On the Shape of things

From holography to elastica

Álvaro Véliz-Osorio (Jagiellonian University)

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Based on 1611.03462 with:

Piermarco Fonda (Leiden)

Vishnu Jejjala (Wits)

...and work in progress

General question

Which shape a manifold is compelled to take when immersed in another one, provided it must extremize some functional?

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Did Virgilio forsee Ryu-Takayanagi?!?!

Geometric setup

Immersion $f : N \to (M, g)$

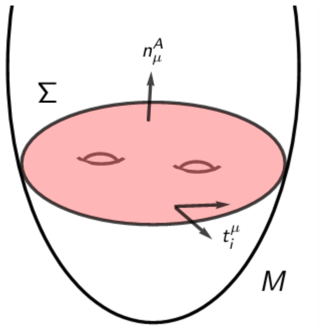
$$\Sigma = \{x^{\mu}(\sigma_i) | \ i = 1, \dots, p\}$$

Indices:

Ambient $\mu,\ \nu = 1, \dots, d$

Tangent $i,\ j = 1, \dots, p$

Normal $A,\ B = 1, \dots, (d - p)$



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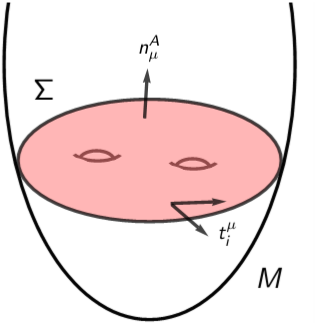

Induced metric

$$h_{ij} = g_{\mu\nu} t^\mu_i t^\nu_j$$

We can associate $\tilde{\nabla}_i$, $\mathcal{R}_{ijkl}$, $\mathcal{R}$, ...

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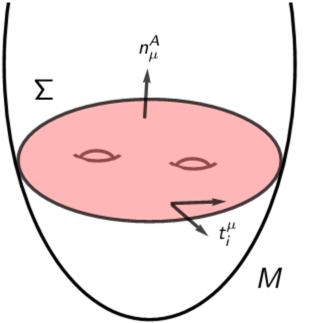

Projecting ambient tensors, example

$$R^A{}_{jik} = R_{\mu\nu\rho\sigma} n^{A\mu} t^\nu_j t^\rho_j t^\sigma_j$$

Extrinsic geometry

As one moves along Σ, how do normal vectors change?

$$t_i^\nu \nabla_\nu n^{\mu A} = K_{ij}^A t^{\mu j} - T_i^{AB} n_B^\mu \ ,$$

Extrinsic curvatures $\quad K_{ij}^A = t_i^\mu t_j^\nu \nabla_\mu n_\nu^A$

Extrinsic torsions $\quad T_i^{AB} = t_i^\mu n^{\nu A} \nabla_\mu n_\nu^B$

Gauss relation

$$\mathcal{R}_{jkil} = R_{jkil} + K_{[ij}^A K_{kl]}^B \eta_{AB}$$

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Contracted

$$\mathcal{R} = R - 2{R_A}^A + {R_{AB}}^{AB} + \mathrm{Tr} K_A \mathrm{Tr} K^A - \mathrm{Tr}(K_A K^A)$$

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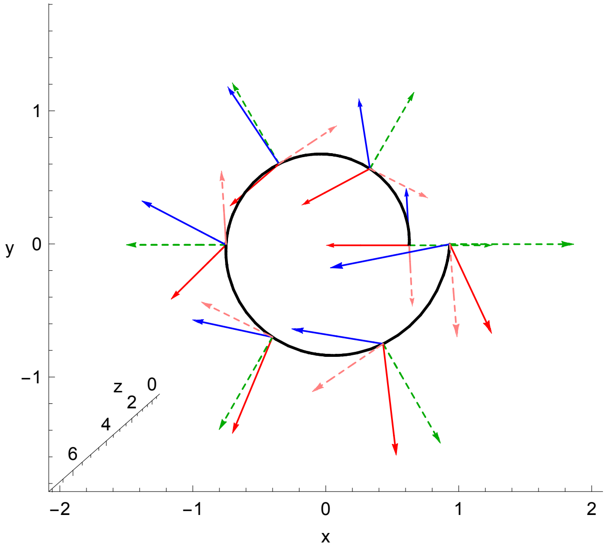
Contains the *Theorema egregium* $\quad \mathcal{R} = 2 \det K_i^{\,j}$

Gauge theory and the normal frame



Gauge theory and normal bundle

Under gauge transformations, $n^B_\mu \to \mathcal{M}^A{}_B n^B_\mu$

$$K^A_{ij} \to \mathcal{M}^A{}_B K^B_{ij} \qquad T_i^{AB} \to \mathcal{M}^C{}_A \mathcal{M}^D{}_B \; T_i^{AB} + \eta^{AB} \mathcal{M}^C{}_A \partial_i \mathcal{M}^D{}_B$$

T_i^{AB} transform as connections

Introduce a covariant derivative

$$\tilde{D}_i{}^A{}_B V^B_{j\dots} \equiv \tilde{\nabla}_i V^B_{j\dots} + T_i^{AB} \eta_{BC} V^C_{j\dots}$$

Gauge theory technology can be imported

Interesting to combine with geometric identities, Ex:

$$F^{AB}_{ij} = K^A_{[ik} K^B_{j]l} h^{kl} - R^{AB}{}_{ij}$$

A simple and beautiful example

A surface in a three dimensional manifold

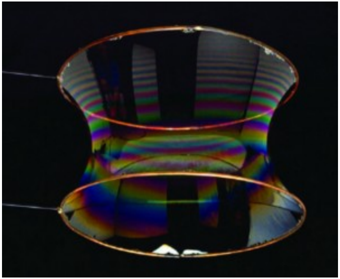
$$S_0[\Sigma] = \lambda_0 \int_\Sigma d^p\sigma \, \sqrt{h} \, 1 = \lambda_0 \, \mathrm{Area}[\Sigma] \, .$$

Moving away in a normal direction

$$\mathcal{L}_n h_{ij} = 2\varepsilon \, K_{ij} \, ,$$

The first *shape equation*

$$\mathrm{Tr} K_{ij} = 0$$



When minimising its area a surface makes each of its point a saddle whose directions have the exact opposite curvature.

A beautiful fact indeed!!

Effective action

Generalizations of the area functional

$$S_0[\Sigma] = \lambda_0 \int_\Sigma d^p\sigma \, \sqrt{h} \, 1 = \lambda_0 \, \mathrm{Area}[\Sigma] \, .$$

Ex: Willmore functional, Canham-Helfrich, Dong functional, ...

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Contributions at second order

$$\lambda_1 \mathcal{R} + \lambda_2 R + \lambda_3 {R_A}^A + \lambda_4 {R_{AB}}^{AB} + \lambda_5 \mathrm{Tr} K_A \mathrm{Tr} K^A + \lambda_6 \mathrm{Tr} K^A K_A$$

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They aren't all independent, since

$$\mathcal{R} = R - 2R_A{}^A + R_{AB}{}^{AB} + \mathrm{Tr} K_A \mathrm{Tr} K^A - \mathrm{Tr}(K_A K^A)$$

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Mission: find the extrema of this functional

Topological terms

For a curve in three dimensions

$$\tau = \frac{1}{2}\epsilon_{AB} T^{AB}$$

For a surface in four dimensions

$$\varphi = \frac{1}{4}\epsilon_{AB}\epsilon^{ij} F_{ij}^{AB}$$

Three-dimensional submanifolds

$$\Phi = \epsilon^{ijk}\eta_{AC}\left(F_{ij\ B}^{A} T_{k}^{BC} - \frac{1}{3} T_{i\ B}^{A} T_{j\ D}^{B} T_{k}^{DC} \right)$$

Important when studying EE for TMG or CS-gravity

Castro, Detournay, Iqbal, Perlmutter '14

Azeyanagi, Logayagam, Ng '15

Ali, Haque, Murugan '16

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Shape equations

Complicated, yet they are completely expressed in terms of geometrical objects.

Simons, Yau, Yano, Chen, Carter, Guven, Capovilla, ...

Many interesting physical applications

Canham, Helfrich, Zhon-Chan, Boisseau-Letelier, Armas, ...

Some of interesting cases:

- Minimal submanifolds $\lambda_0 \neq 0$

$$\mathrm{Tr}\, K^A = 0$$

- Generalized Willmore $\lambda_5 \neq 0$

$$\mathrm{Tr} K_B \left[\mathrm{Tr} K^A \mathrm{Tr} K^B - 2\mathrm{Tr}\left(K^B K^A\right) - 2R^{B}{}_{i}{}^{Ai} \right] - 2\tilde{D}_i{}^B{}_C \tilde{D}^{iCA} \mathrm{Tr} K_B = 0$$

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...in their full glory

Shape equations

For arbitrary dimension and codimension, the extrema of the second order functional obey

$$\mathcal{E}^A = \lambda_0 \mathrm{Tr} K^A + \sum_{n=1}^{6} \lambda_n \mathcal{E}_n^A = 0$$

with

$$\begin{aligned}
\mathcal{E}_1^A &= \mathrm{Tr} K^A \mathcal{R} - 2\mathcal{R}^{ij} K_{ij}^A, \\
\mathcal{E}_2^A &= \mathrm{Tr} K^A R + n_\mu^A \nabla^\mu R, \\
\mathcal{E}_3^A &= \mathrm{Tr} K^A R_B{}^B + 2\tilde{D}_k^{BA} R_B^k + n_C^\mu n^{C\nu} n^{A\delta} \nabla_\delta R_{\mu\nu}, \\
\mathcal{E}_4^A &= \mathrm{Tr} K^A R_{CB}{}^{CB} + 4\tilde{D}_k^{BA} R^{kC}{}_{BC} + n_C^\mu n_B^\nu n^{C\rho} n^{B\sigma} n^{A\delta} \nabla_\delta R_{\mu\nu\rho\sigma}, \\
\mathcal{E}_5^A &= \mathrm{Tr} K_B \left[\mathrm{Tr} K^A \mathrm{Tr} K^B - 2\mathrm{Tr}\left(K^B K^A\right) - 2R^B{}_i{}^{Ai}\right] - 2\tilde{D}_i{}^B{}_C \tilde{D}^{iCA} \mathrm{Tr} K_B, \\
\mathcal{E}_6^A &= \mathrm{Tr} K^A \mathrm{Tr}\left(K_B K^B\right) - 2\left[\tilde{D}_i{}^C{}_B \tilde{D}_j{}^{BA} K_C^{ij} + \mathrm{Tr}\left(K^B K_B K^A\right) + K_B^{ij} R^B{}_j{}^A{}_i\right],
\end{aligned}$$

Shape equations in maximally symmetric spaces

Suppose that the ambient has

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) \qquad R = \text{const}$$

with

$$R = \kappa \frac{d(d-1)}{L^2} \ , \qquad \kappa = 0, \pm 1 \ .$$

Minimal submanifolds are extrema if

$$\lambda_1 = \lambda_6 \qquad \text{or} \qquad \mathcal{R}^{ij} K^A_{ij} = 0 \ .$$

The second condition is true for curves $\mathcal{R}^{ij} = 0$ and surfaces

$$\mathcal{R}^{ij} K^A_{ij} = \frac{\mathcal{R}}{2} \mathrm{Tr} K^A = 0$$

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For $p > 2$, minimal submanifolds are NOT necessarily extrema

Extrema in maximally symmetric surfaces

Considering a curve γ in $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$

Reminiscences

```
In[16]:= eomf = VariationalD[ℒ[ϕ[r]], ϕ[r], r] // FullSimplify
                                                  simplifica completa
```

Out[16]=
$$\frac{1}{L^2\left(L^2+r^4\phi'[r]^2\right)^4\sqrt{\frac{L^2+r^4\phi'[r]^2}{r^2}}}\; r\Big(-L^2\lambda_0\left(L^2+r^4\phi'[r]^2\right)^3\left(3L^2\phi'[r]+r^4\phi'[r]^3+L^2 r\,\phi''[r]\right)+\lambda_5\Big(75L^4r^8\phi'[r]^5-3L^2r^{12}\phi'[r]^7-r^{16}\phi'[r]^9+L^2r^{13}\phi'[r]^6\phi''[r]-5L^4r^4\phi'[r]^3\left(37L^2+r^6\phi''[r]\left(-15\phi''[r]+4r\,\phi^{(3)}[r]\right)\right)+L^6\phi'[r]\left(18L^2-5r^6\phi''[r]\left(27\phi''[r]+4r\,\phi^{(3)}[r]\right)\right)+L^4r^5\phi'[r]^2\left(-267L^2\phi''[r]+30r^6\phi''[r]^3+4L^2r^2\phi^{(4)}[r]\right)+2L^4r^9\phi'[r]^4\left(54\phi''[r]+r\left(-10\phi^{(3)}[r]+r\,\phi^{(4)}[r]\right)\right)+L^6r\left(46L^2\phi''[r]-5r^6\phi''[r]^3+2L^2r\left(10\phi^{(3)}[r]+r\,\phi^{(4)}[r]\right)\right)\Big)\Big)$$

This is the shape equation for a curve in the Poincaré disk

Extrema in maximally symmetric surfaces

Using the shape equations instead

$$2\tilde{\Delta}\mathrm{Tr}k + \mathrm{Tr}k^3 - \left(\frac{\hat{\lambda}_0}{\lambda_5'} - \frac{2\kappa}{L^2}\right)\mathrm{Tr}k = 0$$

In arc-length parametrization

$$2\ddot{k} + k^3 - B\,k = 0\ , \qquad B = \left(\frac{\hat{\lambda}_0}{\lambda_5'} - \frac{2\kappa}{L^2}\right)$$

Two-step splitting

- First find out the extrinsic curvature
- Second invert this curvature to find γ

Extrema in maximally symmetric surfaces

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Perhaps this can be solved (?)

Extrema in maximally symmetric surfaces

Finding the extrinsic curvature

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Simplest solutions, $k = const$

$$\text{Geodesics} \quad k = 0 \quad \text{and} \quad \text{CMC} \quad k^2 = B$$

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Even these cannot be spotted straightaway in the old equation

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Simplest solutions, $k = const$

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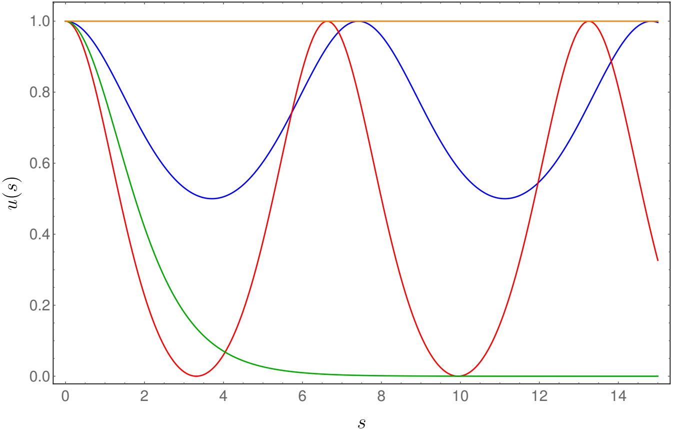
In fact, the full solution can be found !!

$$k^2(s) = \alpha\left[1 - \frac{\alpha-\gamma}{\alpha}\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\alpha-\beta}\,s,\ \frac{\alpha-\gamma}{\alpha-\beta}\right)\right]$$

With $\mathrm{sn}(z,m)$ a Jacobi elliptic function and α, γ and β constants

Langer-Singer '84



Possible behaviour for $u(s) = k^2(s)$

- Orange: Constant mean curvature
- Red: Wavelike
- Blue: Orbitlike
- Green: Asymptotically geodesic

Extrema in $\mathbb{H}^2$

Now, we must invert $k(s)$

This is rather involved, yet attainable analytically

For hyperbolic geometry $\mathbb{H}^2$

$$ds^2 = \frac{1}{z^2}\left(dx^2 + dz^2\right)$$

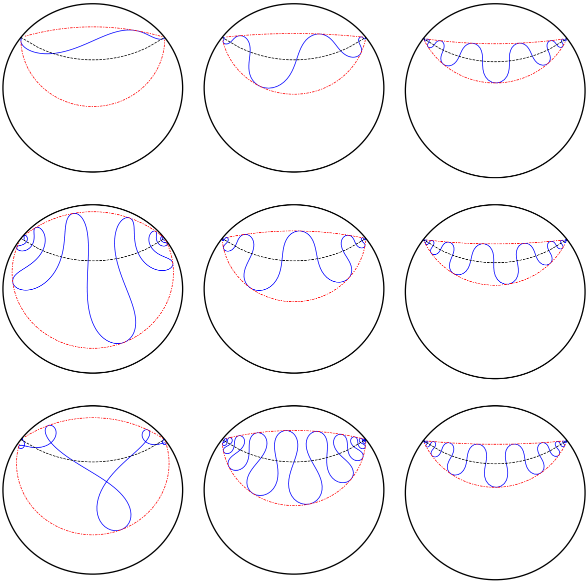
Wavy solutions in $\mathbb{H}^2$ read

$$z(s) = \frac{C}{2+\lambda}\frac{\exp\left[\sqrt{C^2-4(\lambda+1)}\left(\frac{s}{4} - \frac{2(C-2)}{4\sqrt{2C}(C+2)}\Pi\left[n,\varphi(s);m\right]\right)\right]}{\sqrt{(C+2)^2 - 4(C+2+\lambda)\mathrm{sn}^2\left(\sqrt{\frac{C}{2}}s, \frac{C+2+\lambda}{2C}\right)}}$$

Where C and λ are constants, while

$$\varphi(s) = \mathrm{amp}\left(\sqrt{\frac{C}{2}}s, \frac{C+2+\lambda}{2C}\right)$$

Extrema in $\mathbb{H}^2$



Application: holographic entanglement entropy

For field theories with an Einstein gravity dual

Area functional

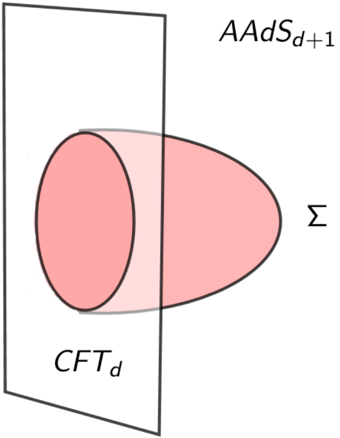
$$S_{\mathrm{eff}}[\Sigma] = \frac{1}{4G_d}\int_\Sigma d^p\sigma\,\sqrt{h}$$

Minimize

$$\mathrm{Tr}\,K^A = 0$$

Evaluate on-shell

$$S_{\mathrm{EE}}(A) = S_{\mathrm{eff}}^{\mathrm{on-shell}}[\Sigma]\,.$$



Ryu, Takayanagi '06

Application: holographic entanglement entropy

If the dual gravitational theory has h.c. corrections

$$\mathcal{L} = -2\Lambda + R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

Then the EE functional

$$S_{\rm eff} = \frac{1}{4G_d} \int_\Sigma d^p\sigma \sqrt{h} \Big[1 + 2c_1 R + c_2 \left(R_A{}^A - \frac{1}{2} \mathrm{Tr} K_A \mathrm{Tr} K^A \right) + 2c_3 \left(R_{AB}{}^{AB} - \mathrm{Tr}(K^A K_A) \right) \Big]$$

Bhattacharrya, Sharma, Sinha; Camps; Dong '13

- To find the EE one must evaluate the functional on an extremum.
- Which of the possible extrema is not settled (minimal??)
- It would be nice to be able to scan the space of extrema.

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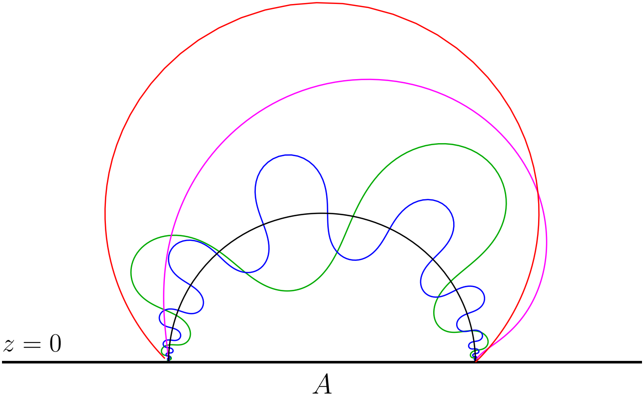
Clearly, the extrema are solutions to the shape equations

Entanglement entropy AdS_3

For a higher-derivative theory (such as NMG)



We know **all** the extrema

Entanglement entropy AdS_3/CFT_2

The EE for an interval is universal

$$S_{\rm EE}(A) = \frac{c}{3}\log\left(\frac{\ell}{\epsilon}\right) + \mathcal{O}(\epsilon)$$

Holzhey, Larsen,Wilczek '94

Using the fact that

$$c = \frac{L}{2G_3}\, g^{\mu\nu}\frac{\partial \mathcal{L}}{\partial R_{\mu\nu}}\,,$$

Saida, Soda '00

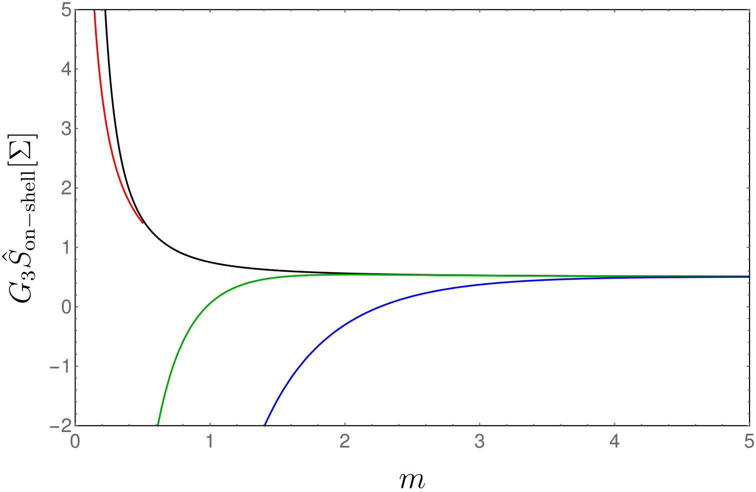
We show that

$$S_{\rm EE}(A) = S^{\rm Geo}_{\rm on-shell}[\Sigma]\,,$$

Are geodesics minimal? Let's compare

$$\hat{S}_{\rm on-shell}[\Sigma] = \ell\frac{d}{d\ell}\, S_{\rm on-shell}[\Sigma]$$

Comparing on-shell values in NMG



The geodesic's on-shell value is the black one (**not** minimal)

Interesting questions

- If not minimality then which criterium?
- What if geodesics are not extrema?
- What about perturbations? or curvature driven flows?
- Is there an information theoretic interpretaton of other extrema in terms of:
  - Length and differential entropy

    Czech, Hayden, Lashkari, Swingle '14
  - The surface/state correspondece

    Miyaji, Takayanagi '15
- How far can we get in higher-dimensional settings?

  Fonda, Véliz-Osorio ..in progress
- How far can we get in less symmetric ambients?

..sometimes you get homework

Anomalies and flows

Chiral anomalies

The c-theorem for parity violating ($c_L \neq c_R$)

$$T(z)T(0) \sim \frac{c_L/2}{z^4} + \frac{T(0)}{z^2} + \frac{\partial T(0)}{z} + \dots$$
$$\bar{T}(\bar{z})T(0) \sim \frac{c_R/2}{z^4} + \frac{T(0)}{z^2} + \frac{\partial T(0)}{z} + \dots$$

Along the flow

$$t\frac{d}{dt}\left(c_L(t) + c_R(t)\right) \leq 0 \qquad t\frac{d}{dt}\left(c_L(t) - c_R(t)\right) = 0$$

Bastianelli, Lindström '96

Holographically with Topologically Massive Gravity

$$c_L = \frac{3\ell}{2G}\left(1 + \frac{1}{\mu\ell}\right) \qquad c_R = \frac{3\ell}{2G}\left(1 - \frac{1}{\mu\ell}\right)$$

Torsion and anomalies

For a theory with chiral anomaly ($c_L \neq c_R$)

$$S_{\text{EE}}[\Sigma] = \int_\Sigma d\sigma \sqrt{h}\,(\mathfrak{m} + \mathfrak{s}\,\tau)$$

Castro, Detournay, Iqbal, Perlmutter '14

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Shape equations

$$\mathfrak{m}\,K_A + \mathfrak{s}\,\epsilon_{AB}\left(\tilde{D}_s{}^B{}_C K^C + R_s{}^B\right) = 0\,,$$

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For MSS spaces

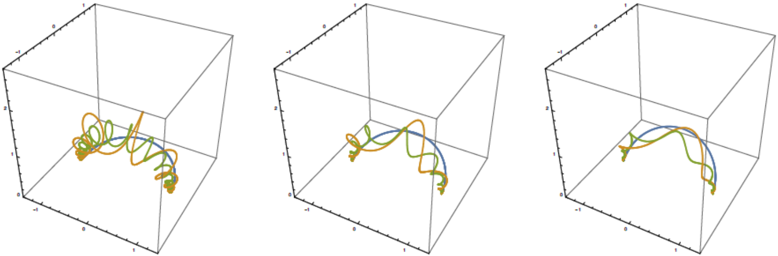
$$\mathfrak{m} k_{FS} - \mathfrak{s}\tau_{FS} k_{FS} = 0 \qquad \mathfrak{s}\dot{k}_{FS} = 0$$

Solutions are nice!!

$$k_{FS} = \text{const} > 0 \qquad \text{and} \qquad \tau_{FS} = \mathfrak{m}/\mathfrak{s}$$

Castro, Fonda, Véliz-Osorio ..in progress

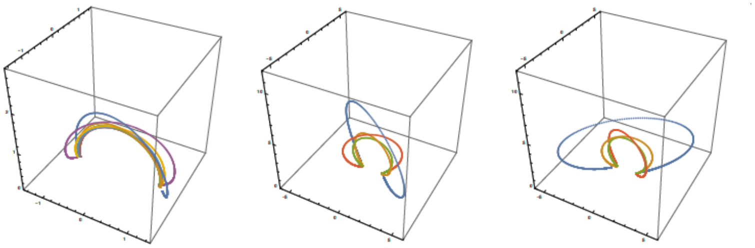
Curly extrema $\mathbb{H}_3$



Helices

$$x^{\mu}(s) = \frac{1}{\cosh\gamma\cosh\alpha s - \sinh\gamma\sin\beta s}\left(\cosh\gamma\sinh\alpha s, \sinh\gamma\cos\beta s, 1\right)$$

Curly extrema AdS$_3$



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The on-shell value

$$S[\Sigma]^{\mathrm{Hel}}_{o.s.} = 2\mathfrak{m}\,\ell_{\mathrm{Hel}}(\Sigma) + \mathfrak{s}(\eta_f - \eta_i) \qquad \ell_{\mathrm{Hel}} \approx |\alpha|^{-1}\log(\ell/\varepsilon)$$

Flows and curly extrema

TMG flows driven by a scalar $\varphi(z)$

$$ds^2 = \frac{dz^2}{z^2 f(z)} + \frac{\eta_{ab} dx^a dx^b}{z^2} .$$

Flowing shape equations

$$\mathfrak{m} K_A + \mathfrak{s}\, \epsilon_{AB} \tilde{D}_s{}^B{}_C K^C = -\mathfrak{s} \epsilon_{AB}\, \partial^B \varphi \partial_s \varphi$$

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Flowing curls are yet to be found

$$\tau_{\mathrm{FS}} = \frac{\mathfrak{m}}{\mathfrak{s}} - \eta^z\, \partial_s \log\left(k_{\mathrm{FS}}\right)$$
$$\dot{k}_{\mathrm{FS}} = \mu^z\, \varphi'(z)^2$$

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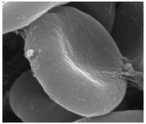
You'll be hearing from us ... stay tunned

سپاس

Epilogue

Elastica

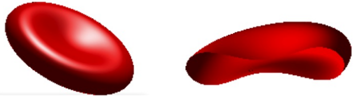
Two interesting set-ups



Canham-Helfrich

$p = 2,\ M = \mathbb{R}^3$

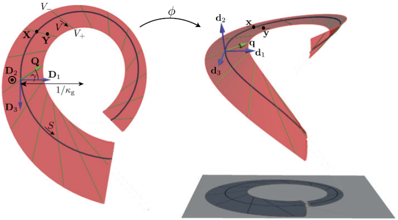
$$\mathcal{L} = \left[\sigma + \frac{k_c}{4}(\mathrm{Tr}K)^2 + \bar{k}_c \det K\right]$$



Sadowsky-Wünderlich

$p = 2,\ M = \mathbb{R}^3$

$$\mathcal{L} = \frac{(k_{\mathrm{FS}}^2 + \tau_{\mathrm{FS}}^2)^2}{k_{\mathrm{FS}}^2}\,.$$



(a) (b)

Elastica

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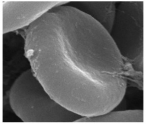

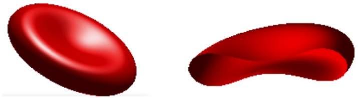

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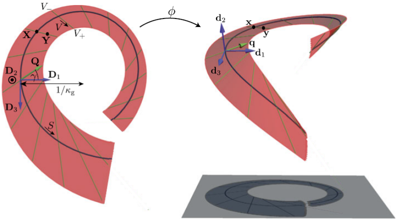
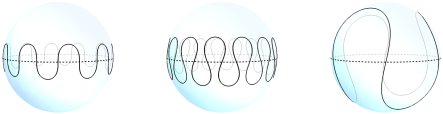
Judicious breaking of gauge invariance



(a) (b)

Interesting problems

- Elastica in (evolving) surfaces



- Embeddings of embeddings, interface theory on surfaces
- Minimal surfaces bounded by elastic lines

Giomi, Mahadevan '11

- Conformal maps: Minimal in $\mathrm{AdS}_d \longleftrightarrow$ Willmore in $\mathbb{R}^3$

Alexakis, Mazzeo '10; Fonda, Seminara, Tonni '15

- Generalized curvature flows

Fonda, Véliz-Osorio...in progress