An Excursion in Higher Derivative Gravity

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Outline

Based on: Kerr-Schild–Kundt Class Metrics

Kerr-Schild–Kundt class metrics:

- M. Gurses, S. Hervik, T. C. S. and B. Tekin, "Anti–de Sitter–Wave Solutions of Higher Derivative Theories," Phys. Rev. Lett. **111**, 101101 (2013).
- M. Gurses, T. C. S. and B. Tekin, "AdS-plane wave and *pp*-wave solutions of generic gravity theories," Phys. Rev. D **90**, no. 12, 124005 (2014).
- M. Gurses, T. C. S. and B. Tekin, "Kerr-Schild–Kundt Metrics are Universal," arXiv:1603.06524 [gr-qc].

AdS-wave solutions of quadratic curvature gravities:

- I. Gullu, M. Gurses, T. C. S. and B. Tekin, "AdS Waves as Exact Solutions to Quadratic Gravity," Phys. Rev. D **83**, 084015 (2011).
- M. Gurses, T. C. S. and B. Tekin, "New Exact Solutions of Quadratic Curvature Gravity," Phys. Rev. D **86**, 024009 (2012).

Based on: Solution Generation and Born-Infeld Gravity

Solution generation technique for (A)dS-waves:

- M. Gurses, T. C. S. and B. Tekin, "From Smooth Curves to Universal Metrics," arXiv:1603.09655 [gr-qc].
- M. Gurses, C. Senturk, T. C. S. and B. Tekin, "Hyperbolic-dS Plane Waves of Generic Gravity Theories," in progress.

A specific higher curvature gravity, the Born-Infeld gravity:

- I. Gullu, T. C. S. and B. Tekin, "Born-Infeld Gravity with a Massless Graviton in Four Dimensions," Phys. Rev. D **91**, no. 4, 044007 (2015).
- I. Gullu, T. C. S. and B. Tekin, "Born-Infeld Gravity with a Unique Vacuum and a Massless Graviton," Phys. Rev. D **92**, no. 10, 104014 (2015).

Why higher curvature/derivative theory?

- **Problem:** Einstein's gravity

$$I = \frac{1}{\kappa} \int d^4x \sqrt{-g}\, R,$$

is *not* renormalizable ('t Hooft and Veltman, 1974; Deser and van Nieuwenhuizen, 1974). ($\kappa \equiv 16\pi G$)

- **Effective field theory perspective:** High energies $\Rightarrow$ Higher curvature and derivative terms;

$$I = \int d^Dx \sqrt{-g} \left\{ \frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) + \sum_{n=3}^{\infty} a_n (\text{Riem, Ric, R, } \nabla\text{Riem, } \ldots)^n \right\}.$$

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A basic question

- What can we say about the solutions of

$$I = \int d^D x \sqrt{-g}\,\mathscr{L}\left(\text{Riem},\ \nabla\text{Riem},\ \ldots, \nabla^n\text{Riem}, \ldots\right),$$

without a further specification on the exact form of the theory?

What we know: pp-wave solves the generic theory

- pp-wave solution for Einstein's gravity

$$I = \frac{1}{\kappa} \int d^D x \sqrt{-g}\, R,$$

is a solution of the generic higher derivative gravity theory (Güven, 1987; Amati and Klimcik, 1989; Horowitz and Steif, 1990)

$$I = \int d^D x \sqrt{-g} \left\{ \frac{1}{\kappa} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) + \sum_{n=3}^{\infty} a_n \left(\text{Riem, Ric, R, } \nabla\text{Riem, } \ldots \right)^n \right\}.$$

What is pp-wave?

- *pp-waves* $\equiv$ *plane-fronted gravitational waves with parallel rays*
- pp-wave metric in Kerr-Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + 2V\lambda_\mu\lambda_\nu,$$

where

$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = 0, \qquad \lambda^\mu\partial_\mu V = 0.$$

Kerr-Schild–Kundt (KSK) metrics

- Kerr-Schild metrics

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu,$$

where $\bar{g}_{\mu\nu}$ is the (A)dS metric

$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}, \qquad \xi_\mu\lambda^\mu = 0, \qquad \lambda^\mu\partial_\mu V = 0.$$

- With $\nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}$ and $\xi_\mu\lambda^\mu = 0$, λ_μ is divergenceless, shear-free, and non-twisting $\Rightarrow$ Kundt spacetime

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Properties of KSK metrics

- Type-N Weyl

$$C_{\mu\alpha\nu\beta} = \lambda_\mu \lambda_\nu \Omega_{\alpha\beta} + \lambda_\alpha \lambda_\beta \Omega_{\mu\nu} - \lambda_\mu \lambda_\beta \Omega_{\alpha\nu} - \lambda_\alpha \lambda_\nu \Omega_{\mu\beta},$$

where $\lambda^\alpha \Omega_{\alpha\beta} = 0$, $\Omega^\alpha_\alpha = 0$.

- Type-N traceless-Ricci

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{D} g_{\mu\nu} R = -\left(\Box + \frac{2}{\ell^2}\right)(\lambda_\mu \lambda_\nu V) = \rho\, \lambda_\mu \lambda_\nu,$$

where $\Box \equiv \nabla^\mu \nabla_\mu$.

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Field Equations of Generic Higher Derivative Gravity

Theorem

For this special spacetime, any second rank symmetric tensor constructed from the Riemann tensor and its covariant derivatives can be written as a linear combination of $g_{\mu\nu}$, $S_{\mu\nu}$, and higher orders of $S_{\mu\nu}$(such as, for example, $\Box^n S_{\mu\nu}$). (Gurses, Hervik, S. and Tekin, 2013)

Corollary

The field equations for generic higher derivative gravity has the form

$$E_{\mu\nu} = e g_{\mu\nu} + \left(\Box - \bar{b}\right)\left(\Box - b\right)\ldots\left(\Box - a\right) S_{\mu\nu} = 0,$$

whose trace determines cosmological constant and the traceless part is solved by

$$\left(\Box - a\right) S_{\mu\nu} = 0,$$

that is AdS-wave solutions of Einstein's gravity, $S_{\mu\nu} = 0$, and quadratic curvature gravity, $\left(\Box - a\right)\left(\lambda_\mu \lambda_\nu V\right) = 0$.

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The Proof: Basic Idea

- Weyl tensor and traceless-Ricci tensor have two free index λ.
- Any contraction of the λ vector with any other tensorial form always yields a free-index λ vector or zero.
- To have a nonzero two-tensor from the contractions of the Riemann tensor and its derivatives, one should start with a tensorial form involving 2 λ's.
- So, one has the possibilities $R_{\alpha\beta\rho\sigma}$ yielding $R_{\mu\nu}$ and $\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_n}R_{\alpha\beta\rho\sigma}$ yielding the $\Box^{n/2}S_{\mu\nu}$ and lower orders.

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Field Equations for Einstein's Gravity and Quadratic Curvature Gravity

For KSK class of metrics,

- the field equations of cosmological Einstein's gravity

$$I = \frac{1}{\kappa}\int d^D x \sqrt{-g}\,(R - 2\Lambda_0),$$

reduces to

$$S_{\mu\nu} = \left(\Box + \frac{2}{\ell^2}\right)(V\lambda_\mu\lambda_\nu) = \left(\bar{\Box} + \frac{2}{\ell^2}\right)(V\lambda_\mu\lambda_\nu) = 0.$$

- the field equations of quadratic curvature gravity

$$I = \int d^D x \sqrt{-g}\left\{\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)\right\},$$

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$$(\Box - a)\,S_{\mu\nu} = \left(\Box + \frac{2}{\ell^2} - M^2\right)S_{\mu\nu} = \left(\bar{\Box} + \frac{2}{\ell^2} - M^2\right)S_{\mu\nu} = 0,$$

where $M^2 = -\frac{1}{\beta}\left(\frac{1}{\kappa} + \frac{4\Lambda D}{D-2}\alpha + \frac{4\Lambda}{D-1}\beta + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)}\gamma\right) + \frac{2}{\ell^2}.$

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AdS-waves solve the generic theory

- *AdS-waves are members of KSK class* $\Rightarrow$ *AdS-waves are solutions to any higher derivative theory*

- AdS-wave solutions of cosmological Einstein's gravity

$$I = \frac{1}{\kappa}\int d^D x \sqrt{-g}\,(R - 2\Lambda_0),$$

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AdS-plane and AdS-spherical Waves

Metric forms: *(for simplicity, let us have 4D)*

- AdS-plane wave or Siklos metric form:

$$ds^2 = \frac{\ell^2}{z^2}\left(2dudv + dy^2 + dz^2\right) + 2V(u,y,z)\,\mathrm{d}u^2.$$

- AdS-spherical wave:

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(\frac{4dudv}{(u+v)^2} + d\theta^2 + \sin^2\theta\, d\phi^2\right) + 2V(u,\Omega_{D-2})du^2.$$

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4D AdS-plane wave solution

- Einstein's gravity solution is (Kaigorodov, 1963; see also Chamblin and Gibbons, 2000)

$$V(u,y,z) = \frac{1}{\sqrt{z}}[c_1(u)I_\nu(az) + c_2(u)K_\nu(az)]\sin(ay + c_3(u)),$$

where $\nu = \frac{3}{2}$ and a is an arbitrary constant; and for $a = 0$,

$$V(u,z) = c(u)z.$$

- Quadratic curvature gravity solution found as (Alishahiha and Fareghbal, 2011; Gullu, Gurses, S. and Tekin, 2011)

$$V(u,y,z) = \frac{1}{\sqrt{z}}[c_1(u)I_\nu(az) + c_2(u)K_\nu(az)]\sin(ay + c_3(u)),$$

where $\nu = \frac{1}{2}\sqrt{9+4\ell^2M^2}$ and a is an arbitrary constant; and for $a = 0$,

$$V(u,z) = \frac{1}{\sqrt{z}}\left(c_2(u)z^\nu + c_3(u)z^{-\nu}\right).$$

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4D AdS-spherical wave solution

- Einstein's gravity solution (Gurses, S. and Tekin, 2012)

$$V(u,\theta,\phi) = \left[c_1(u) \left(\tan\frac{\theta}{2} \right)^{a} \sec\theta \,(a+\sec\theta) + c_2(u) \left(\tan\frac{\theta}{2} \right)^{-a} \sec\theta \,(a-\sec\theta) \right] \times \left[c_3(u)\cos(a\phi) + c_4(u)\sin(a\phi) \right],$$

where a is an arbitrary constant; and for $a=0$,

$$V(u,\theta,\phi) = \frac{1}{\cos^2\theta}\left[1 + c_2(u)\left(\cos\theta + \log\left[\tan\left(\frac{\theta}{2}\right)\right]\right)\right](c_3(u) + c_4(u)\phi).$$

Questions

- Are there any other member of KSK class?
- Is there any dS-wave?
- Is there a systematic way to construct KSK metrics solving a generic higher derivative theory?

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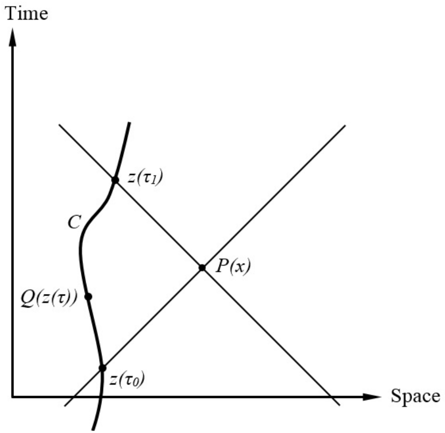
A solution generation technique

- KSK metrics in D-dimensions can be generated from curves in one less dimensions. (Gurses, S and Tekin, 2016)

Curves in D-Dimensional Minkowski



- The distance between the points $P(x^\mu)$ and $Q(z^\mu)$

$$\Omega^2 = \eta_{\mu\nu}\left(x^\mu - z^\mu(\tau)\right)\left(x^\nu - z^\nu(\tau)\right).$$

Null vector field out of the curve

- Differentiating $\Omega(\tau_0) = 0$ with respect to x_μ yields a null vector:

$$\ell_\mu \equiv \partial_\mu \tau_0 = \frac{x_\mu - z_\mu(\tau_0)}{R},$$

where $R \equiv \dot{z}^\alpha(\tau_0)\left(x_\alpha - z_\alpha(\tau_0)\right)$ with $\dot{z}^\alpha(\tau_0) \equiv \partial_{\tau_0} z^\alpha(\tau_0)$.

- One more derivative:

$$\partial_\nu \ell_\mu = \frac{1}{R}\left(\eta_{\mu\nu} - \dot{z}_\mu\,\ell_\nu - \dot{z}_\nu\,\ell_\mu - (A - \varepsilon)\,\ell_\mu\,\ell_\nu\right),$$

with $A \equiv \ddot{z}^\mu\left(x_\mu - z_\mu\right)$ and $\varepsilon \equiv \dot{z}^\mu\,\dot{z}_\mu$.

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- One more derivative:
$$\partial_\nu \ell_\mu = \frac{1}{R}\left(\eta_{\mu\nu} - \dot{z}_\mu \ell_\nu - \dot{z}_\nu \ell_\mu - (A - \varepsilon)\, \ell_\mu \ell_\nu\right),$$
with $A \equiv \ddot{z}^\mu \left(x_\mu - z_\mu\right)$ and $\varepsilon \equiv \dot{z}^\mu \dot{z}_\mu$.

Null vector field of KSK class

- Remember $\nabla_\mu \lambda_\nu = \xi_{(\mu}\lambda_{\nu)}$ and consider AdS and dS metrics as
$$d\bar{s}^2 = \frac{\ell^2}{z^2}\left(-dt^2 + \sum_{m=1}^{d-2}(dx^m)^2 + dz^2\right),$$
and
$$d\bar{s}^2 = \frac{\ell^2}{t^2}\left(-dt^2 + \sum_{m=1}^{d-1}(dx^m)^2\right).$$

- Consider $\partial_\nu \lambda_\mu$ in these coordinates
$$\partial_\nu \lambda_\mu = a\,\eta_{\mu\nu} + \lambda_\mu \left(\frac{1}{2}\xi_\nu - \zeta_\nu\right) + \lambda_\nu \left(\frac{1}{2}\xi_\mu - \zeta_\mu\right),$$
where $a = \frac{\lambda_z}{z}$, $\zeta_\nu = \frac{1}{z}\delta^z_\nu$ for AdS and $a = \frac{\lambda_t}{t}$, $\zeta_\nu = \frac{1}{t}\delta^t_\nu$ for dS.

Null vector field of KSK class

- Remember $\nabla_\mu \lambda_\nu = \xi_{(\mu}\lambda_{\nu)}$ and consider AdS and dS metrics as

$$d\bar{s}^2 = \frac{\ell^2}{z^2}\left(-dt^2 + \sum_{m=1}^{d-2}(dx^m)^2 + dz^2\right),$$

and

$$d\bar{s}^2 = \frac{\ell^2}{t^2}\left(-dt^2 + \sum_{m=1}^{d-1}(dx^m)^2\right).$$

- Consider $\partial_\nu \lambda_\mu$ in these coordinates

$$\partial_\nu \lambda_\mu = a\,\eta_{\mu\nu} + \lambda_\mu\left(\frac{1}{2}\xi_\nu - \zeta_\nu\right) + \lambda_\nu\left(\frac{1}{2}\xi_\mu - \zeta_\mu\right),$$

where $a = \frac{\lambda_z}{z}$, $\zeta_\nu = \frac{1}{z}\delta_\nu^z$ for AdS and $a = \frac{\lambda_t}{t}$, $\zeta_\nu = \frac{1}{t}\delta_\nu^t$ for dS.

The prescription

- Observing the similarity:

$$\partial_\nu \ell_\mu = \frac{1}{R}\left(\eta_{\mu\nu} - \dot{z}_\mu \,\ell_\nu - \dot{z}_\nu \,\ell_\mu - (A-\varepsilon)\,\ell_\mu \,\ell_\nu\right),$$

and

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suggests the prescription:

1. Take the vectors ℓ_μ and λ_μ to be equal and derive the corresponding vector ξ_μ
2. Set $\lambda^\mu \xi_\mu = 0$, and obtain the constraint on $z^\mu(\tau)$.

- 2nd step constrains $z^\mu(\tau)$ curves to live in one less dimension.

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A new solution: dS-hyperbolic wave

- With this method, one can generate AdS-plane and AdS-spherical waves. But, also a new solution: dS-hyperbolic wave.
- Consider the curve $z^\mu = \tau\delta^\mu_x$ and follow the prescription.

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4D dS-hyperbolic wave solution

- 4D (Gurses, Senturk, S and Tekin, in progress):

$$ds^2 = \frac{\ell^2}{\cosh^2\theta}\left(\frac{du^2+2dudr}{r^2}+d\theta^2+\sinh^2\theta\, d\phi^2\right)+2V(u,\theta,\phi)\,du^2,$$

where

$$\begin{aligned} V(u,\theta,\phi) = \Bigg[& c_1(u)\left(\tanh\frac{\theta}{2}\right)^{a}\operatorname{sech}\theta\,(a+\operatorname{sech}\theta) \\ & + c_2(u)\left(\tanh\frac{\theta}{2}\right)^{-a}\operatorname{sech}\theta\,(a-\operatorname{sech}\theta)\Bigg] \\ & \times\left[c_3(u)\cos(a\phi)+c_4(u)\sin(a\phi)\right], \end{aligned}$$

where a is an arbitrary constant; and for $a=0$,

$$V(u,\theta,\phi) = \frac{1}{\cosh^2\theta}\left[1+c_2(u)\left(\cosh\theta+\log\left[\tanh\left(\frac{\theta}{2}\right)\right]\right)\right](c_3(u)+c_4(u)\phi).$$

3D dS-hyperbolic wave solution

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$$ds^2 = \frac{\ell^2}{\cosh^2\theta}\left(\frac{du^2 + 2dudr}{r^2} + d\theta^2\right) + 2V(u,\theta)\,du^2,$$

where

$$V(u,\theta) = \frac{1}{\cosh\theta}\left[c_1(u)\cos(p\,\mathrm{gd}(\theta)) + c_2(u)\sin(p\,\mathrm{gd}(\theta))\right],$$

with $p = \sqrt{\frac{1}{2} - \frac{m^2}{k^2}}$ and $\mathrm{gd}(\theta)$ is the Gudermannian function defined as

$$\mathrm{gd}(x) = \arctan(\sinh x), \qquad \text{for real } x.$$

Born-Infeld Gravity and Desired Properties

- Born-Infeld (BI) Gravity: the Lagrangian density

$$\mathscr{L} = \sqrt{\det\left(g_{\mu\nu} + \gamma A_{\mu\nu}\right)}.$$

- Imposing the requirements:
  1. The theory reduces to the cosmological Einstein's theory at the lowest order in the small curvature expansion about the flat space or the (anti)-de Sitter [(A)dS], and to the Einstein–Gauss-Bonnet (EGB) theory at the quadratic order.
  2. The theory describes only massless gravitons around its flat or (A)dS vacuum at any finite (truncated) order in the curvature expansion or as a full theory (namely, if all powers of curvature are included).
  3. The theory has a unique viable vacuum: namely, there is a single maximally symmetric solution for which the massless spin-2 excitation about this solution is unitary.

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The 4D BI-gravity with a unique vacuum and massless graviton

- The minimal (at most quadratic curvature in $A_{\mu\nu}$) BI-gravity is (Gullu, S and Tekin, 2015)

$$I = \frac{2}{\kappa\gamma}\int d^n x \left[\sqrt{-\det\left(g_{\mu\nu}+\gamma A_{\mu\nu}\right)} - \left(\gamma\Lambda_0+1\right)\sqrt{-\det g}\right],$$

where

$$\begin{aligned} A_{\mu\nu} =& R_{\mu\nu} + \beta S_{\mu\nu} \\ &+ \gamma\left(a_2 C_{\mu\rho\nu\sigma}R^{\rho\sigma} + \frac{\beta+1}{4}R_{\mu\rho}R^{\rho}_{\nu} + \left(\frac{\beta\left(\beta+2\right)}{2} - 2 - b_3\right)S_{\mu\rho}S^{\rho}_{\nu}\right) \\ &+ \frac{\gamma}{4}g_{\mu\nu}\left(\frac{9}{8}C_{\rho\sigma\lambda\gamma}C^{\rho\sigma\lambda\gamma} - \frac{\beta}{4}R_{\rho\sigma}R^{\rho\sigma} + b_3 S_{\rho\sigma}S^{\rho\sigma}\right). \end{aligned}$$

- Although the theory is infinite order in curvature, it has two (A)dS vacua for $\Lambda_0 < \frac{11}{16\gamma}$ satisfying $\Lambda < \frac{1}{\gamma}$.
- The spectrum of the theory consists of *only* a massless graviton which is (tree-level) unitary about *only* one of these (A)dS vacua.

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(A)dS-wave solutions of the 4D BI-gravity

- Since the spectrum of the 4D BI gravity consists of a massless graviton, its linearized field equations involves the same operator as the Einstein's theory.

- Since KSK class of metrics linearizes the field equations, we know *without doing any calculation* that 4D BI-gravity has the Einsteinian (A)dS-wave solutions of:
  - AdS-plane wave: $ds^2 = \frac{\ell^2}{z^2}\left(2dudv + dy^2 + dz^2\right) + 2c(u)\,zdu^2$.
  - AdS-spherical wave:

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(\frac{4dudv}{(u+v)^2} + d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

$$+\frac{2}{\cos^2\theta}\left[1 + c_2(u)\left(\cos\theta + \log\left[\tan\left(\frac{\theta}{2}\right)\right]\right)\right]\left(c_3(u) + c_4(u)\,\phi\right)du^2.$$

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Conclusion

- (A)dS-waves are solutions of a generic higher order gravity theory.
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