KS, so it seems possible to determine the density function of the simple p-brane fluid by using our previous machinery**.

Gravitational Dirac p-currents and brane densities in background geometries

- Assuming a fixed background (M, g) containing a KS ψ it is possible as a first approximation to make the following identification:

$$\mathfrak{B}_i \Pi = \mathcal{J}_i((\psi\overline{\psi})_{p+1}) \quad ; \quad \Pi = \frac{\widetilde{\Lambda}}{|S_0(\widetilde{\Lambda} \, . \, \widetilde{\Lambda}^{\xi})|^{\frac{1}{2}}}$$

Π is the Clifford-normalized **timelike** $(p+1)$-form.

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- The $(\mathbf{p}+\mathbf{1})$**-multivector**

$$\Lambda := V_1 \wedge V_2 \wedge \ldots \wedge V_{p+1}$$

is constructed from foliating vector fields.

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- A straightforward calculation yields

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- Then with some rearrangement the explicit form for the KS sourced SEM tensor will be:

$$T_i = -[i_{\tilde{\Pi}}\mathcal{J}_i((\psi\overline{\psi})_{p+1}))]S_0(i_{X_a}\Pi \wedge (i_{X_b}\Pi)^{\xi})e^a \otimes e^b$$

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- **This is the primary result of our proposition for the most simple case!**

Questions and Future Insights

- First of all in order to check our construction we ought to apply it in a **background spacetime** which is known to **admit a KS**.

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