Entanglement in Field Space

Ali Mollabashi

School of Physics

Institute for Research in Fundamental Sciences (IPM)

IPM , 8 May 2017

Outline

Introduction

Quantum Entanglement

- Consider a many-body system S

$$\mathcal{H}_S = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

Quantum Entanglement

- Consider a many-body system S

$$\mathcal{H}_S = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

- Decomposition of $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$

Quantum Entanglement

- Consider a many-body system S

$$\mathcal{H}_S = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

- Decomposition of $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$
- Reduced density matrix (for A)

$$\rho_A \equiv \mathrm{Tr}_{A^c}[\rho] = \sum_{i=1}^{\dim[\mathcal{H}_{A^c}]} \langle \Psi_i^{A^c} | \rho | \Psi_i^{A^c} \rangle$$

Quantum Entanglement

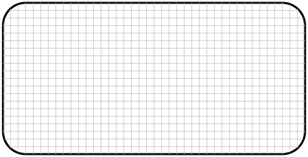
- Consider a many-body system S

$$\mathcal{H}_S = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

- Decomposition of $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$
- Reduced density matrix (for A)

$$\rho_A \equiv \mathrm{Tr}_{A^c}[\rho] = \sum_{i=1}^{\dim[\mathcal{H}_{A^c}]} \langle \Psi_i^{A^c} | \rho | \Psi_i^{A^c} \rangle$$

$\mathcal{H}_S$



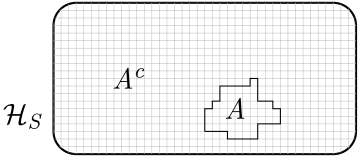
Quantum Entanglement

- Consider a many-body system S

$$\mathcal{H}_S = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

- Decomposition of $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$
- Reduced density matrix (for A)

$$\rho_A \equiv \mathrm{Tr}_{A^c}[\rho] = \sum_{i=1}^{\dim[\mathcal{H}_{A^c}]} \langle \Psi_i^{A^c} | \rho | \Psi_i^{A^c} \rangle$$



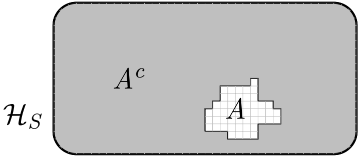
Quantum Entanglement

- Consider a many-body system S

$$\mathcal{H}_S = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

- Decomposition of $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$
- Reduced density matrix (for A)

$$\rho_A \equiv \mathrm{Tr}_{A^c}[\rho] = \sum_{i=1}^{\dim[\mathcal{H}_{A^c}]} \langle \Psi_i^{A^c} | \rho | \Psi_i^{A^c} \rangle$$



Some Physical Measures for Entanglement

- Entanglement entropy (von-Neumann entropy for ρ_A):

$$S_A = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right)$$

Some Physical Measures for Entanglement

- Entanglement entropy (von-Neumann entropy for ρ_A):

$$S_A = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right)$$

- Renyi entropy (replica method)

$$S_A^{(n)} = \frac{1}{1-n}\mathrm{Tr}\left[\rho_A^n\right], \qquad n > 1$$

where $\lim_{n\to 1} S_A^{(n)} = S_A$

Some Physical Measures for Entanglement

- Entanglement entropy (von-Neumann entropy for ρ_A):

$$S_A = -\mathrm{Tr}_A \left(\rho_A \log \rho_A\right)$$

- Renyi entropy (replica method)

$$S_A^{(n)} = \frac{1}{1-n} \mathrm{Tr}\left[\rho_A^n\right], \qquad n > 1$$

where $\lim_{n\to 1} S_A^{(n)} = S_A$

- Mutual information:

$$I\left(A, B\right) = S_A + S_B - S_{A\cup B}$$

Some Physical Measures for Entanglement

- Entanglement entropy (von-Neumann entropy for ρ_A):

$$S_A = -\mathrm{Tr}_A \left(\rho_A \log \rho_A\right)$$

- Renyi entropy (replica method)

$$S_A^{(n)} = \frac{1}{1-n} \mathrm{Tr}\left[\rho_A^n\right], \qquad n > 1$$

where $\lim_{n\to 1} S_A^{(n)} = S_A$

- Mutual information:

$$I\left(A, B\right) = S_A + S_B - S_{A\cup B}$$

- Tripartite information:

$$I^{(3)}\left(A, B, C\right) = S_A + S_B + S_C - S_{A\cup B} - S_{A\cup C} - S_{B\cup C} + S_{A\cup B\cup C}$$

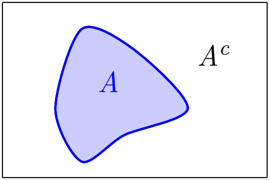
Geometrical (Spatial) Entanglement

- QFT on $\mathbb{R} \times \mathcal{M}_d$, divide $\mathcal{M}_d$ to A and A^c

Geometrical (Spatial) Entanglement

- QFT on $\mathbb{R} \times \mathcal{M}_d$, divide $\mathcal{M}_d$ to A and A^c
- In **local** field theories

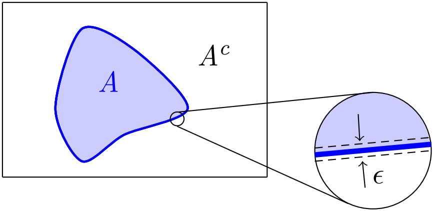
$$\text{decomposition of } \mathcal{M}_d \quad \Leftrightarrow \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$



Geometrical (Spatial) Entanglement

- QFT on $\mathbb{R} \times \mathcal{M}_d$, divide $\mathcal{M}_d$ to A and A^c
- In **local** field theories

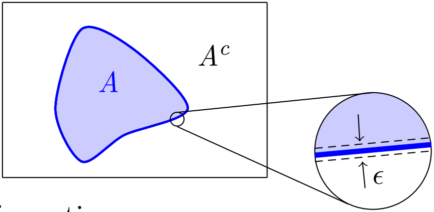
$$\text{decomposition of } \mathcal{M}_d \quad \Leftrightarrow \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$



Geometrical (Spatial) Entanglement

- QFT on $\mathbb{R} \times \mathcal{M}_d$, divide $\mathcal{M}_d$ to A and A^c
- In **local** field theories

$$\text{decomposition of } \mathcal{M}_d \quad \Leftrightarrow \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$



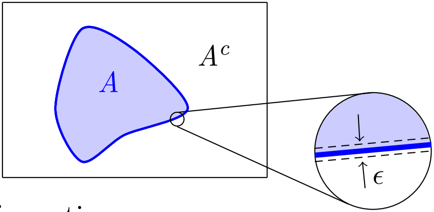
- Universal information

$$S_A = g_{d-1}(\partial A)\epsilon^{-(d-1)} + \cdots + S_0(\partial A)$$

Geometrical (Spatial) Entanglement

- QFT on $\mathbb{R}\times\mathcal{M}_d$, divide $\mathcal{M}_d$ to A and A^c
- In **local** field theories

$$\text{decomposition of } \mathcal{M}_d \quad \Leftrightarrow \quad \mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_{A^c}$$



- Universal information

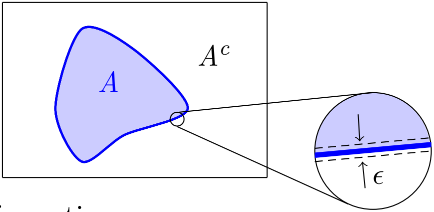
$$S_A = g_{d-1}(\partial A)\epsilon^{-(d-1)}+\cdots+S_0(\partial A)$$

- Celebrated area law [Bombelli-Koul-Lee-Sorkin '86, Srednicki '93]

Geometrical (Spatial) Entanglement

- QFT on $\mathbb{R} \times \mathcal{M}_d$, divide $\mathcal{M}_d$ to A and A^c
- In **local** field theories

$$\text{decomposition of } \mathcal{M}_d \quad \Leftrightarrow \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$



- Universal information

$$S_A = g_{d-1}(\partial A)\epsilon^{-(d-1)} + \cdots + S_0(\partial A)$$

- Celebrated area law [Bombelli-Koul-Lee-Sorkin '86, Srednicki '93]
- Holographic Picture [Ryu-Takayanagi '06]

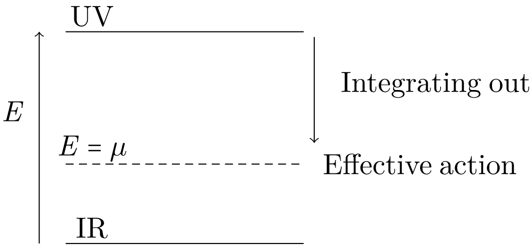
Momentum Space Entanglement

- The Wilsonian effective action carries the information of reduced density matrix

  [Balasubramanian-McDermott-Van Raamsdonk '11]

- Hilber space decomposition

$$\mathcal{H} = \bigotimes_{\vec{p}} \mathcal{H}_{\vec{p}} = \mathcal{H}_{|\vec{p}|<\mu} \otimes \mathcal{H}_{|\vec{p}|>\mu}$$



Field Space Entanglement

- Entanglement between fields (due to interaction) [Furukawa, Kim, Xu, Chen, Fradkin, Lundgren, Mollabashi, Shiba, Takayanagi, Taylor]

Field Space Entanglement

- Entanglement between fields (due to interaction) [Furukawa, Kim, Xu, Chen, Fradkin, Lundgren, Mollabashi, Shiba, Takayanagi, Taylor]
- Simplest model [Mollabashi-Shiba-Takayanagi '14]

$$\mathcal{L} = \mathcal{L}_1\left[\phi(x)\right] + \mathcal{L}_2\left[\psi(x)\right] + \mathcal{L}_{\text{int.}}\left[\phi(x), \psi(x)\right]$$

Field Space Entanglement

- Entanglement between fields (due to interaction) [Furukawa, Kim, Xu, Chen, Fradkin, Lundgren, Mollabashi, Shiba, Takayanagi, Taylor]
- Simplest model [Mollabashi-Shiba-Takayanagi '14]

$$\mathcal{L} = \mathcal{L}_1 [\phi(x)] + \mathcal{L}_2 [\psi(x)] + \mathcal{L}_{\text{int.}} [\phi(x), \psi(x)]$$

- Hilbert space decomposition $\mathcal{H} = \mathcal{H}_\phi \otimes \mathcal{H}_\psi$

Field Space Entanglement

- Entanglement between fields (due to interaction) [Furukawa, Kim, Xu, Chen, Fradkin, Lundgren, Mollabashi, Shiba, Takayanagi, Taylor]
- Simplest model [Mollabashi-Shiba-Takayanagi '14]

$$\mathcal{L} = \mathcal{L}_1 [\phi(x)] + \mathcal{L}_2 [\psi(x)] + \mathcal{L}_{\text{int.}} [\phi(x), \psi(x)]$$

- Hilbert space decomposition $\mathcal{H} = \mathcal{H}_\phi \otimes \mathcal{H}_\psi$
- Reduced density matrix

$$\rho_\phi = \text{Tr}_{\mathcal{H}_\psi} [\rho_{\text{tot.}}]$$

Field Space Entanglement

- Entanglement between fields (due to interaction) [Furukawa, Kim, Xu, Chen, Fradkin, Lundgren, Mollabashi, Shiba, Takayanagi, Taylor]
- Simplest model [Mollabashi-Shiba-Takayanagi '14]

$$\mathcal{L} = \mathcal{L}_1\left[\phi(x)\right] + \mathcal{L}_2\left[\psi(x)\right] + \mathcal{L}_{\text{int.}}\left[\phi(x), \psi(x)\right]$$

- Hilbert space decomposition $\mathcal{H} = \mathcal{H}_\phi \otimes \mathcal{H}_\psi$
- Reduced density matrix

$$\rho_\phi = \text{Tr}_{\mathcal{H}_\psi}\left[\rho_{\text{tot.}}\right]$$

- Entanglement and Renyi Entropies

$$S_{\text{ent.}} = -\text{Tr}\left[\rho_\phi \log \rho_\phi\right] \quad , \quad S^{(n)} = \frac{1}{1-n} \log \text{Tr}\left[\rho_\phi^n\right]$$

Field Space Entanglement

- Direct generalization

$$\mathcal{L} = \mathcal{L}_1 \left[\phi_1(x)\right] + \mathcal{L}_2 \left[\phi_2(x)\right] + \cdots + \mathcal{L}_N \left[\phi_N(x)\right] + \mathcal{L}_{\text{int.}} \left[\phi_i(x)\right]$$

Field Space Entanglement

- Direct generalization

$$\mathcal{L} = \mathcal{L}_1\left[\phi_1(x)\right] + \mathcal{L}_2\left[\phi_2(x)\right] + \cdots + \mathcal{L}_N\left[\phi_N(x)\right] + \mathcal{L}_{\text{int.}}\left[\phi_i(x)\right]$$

- Hilbert space decomposition

$$\begin{aligned}\mathcal{H}_{\text{tot.}} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \\ &= \mathcal{H}_{(m)} \otimes \mathcal{H}_{(N-m)}\end{aligned}$$

Field Space Entanglement

- Direct generalization

$$\mathcal{L} = \mathcal{L}_1\left[\phi_1(x)\right] + \mathcal{L}_2\left[\phi_2(x)\right] + \cdots + \mathcal{L}_N\left[\phi_N(x)\right] + \mathcal{L}_{\text{int.}}\left[\phi_i(x)\right]$$

- Hilbert space decomposition

$$\begin{aligned}\mathcal{H}_{\text{tot.}} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \\ &= \mathcal{H}_{(m)} \otimes \mathcal{H}_{(N-m)}\end{aligned}$$

- Reduced density matrix

$$\rho(m, N) = \text{Tr}_{\mathcal{H}_{(N-m)}}\left[\rho_{\text{tot.}}\right]$$

Field Space Entanglement

- Direct generalization

$$\mathcal{L} = \mathcal{L}_1\left[\phi_1(x)\right] + \mathcal{L}_2\left[\phi_2(x)\right] + \cdots + \mathcal{L}_N\left[\phi_N(x)\right] + \mathcal{L}_{\text{int.}}\left[\phi_i(x)\right]$$

- Hilbert space decomposition

$$\begin{aligned}\mathcal{H}_{\text{tot.}} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \\ &= \mathcal{H}_{(m)} \otimes \mathcal{H}_{(N-m)}\end{aligned}$$

- Reduced density matrix

$$\rho(m,N) = \text{Tr}_{\mathcal{H}_{(N-m)}}\left[\rho_{\text{tot.}}\right]$$

- Entanglement and Renyi entropies

$$S_{\text{ent.}} = -\text{Tr}\left[\rho(m,N)\log\rho(m,N)\right] \quad , \quad S^{(n)} = \frac{1}{1-n}\log\text{Tr}\left[\rho^n(m,N)\right]$$

Kinetic Mixing Gaussian Models

Gaussian Models

- Most general wave functional

$$\Psi[\phi_i] = \mathcal{N} \exp\left\{ -\frac{1}{2} \int dx^{d-1} dy^{d-1} \sum_{i,j=1}^{N} \phi_i(x) G_{ij}(x,y) \phi_j(y) \right\}$$

Gaussian Models

- Most general wave functional

$$\Psi[\phi_i] = \mathcal{N} \exp\left\{ -\frac{1}{2} \int dx^{d-1} dy^{d-1} \sum_{i,j=1}^{N} \phi_i(x) G_{ij}(x,y) \phi_j(y) \right\}$$

- Density matrix $\rho_{\text{tot.}}[\phi_i', \phi_i] = \Psi^*[\phi_i'] \Psi[\phi_i]$

Gaussian Models

- Most general wave functional

$$\Psi[\phi_i] = \mathcal{N} \exp\left\{ -\frac{1}{2} \int dx^{d-1} dy^{d-1} \sum_{i,j=1}^{N} \phi_i(x) G_{ij}(x,y) \phi_j(y) \right\}$$

- Density matrix $\rho_{\text{tot.}}[\phi'_i, \phi_i] = \Psi^*[\phi'_i]\Psi[\phi_i]$
- Reduced density matrix

$$\rho(m,N)[\phi'_{m+1}, \phi_{m+1}, \cdots, \phi'_N, \phi_N] = \int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_{N-m} \Psi^*[\phi'_i]\Psi[\phi_i]$$

Gaussian Models

- Most general wave functional

$$\Psi[\phi_i] = \mathcal{N} \exp\left\{ -\frac{1}{2} \int dx^{d-1} dy^{d-1} \sum_{i,j=1}^{N} \phi_i(x) G_{ij}(x,y) \phi_j(y) \right\}$$

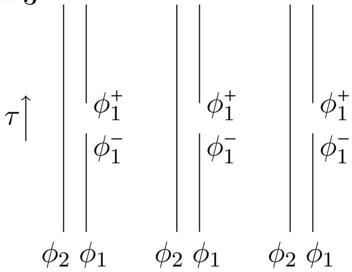
- Density matrix $\rho_{\text{tot.}}[\phi_i', \phi_i] = \Psi^*[\phi_i'] \Psi[\phi_i]$
- Reduced density matrix

$$\rho(m,N)[\phi'_{m+1}, \phi_{m+1}, \cdots, \phi'_N, \phi_N] = \int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_{N-m} \Psi^*[\phi_i'] \Psi[\phi_i]$$

- Replication $\Rightarrow$ $\text{Tr}\left[\rho^n(m,N)\right]$

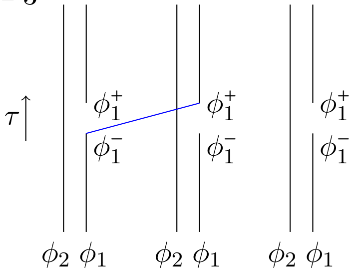
Replication

- $N = 2,\ m = 1,\ n = 3$



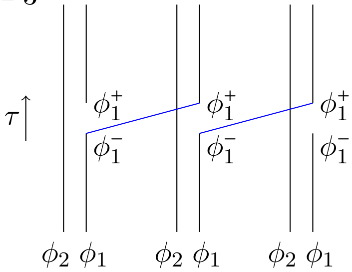
Replication

- $N = 2,\ m = 1,\ n = 3$



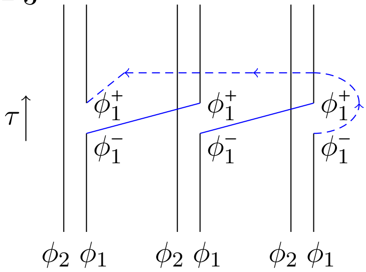
Replication

- $N = 2,\ m = 1,\ n = 3$



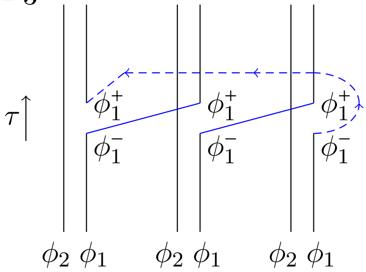
Replication

- $N = 2,\ m = 1,\ n = 3$

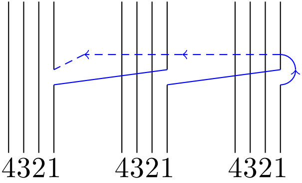


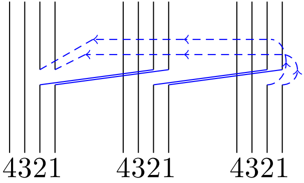
Replication

- $N = 2,\ m = 1,\ n = 3$



- $N = 4,\ m = 3, 2,\ n = 3$





Our (Gaussian) Models

1. Infinite-Range model

$$S = \frac{1}{2} \int d^d x \left[\sum_{i=1}^{N} (\partial_\mu \phi_i)^2 + \lambda \sum_{i<j\leq N}^{N} \partial_\mu \phi_i \partial^\mu \phi_j \right]$$

Our (Gaussian) Models

1. Infinite-Range model

$$S = \frac{1}{2}\int d^d x\left[\sum_{i=1}^{N}(\partial_\mu\phi_i)^2 + \lambda\sum_{i<j\leq N}^{N}\partial_\mu\phi_i\partial^\mu\phi_j\right]$$

2. Nearest-Neighbour model

$$S = \frac{1}{2}\int d^d x\left[\sum_{i=1}^{N}(\partial_\mu\phi_i)^2 + \lambda\sum_{\langle i,j\rangle}\partial_\mu\phi_i\partial^\mu\phi_j\right]$$

Our (Gaussian) Models

1. Infinite-Range model

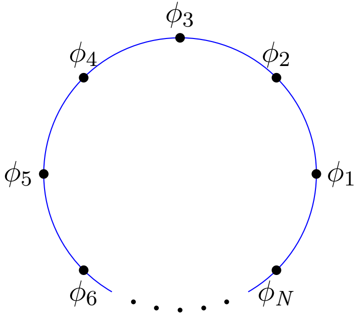
$$S = \frac{1}{2}\int d^dx \left[\sum_{i=1}^{N}(\partial_\mu\phi_i)^2 + \lambda\sum_{i<j\leq N}^{N}\partial_\mu\phi_i\partial^\mu\phi_j\right]$$

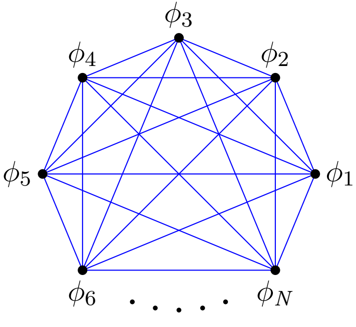
2. Nearest-Neighbour model

$$S = \frac{1}{2}\int d^dx \left[\sum_{i=1}^{N}(\partial_\mu\phi_i)^2 + \lambda\sum_{\langle i,j\rangle}\partial_\mu\phi_i\partial^\mu\phi_j\right]$$

$$G_{\rm IR} \sim \begin{pmatrix} 2 & \lambda & \lambda & \cdots & \lambda \\ \lambda & 2 & \lambda & \cdots & \lambda \\ \lambda & \lambda & 2 & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \cdots & 2 \end{pmatrix} \quad , \quad G_{\rm NN} \sim \begin{pmatrix} 2 & \lambda & 0 & \cdots & 0 & \lambda \\ \lambda & 2 & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & \lambda \\ \lambda & 0 & 0 & \cdots & \lambda & 2 \end{pmatrix}$$

Field Space Visualization





Entanglement Measures in Field Space

Infinite-Range Model

- Reduced density matrix

$$\mathrm{Tr}\left[\rho^n(m,N)\right] = \mathcal{N}\prod_i \prod_{r=1}^{n}\left[1+f(m,N)\cos\left(\frac{2\pi r}{n}\right)\right]$$

where

$$f(m,N) = \frac{4(N-m)\,Y(m)}{4(N-m)\,Y(m)+(N-m)\lambda+2-\lambda}$$

$$Y(m) = -\frac{1}{4}\left(\frac{\lambda}{2}\right)^2\cdot\frac{2m}{2+(m-1)\lambda}$$

Nearest-Neighbour Model

- Reduced density matrix

$$\mathrm{Tr}\left[\rho^n(m,N)\right] = \mathcal{N}\prod_i \prod_{r=1}^{n}\left[1+\frac{2\,Y_-(m)g_-(N-m-1)}{2\,Y_-(m)g_-(N-m-1)-g_-(N-m+1)}\cos\left(\frac{2\pi r}{n}\right)\right]\times \prod_{s=1}^{n}\left[1+\frac{2\,Y_+(m)g_+(N-m-1)}{2\,Y_+(m)g_+(N-m-1)+g_+(N-m+1)}\cos\left(\frac{2\pi s}{n}\right)\right]$$

where

$$g_\pm(N) = \prod_{s=1}^{\frac{N}{2}}\left[1-\lambda\cos\left(\frac{d_\pm(N)+2s}{N}\pi\right)\right], \qquad d_\pm(N) = \sin^2\left(\frac{2N+1\mp 1}{4}\pi\right)-1,$$

$$Y(m) = -\frac{1}{4}\left(-\frac{\lambda}{2}\right)^{m+1}\cdot\frac{1}{Z(m+1)}, \qquad Y_d(m) = -\frac{1}{4}\left(-\frac{\lambda}{2}\right)^{2}\cdot\frac{Z(m)}{Z(m+1)},$$

$$Y_\pm(m) = Y(m)\pm Y_d(m), \qquad Z(m) = \prod_{r=1}^{m-1}\left[1-\lambda\cos\left(\frac{r}{m}\pi\right)\right].$$

Entanglement & Renyi Entropies

- Entropy calculation

$$S^{(n)} \equiv \sum_i s^{(n)}(\xi) = s^{(n)}(\xi) \sum_{\vec{k}} 1, \quad S \equiv \sum_i s(\xi) = s(\xi) \sum_{\vec{k}} 1$$

where

$$s^{(n)}(\xi) = \frac{n \ln(1-\xi) - \ln(1-\xi^n)}{1-n}$$
$$s(\xi) = \left[-\ln(1-\xi) - \frac{\xi}{1-\xi} \ln \xi \right]$$

- Regularization by $e^{-\epsilon|k|}$ on a $(d-1)$-torus with size L

$$\sum_{\vec{k}\neq 0} 1 \sim \left(\sum_{k \neq 0} e^{-\epsilon|k|} \right)^{d-1} = c_{d,d-1} \left(\frac{L}{\epsilon} \right)^{d-1} + c_{d,d-2} \left(\frac{L}{\epsilon} \right)^{d-2} + \cdots + c_{d,0}$$

Entanglement Inequalities

- Positivity of EE $S_A \geq 0$

Entanglement Inequalities

- Positivity of EE $S_A \geq 0$
- Renyi Inequalities

$$\frac{\partial}{\partial n} S^{(n)} \leq 0, \qquad \frac{\partial}{\partial n}\Big((n-1)S^{(n)}\Big) \geq 0,$$

$$\frac{\partial}{\partial n}\left(\frac{n-1}{n} S^{(n)}\right) \geq 0, \qquad \frac{\partial^2}{\partial n^2}\Big((n-1)S^{(n)}\Big) \leq 0.$$

Entanglement Inequalities

- Positivity of EE $S_A \geq 0$
- Renyi Inequalities

$$\frac{\partial}{\partial n} S^{(n)} \leq 0, \qquad \frac{\partial}{\partial n}\Big((n-1)S^{(n)}\Big) \geq 0,$$
$$\frac{\partial}{\partial n}\left(\frac{n-1}{n} S^{(n)}\right) \geq 0, \qquad \frac{\partial^2}{\partial n^2}\Big((n-1)S^{(n)}\Big) \leq 0.$$

- Positivity of $I(A,B) = S_A + S_B - S_{A\cup B}$ (subadditivity)

Entanglement Inequalities

- Positivity of EE $S_A \geq 0$
- Renyi Inequalities

$$\frac{\partial}{\partial n} S^{(n)} \leq 0, \qquad \frac{\partial}{\partial n}\Big((n-1)S^{(n)}\Big) \geq 0,$$

$$\frac{\partial}{\partial n}\left(\frac{n-1}{n} S^{(n)}\right) \geq 0, \qquad \frac{\partial^2}{\partial n^2}\Big((n-1)S^{(n)}\Big) \leq 0.$$

- Positivity of $I(A,B) = S_A + S_B - S_{A\cup B}$ (subadditivity)
- Araki-Lieb inequality: $S_A \leq S_{A\cup B} + S_B$

Entanglement Inequalities

- Positivity of EE $S_A \geq 0$
- Renyi Inequalities

$$\frac{\partial}{\partial n} S^{(n)} \leq 0, \qquad \frac{\partial}{\partial n}\left((n-1)S^{(n)}\right) \geq 0,$$
$$\frac{\partial}{\partial n}\left(\frac{n-1}{n} S^{(n)}\right) \geq 0, \qquad \frac{\partial^2}{\partial n^2}\left((n-1)S^{(n)}\right) \leq 0.$$

- Positivity of $I(A,B) = S_A + S_B - S_{A\cup B}$ (subadditivity)
- Araki-Lieb inequality: $S_A \leq S_{A\cup B} + S_B$
- Strong subadditivity

$$S_{A\cup B\cup C} + S_B \leq S_{A\cup B} + S_{B\cup C}$$
$$S_A + S_C \leq S_{A\cup B} + S_{B\cup C}$$

Entanglement Inequalities

- Positivity of EE $S_A \geq 0$
- Renyi Inequalities

$$\frac{\partial}{\partial n} S^{(n)} \leq 0, \qquad \frac{\partial}{\partial n}\Big((n-1)S^{(n)}\Big) \geq 0,$$
$$\frac{\partial}{\partial n}\left(\frac{n-1}{n} S^{(n)}\right) \geq 0, \qquad \frac{\partial^2}{\partial n^2}\Big((n-1)S^{(n)}\Big) \leq 0.$$

- Positivity of $I(A,B) = S_A + S_B - S_{A\cup B}$ (subadditivity)
- Araki-Lieb inequality: $S_A \leq S_{A\cup B} + S_B$
- Strong subadditivity

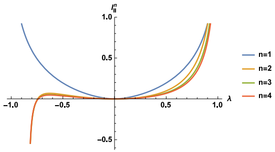
$$S_{A\cup B\cup C} + S_B \leq S_{A\cup B} + S_{B\cup C}$$
$$S_A + S_C \leq S_{A\cup B} + S_{B\cup C}$$

- Positivity of n-partite information ($n \leq 5$)

Entanglement Inequalities: Two Remarkable Notes

- Sign of mutual Renyi information?

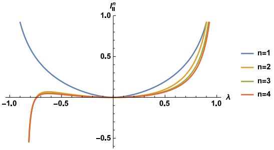
$$I^{(n)}(A,B) = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$$



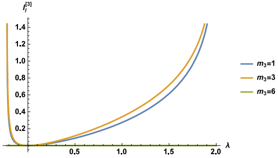
Entanglement Inequalities: Two Remarkable Notes

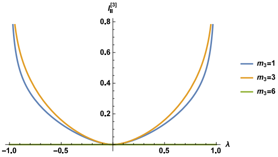
- Sign of mutual Renyi information?

$$I^{(n)}(A,B) = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$$



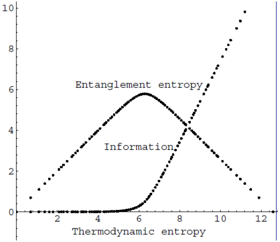
- MI is NOT monogamous ⇒ Existence of holographic dual?
  [Hayden-Headrick-Maloney 11]

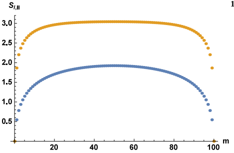


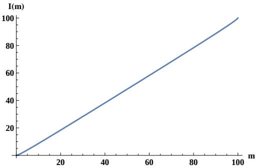


A Model for Black Hole Radiation

- Two subsystems with m & n-dim. Hilbert spaces in a pure state [Page '93]
- EE is symmetric as a function of $\log m$
- Information $I = \log m - S_{EE}$ leaks out of the BH after EE starts decreasing
- $I = m - S_{EE} \quad \Rightarrow \quad$ leakage from the beginning







How About Massive Models?

- Generic massive model (Not solved yet !)

$$S = \frac{1}{2}\int d^dx \left[\sum_{i=1}^{N}(\partial_\mu\phi_i)^2 + \sum_{i<j\leq N}^{N} m_{ij}\ \phi_i\phi_j\right]$$

- Case $N = 2$

$$S = \frac{1}{2}\int d^dx \left[(\partial_\mu\phi)^2 + (\partial_\mu\psi)^2 - (\phi_1, \phi_2)\begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22}\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}\right]$$

In the ground state

$$\frac{S_\phi}{V_{d-1}} \propto \begin{cases} \text{finite} & \textit{for} \quad d \leq 4 \\ \frac{1}{2}(\ln\Lambda)^2 & \textit{for} \quad d = 5 \\ \frac{1}{d-5}\Lambda^{d-5}\ln\Lambda & \textit{for} \quad d \geq 6 \end{cases}$$

Suppression of volume law divergence

Geometric versus Field Space Entanglement

- Starting in $d+1$-dimensions

$$S^{(d+1)} = \frac{1}{2} \int dt \; d\vec{x}_{d-1} \; d\theta \left[(\partial_\mu \phi)^2 + (\partial_\theta \phi)^2 \right].$$

with

$$\phi(\theta, x^\mu) = \{\phi_n(x^\mu) = \phi(n\Delta, x^\mu)\} \quad , \quad n = 0, 1, \cdots, \frac{2\pi}{\Delta} (\equiv N)$$

- Using

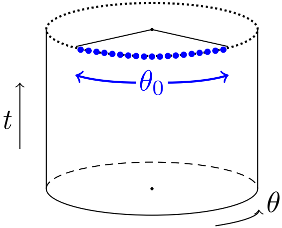
$$\int d\theta \left[(\partial_\mu \phi)^2 + (\partial_\theta \phi)^2 \right] = \Delta \sum_{n=0}^{N} \left[(\partial_\mu \phi_n)^2 + \left(\frac{\phi_{n+1} - \phi_n}{\Delta} \right)^2 \right].$$

leads to

$$S^{(d)} = \frac{\Delta}{2} \int dt \; d\vec{x} \; \sum_{n=0}^{N} \left[(\partial_\mu \phi_n)^2 + \left(\frac{\phi_{n+1} - \phi_n}{\Delta} \right)^2 \right]$$

Geometric versus Field Space Entanglement

- Consider a geometrical entangling region as $0 < \theta < \theta_0$



$$\mathcal{H} = \mathcal{H}_{\theta>\theta_0} \otimes \mathcal{H}_{\theta<\theta_0}$$

- Geometric entanglement between localized d.o.f.'s with $\theta > \theta_0$ and $\theta < \theta_0$ in $d+1$-dimensions is translated to field space entanglement between *massive* fields ϕ_i's with $i < n_0$ and $i > n_0$ $(\theta_0 = n_0 \Delta)$

Holographic Picture

SU(N/2)×SU(N/2) Solution

- Consider N D3-brane located at $\vec{y}_a$, $a = 1, 2, \cdots, N$
- Type IIB SUGRA solution (in NHL)

$$ds^2 = f^{-1/2}(\vec{y})\, dx^\mu dx_\mu + f^{1/2}(\vec{y})\, dy^i dy^i \quad , \quad f(\vec{y}) = \sum_{a=1}^{N} \frac{R^4}{|\vec{y} - \vec{y}_a|^4}$$

SU($N/2$)×SU($N/2$) Solution

- Consider N D3-brane located at $\vec{y}_a,\ a = 1, 2, \cdots, N$
- Type IIB SUGRA solution (in NHL)

$$ds^2 = f^{-1/2}(\vec{y})\, dx^\mu dx_\mu + f^{1/2}(\vec{y})\, dy^i dy^i \quad , \quad f(\vec{y}) = \sum_{a=1}^{N} \frac{R^4}{|\vec{y} - \vec{y}_a|^4}$$

- Consider two stack of n and m D3-branes at $\vec{Y}_{1,2}$

$$f(\vec{y}) = \frac{n}{N} \frac{R^4}{|\vec{y} - \vec{Y}_1|^4} + \frac{m}{N} \frac{R^4}{|\vec{y} - \vec{Y}_2|^4}$$

SU(N/2)×SU(N/2) Solution

- Consider N D3-brane located at $\vec{y}_a$, $a = 1, 2, \cdots, N$
- Type IIB SUGRA solution (in NHL)

$$ds^2 = f^{-1/2}(\vec{y})\, dx^\mu dx_\mu + f^{1/2}(\vec{y})\, dy^i dy^i \quad , \quad f(\vec{y}) = \sum_{a=1}^{N} \frac{R^4}{|\vec{y} - \vec{y}_a|^4}$$

- Consider two stack of n and m D3-branes at $\vec{Y}_{1,2}$

$$f(\vec{y}) = \frac{n}{N}\frac{R^4}{|\vec{y} - \vec{Y}_1|^4} + \frac{m}{N}\frac{R^4}{|\vec{y} - \vec{Y}_2|^4}$$

- Considering $d\vec{y}^2 = dy^2 + du^2 + u^2 d\Omega_4^2$ where $\vec{Y}_{1,2} = (\pm d, \vec{0})$

$$f = \frac{n}{N}\frac{R^4}{\left[(y-d)^2 + u^2\right]^2} + \frac{m}{N}\frac{R^4}{\left[(y+d)^2 + u^2\right]^2}$$

SU(N/2)×SU(N/2) Solution

- Coordinate transformation

$$u = r\sin\theta \quad , \quad y = y_m + r\cos\theta \quad , \quad y_m = d\cdot\frac{m-n}{m+n}$$

which leads to

$$d\vec{y}^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\Omega_4^2 = dr^2 + r^2 d\Omega_5^2$$

SU(N/2)×SU(N/2) Solution

- Coordinate transformation

$$u = r\sin\theta \quad , \quad y = y_m + r\cos\theta \quad , \quad y_m = d\cdot\frac{m-n}{m+n}$$

which leads to

$$d\vec{y}^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\Omega_4^2 = dr^2 + r^2 d\Omega_5^2$$

- For $n = m = \frac{N}{2}$, SSB from SU(N) to

$$\mathrm{SU}(N/2)\times\mathrm{SU}(N/2)$$

SU(N/2)×SU(N/2) Solution

- Coordinate transformation

$$u = r\sin\theta \quad , \quad y = y_m + r\cos\theta \quad , \quad y_m = d\cdot\frac{m-n}{m+n}$$

which leads to

$$d\vec{y}^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\Omega_4^2 = dr^2 + r^2 d\Omega_5^2$$

- For $n = m = \frac{N}{2}$, SSB from SU(N) to

$$\mathrm{SU}(N/2)\times\mathrm{SU}(N/2)$$

- Mass of gauge bosons in $\frac{\mathrm{SU}(N)}{\mathrm{SU}(N/2)\otimes\mathrm{SU}(N/2)}$

$$M \sim d$$

- For $\Lambda \ll d$, we neglect the role of massive bosons

HEE of Two SU($N/2$) Gauge Theories

- **Argument:** EE between these CFTs is given by

$$S_{\text{ent.}} = \frac{\text{Area}(\gamma)}{4\,G_N^{(10)}},$$

γ: Minimal surface separating the two stacks of branes

HEE of Two SU($N/2$) Gauge Theories

- **Argument:** EE between these CFTs is given by

$$S_{\text{ent.}} = \frac{\text{Area}(\gamma)}{4\,G_N^{(10)}},$$

γ: Minimal surface separating the two stacks of branes

- γ is 8-dim surface: $\gamma : t = 0 \ , \ \theta = \frac{\pi}{2}$
- Entanglement entropy

$$S_{\text{ent.}} = \frac{V_3 \text{Vol}(\text{S}^4)}{4\,G_N^{(10)}} \int_0^{r_{UV}} R^2 \frac{r^4}{r^2 + d^2}\, dr$$

HEE of Two SU($N/2$) Gauge Theories

- **Argument:** EE between these CFTs is given by

$$S_{\text{ent.}} = \frac{\text{Area}(\gamma)}{4\,G_N^{(10)}},$$

γ: Minimal surface separating the two stacks of branes

- γ is 8-dim surface: $\quad \gamma : t = 0 \;\; , \;\; \theta = \frac{\pi}{2}$
- Entanglement entropy

$$S_{\text{ent.}} = \frac{V_3 \text{Vol}(S^4)}{4\,G_N^{(10)}} \int_0^{r_{UV}} R^2 \frac{r^4}{r^2 + d^2} dr$$

- With $r_{\text{UV}} = \Lambda R^2$ and $r_{\text{UV}} \ll d$

$$S_{\text{ent.}} \simeq \frac{16 N^2 V_3}{15\pi^3} \lambda g \Lambda^3$$

HEE of Two SU($N/2$) Gauge Theories

- **Argument:** EE between these CFTs is given by

$$S_{\text{ent.}} = \frac{\text{Area}(\gamma)}{4G_N^{(10)}},$$

γ: Minimal surface separating the two stacks of branes

- γ is 8-dim surface: $\gamma : t = 0 \;\; , \;\; \theta = \frac{\pi}{2}$
- Entanglement entropy

$$S_{\text{ent.}} = \frac{V_3 \text{Vol}(\text{S}^4)}{4G_N^{(10)}} \int_0^{r_{UV}} R^2 \frac{r^4}{r^2 + d^2} dr$$

- With $r_{\text{UV}} = \Lambda R^2$ and $r_{\text{UV}} \ll d$

$$S_{\text{ent.}} \simeq \frac{16 N^2 V_3}{15\pi^3} \lambda g \Lambda^3$$

- Based on the observation we can generalize RT:

A proposal for generalized HEE

- The gravity dual is described by $\mathrm{M}_{q+d+1} = \mathrm{Y}^{AdS}_{d+1} \times \mathrm{X}_q$

$$\mathrm{Y}^{AdS}_{d+1} : (d+1) \text{ dimensional asymptotically AdS space}$$
$$\mathrm{X}_q : q \text{ dimensional internal space}$$

A proposal for generalized HEE

- The gravity dual is described by $\mathrm{M}_{q+d+1} = \mathrm{Y}_{d+1}^{AdS} \times \mathrm{X}_q$

$$\mathrm{Y}_{d+1}^{AdS} : (d+1) \text{ dimensional asymptotically AdS space}$$
$$\mathrm{X}_q : q \text{ dimensional internal space}$$

- In Poincare coor. $\partial \mathrm{M}_{q+d+1} = \mathbb{R}^{1,d-1} \times \mathrm{X}_q$

A proposal for generalized HEE

- The gravity dual is described by $\mathrm{M}_{q+d+1} = \mathrm{Y}^{AdS}_{d+1} \times \mathrm{X}_q$

$$\mathrm{Y}^{AdS}_{d+1} : (d+1)\ \text{dimensional asymptotically AdS space}$$
$$\mathrm{X}_q : q\ \text{dimensional internal space}$$

- In Poincare coor. $\partial \mathrm{M}_{q+d+1} = \mathbb{R}^{1,d-1} \times \mathrm{X}_q$
- Consider A such that ∂A divides $\mathbb{R}^{d-1} \times \mathrm{X}_q$ into two parts

A proposal for generalized HEE

- The gravity dual is described by $\mathrm{M}_{q+d+1} = \mathrm{Y}_{d+1}^{AdS} \times \mathrm{X}_q$

$$\mathrm{Y}_{d+1}^{AdS} : (d+1) \text{ dimensional asymptotically AdS space}$$
$$\mathrm{X}_q : q \text{ dimensional internal space}$$

- In Poincare coor. $\partial \mathrm{M}_{q+d+1} = \mathbb{R}^{1,d-1} \times \mathrm{X}_q$
- Consider A such that ∂A divides $\mathbb{R}^{d-1} \times \mathrm{X}_q$ into two parts
- EE of A is given by

$$S_{\text{ent}} = \frac{\text{Area}(\gamma)}{4 G_N},$$

γ: the minimal (extremal) surface i.e. $\partial\gamma = \partial A$

A proposal for generalized HEE

- The gravity dual is described by $\mathrm{M}_{q+d+1} = \mathrm{Y}^{AdS}_{d+1} \times \mathrm{X}_q$

$$\mathrm{Y}^{AdS}_{d+1} : (d+1) \text{ dimensional asymptotically AdS space}$$
$$\mathrm{X}_q : q \text{ dimensional internal space}$$

- In Poincare coor. $\partial \mathrm{M}_{q+d+1} = \mathbb{R}^{1,d-1} \times \mathrm{X}_q$
- Consider A such that ∂A divides $\mathbb{R}^{d-1} \times \mathrm{X}_q$ into two parts
- EE of A is given by

$$S_{\text{ent}} = \frac{\text{Area}(\gamma)}{4 G_N},$$

γ: the minimal (extremal) surface i.e. $\partial\gamma = \partial A$

- If ∂A wraps X_q completely $\rightarrow$ RT proposal

Examples of GHEE

1. $\mathrm{AdS}_5\times\mathrm{S}^5$: $A(B)$: Northern (southern) hemisphere of S^5

$$S_{\text{ent.}} = \frac{4}{9\pi^2}\frac{N^2\,V_3}{\epsilon^3} < S_{\max}$$

Examples of GHEE

1. $AdS_5 \times S^5$: $A(B)$: Northern (southern) hemisphere of S^5

$$S_{\text{ent.}} = \frac{4}{9\pi^2}\frac{N^2 V_3}{\epsilon^3} < S_{\text{max}}$$

2. AdS_5-Schld$\times S^5$

$$S_{\text{ent.}}^{\text{reg.}} = 1.83 \cdot N^2 V_3 T^3 \quad , \quad S_{\text{thermal}}^{\text{free}} \simeq 1.64 \cdot N^2 V_3 T^3$$

Examples of GHEE

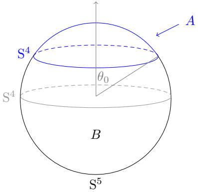
1. $AdS_5 \times S^5$: $A(B)$: Northern (southern) hemisphere of S^5

$$S_{\text{ent.}} = \frac{4}{9\pi^2} \frac{N^2 V_3}{\epsilon^3} < S_{\max}$$

2. AdS_5-Schld$\times S^5$

$$S_{\text{ent.}}^{\text{reg.}} = 1.83 \cdot N^2 V_3 T^3 \quad , \quad S_{\text{thermal}}^{\text{free}} \simeq 1.64 \cdot N^2 V_3 T^3$$

3. $AdS_5 \times S^5$: A a spherical cap of S^5 (∂A: $\theta = \theta_0$)



Examples of GHEE

- Entanglement entropy

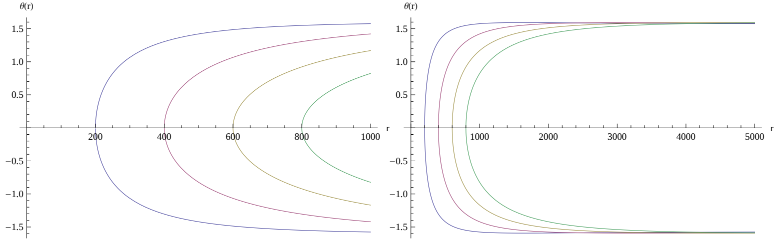
$$S_{\text{ent}} = \frac{R^2 V_3 \text{Vol}(S^4)}{4 G_N^{(10)}} \int_{r_*}^{r_{UV}} r^2 \sin^4\theta(r) \sqrt{1 + r^2 \dot{\theta}^2(r)}\, dr$$

Examples of GHEE

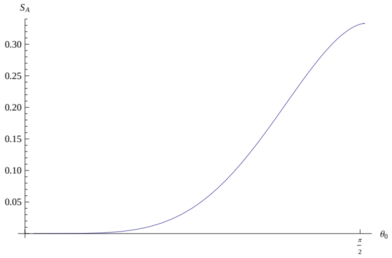
- Entanglement entropy

$$S_{\text{ent}} = \frac{R^2 V_3 \text{Vol}(S^4)}{4 G_N^{(10)}} \int_{r_*}^{r_{UV}} r^2 \sin^4\theta(r) \sqrt{1 + r^2 \dot{\theta}^2(r)}\, dr$$

- In AdS×X(compact): the boundary of minimal surfaces ending on ∂AdS in X is an extremal surface. [Karch-Graham 14]



Examples of GHEE



- Entanglement between SU(n) and SU(m) subsectors of SU(N)

$$\kappa \equiv \frac{m}{N} = \frac{8}{3\pi}\int_0^{\theta_0} d\theta \sin^4\theta$$

Cut-off Surface

- Cut-off: (intuitively) the level sets of the scale factor multiplying Mink. factor [Karch-Uhlemann 15]

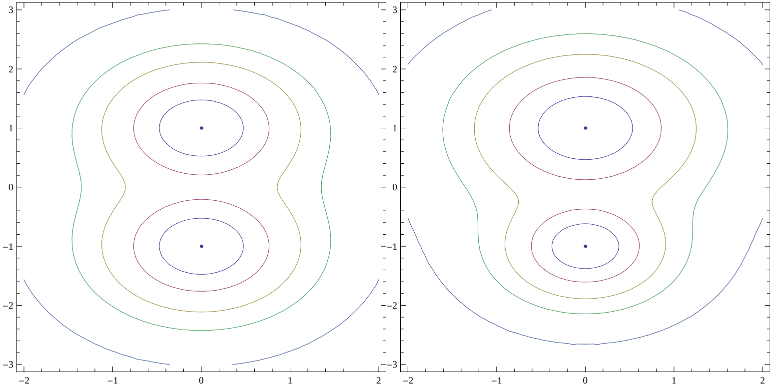


Figure : Left: $\kappa = 1/2$, Right: $\kappa = 1/5$.

Irreducible EE

- Minimal surfaces
  - EE is due to split: a subsector CFT_1 and a part of CFT_2
  - Irreducible EE: EE between U(n) and U(m) SYM due to W interaction

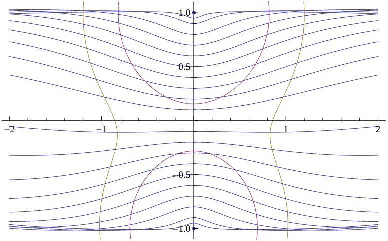


Figure : $d = 1$, $\kappa = 1/3$, $f = \Lambda^4 = \{\frac{2}{3}, 3\}$.

Conclusions & Further Directions

- Field space entanglement is as natural as geometric or momentum space entanglement.

Conclusions & Further Directions

- Field space entanglement is as natural as geometric or momentum space entanglement.
- How about excited states?

Conclusions & Further Directions

- Field space entanglement is as natural as geometric or momentum space entanglement.
- How about excited states?
- Entanglement pattern for higher spins (spin-$\frac{1}{2}$, spin-1, $\cdots$) in the field space.

Conclusions & Further Directions

- Field space entanglement is as natural as geometric or momentum space entanglement.
- How about excited states?
- Entanglement pattern for higher spins (spin-$\frac{1}{2}$, spin-1, $\cdots$) in the field space.
- The effect of massive interactions (still Gaussian).
  Relation to geometric entanglement in one higher dimension.

Conclusions & Further Directions

- Field space entanglement is as natural as geometric or momentum space entanglement.
- How about excited states?
- Entanglement pattern for higher spins (spin-$\frac{1}{2}$, spin-1, $\cdots$) in the field space.
- The effect of massive interactions (still Gaussian).
  Relation to geometric entanglement in one higher dimension.
- Universal behavior during time evolution in field space entanglement?

Conclusions & Further Directions

- Field space entanglement is as natural as geometric or momentum space entanglement.
- How about excited states?
- Entanglement pattern for higher spins (spin-$\frac{1}{2}$, spin-1, $\cdots$) in the field space.
- The effect of massive interactions (still Gaussian).
  Relation to geometric entanglement in one higher dimension.
- Universal behavior during time evolution in field space entanglement?
- Holographic picture: dividing the internal space may address entanglement between interacting CFTs.