Berry Phases of Boundary Gravitons

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(ETH Zurich)

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Based on arXiv 1703.06142 + work in progress

Motivation

Parameter-dependent system

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- Vary parameters

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- Berry phases in reps of G ? [Jordan 1988, Boya *et al.* 2001]

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- Example : Virasoro, BMS

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Application to **aspt symmetries** ?

- Example : Virasoro, BMS
- Generalize **Thomas precession**

Plan of the talk

1. **Berry phases in group reps**

2. Virasoro group

3. Virasoro Berry phases

PLAN OF THE TALK

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A. Berry phases

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A. Berry phases

B. Application to unitary reps

Berry phases

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- Coordinates on manifold $\mathcal{M}$

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BERRY PHASES

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- What is $\theta(t)$?

Berry phases

$$\theta(T) = -\int_0^T dt\, E_n(\gamma(t)) + \cdots$$

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Berry phases

$$\theta(T) = \underbrace{-\int_0^T dt\, E_n(\gamma(t))}_{\text{Dynamical phase}} + i\int_0^T dt\, \langle \phi_n(\gamma(t)) | \frac{\partial}{\partial t} | \phi_n(\gamma(t)) \rangle$$

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Closed paths ?

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$$B_n[\gamma] = i\oint_0^T dt\, \langle\phi_n(\gamma(t))|\frac{\partial}{\partial t}|\phi_n(\gamma(t))\rangle$$

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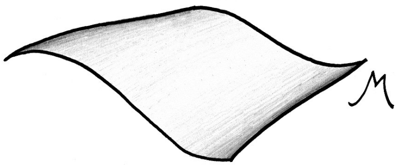
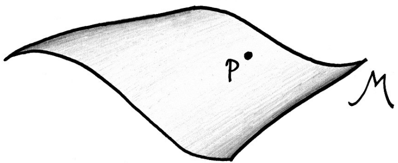
$$B_n[\gamma] = i \oint_0^T dt \left\langle \phi_n(\gamma(t)) \middle| \frac{\partial}{\partial t} \middle| \phi_n(\gamma(t)) \right\rangle = i \oint_\gamma \langle \phi_n(\cdot) | d | \phi_n(\cdot) \rangle$$

Berry phases

Berry phases



Berry phases



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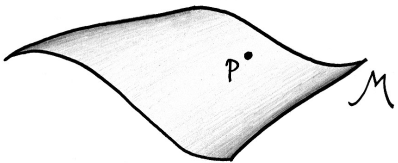
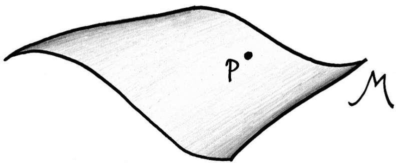


$H(p)$ $E_n(p)$

Berry phases



$$H(p)|\phi\rangle = E_n(p)|\phi\rangle$$

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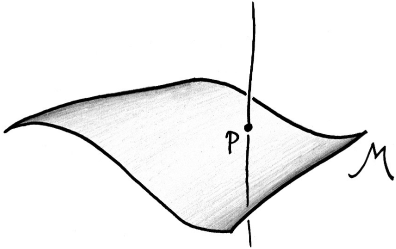

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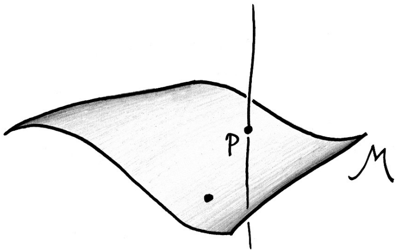

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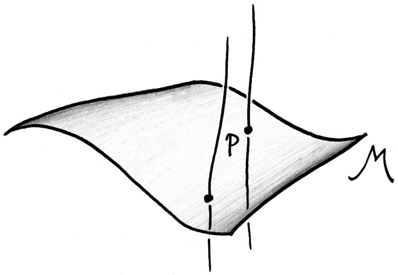

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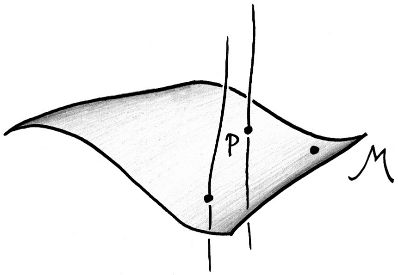

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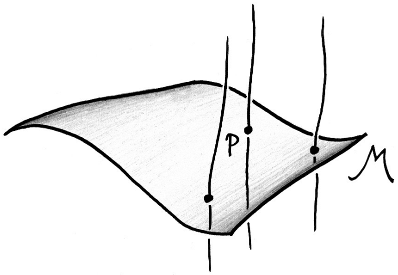

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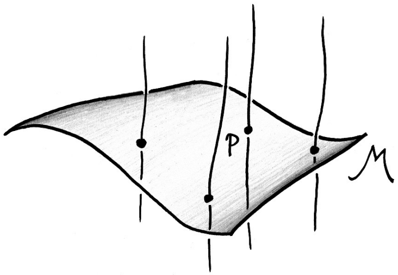

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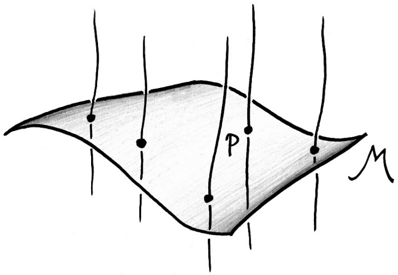

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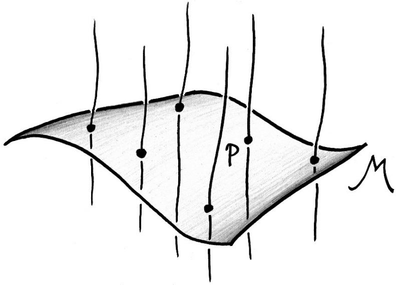

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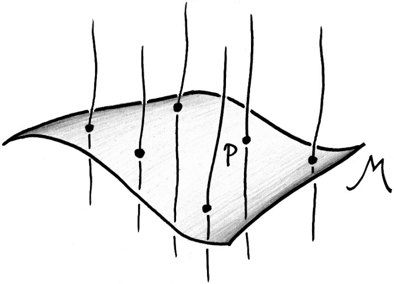

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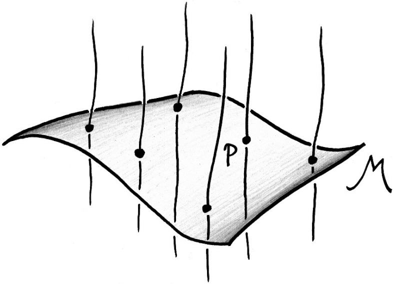

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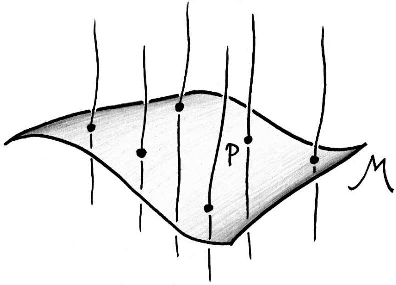

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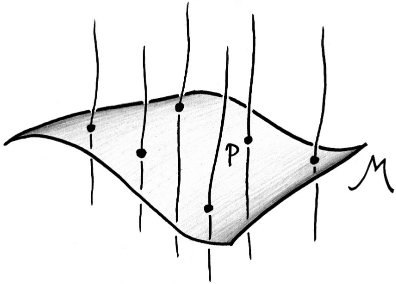

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- States $|\phi_n(\cdot)\rangle$ are a section

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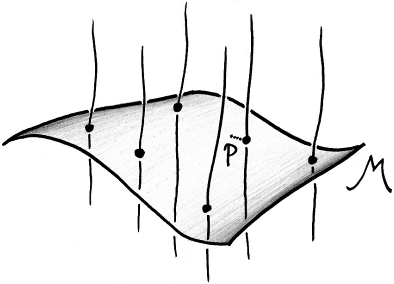

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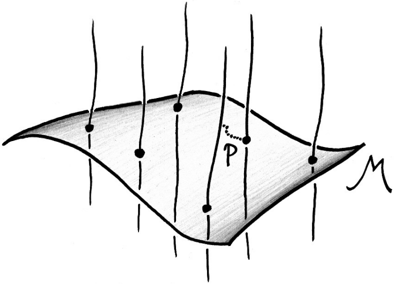

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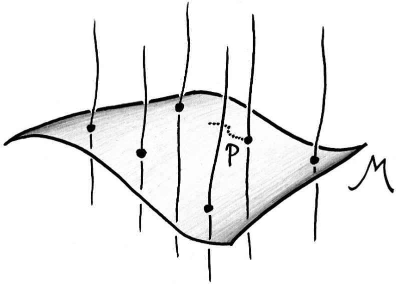

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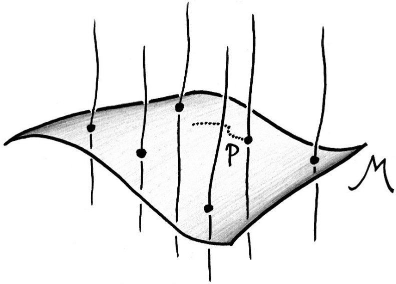

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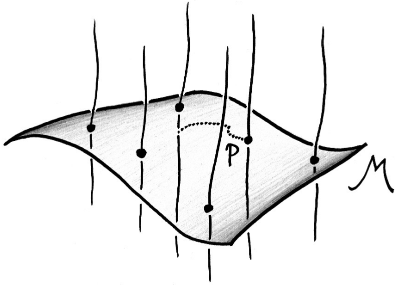

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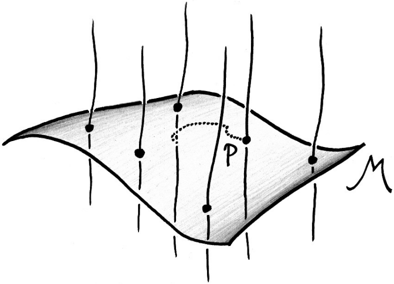

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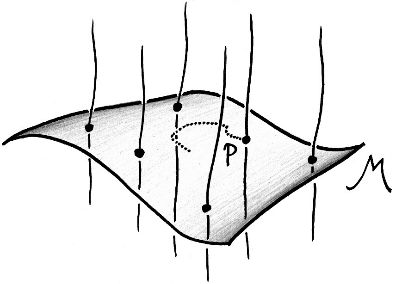

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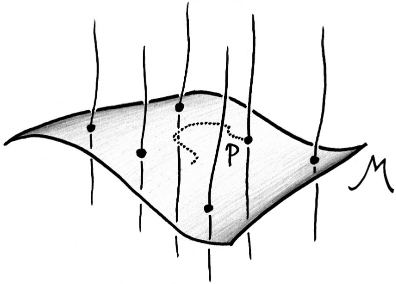

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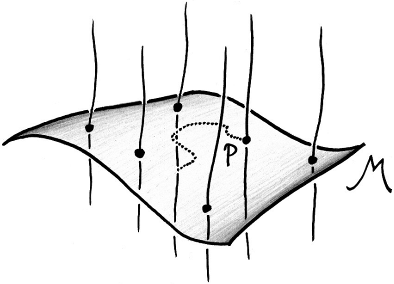

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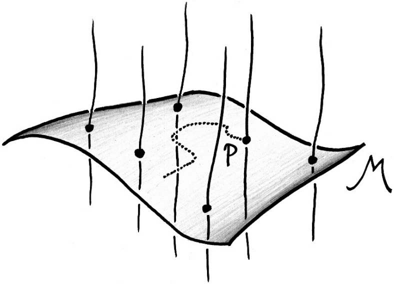

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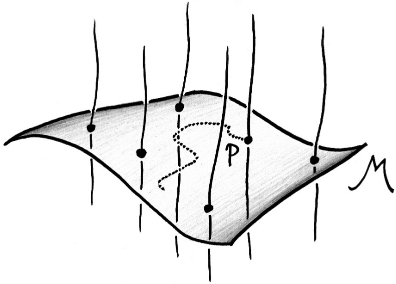

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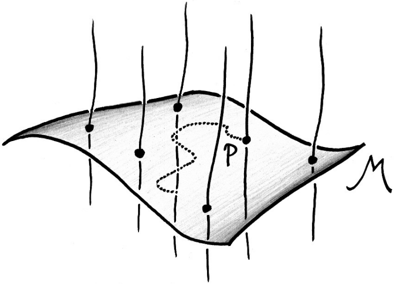

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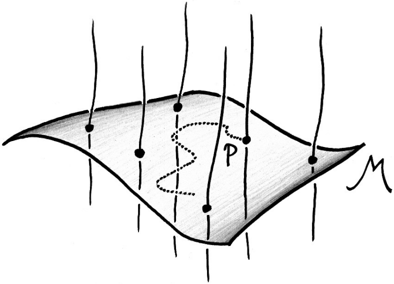

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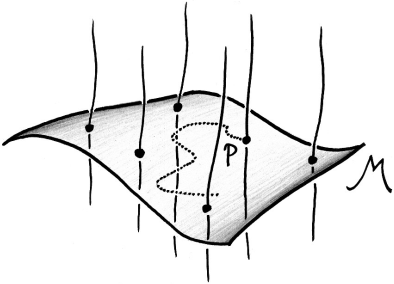

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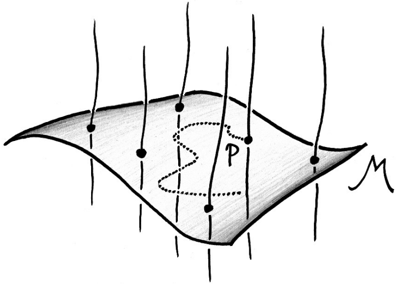

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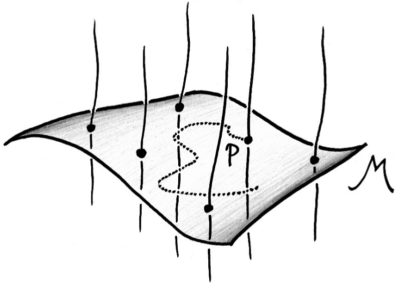

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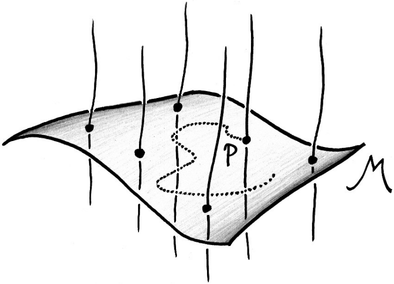

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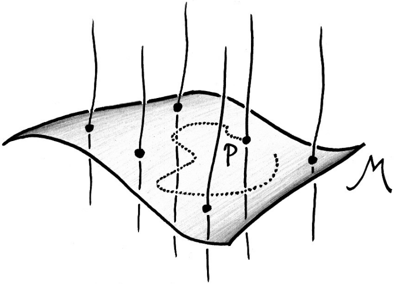

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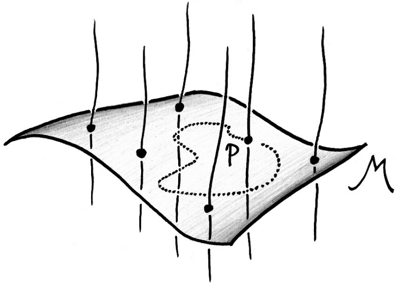

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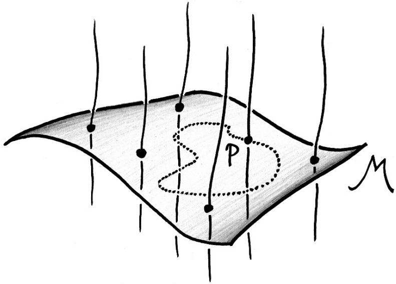

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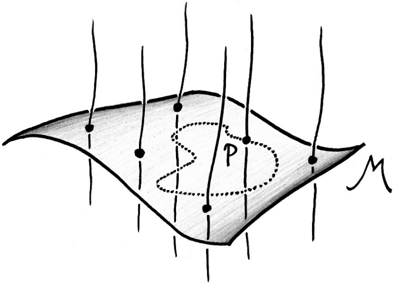

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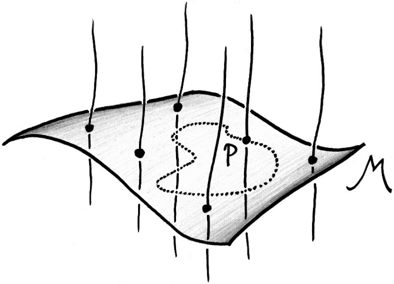

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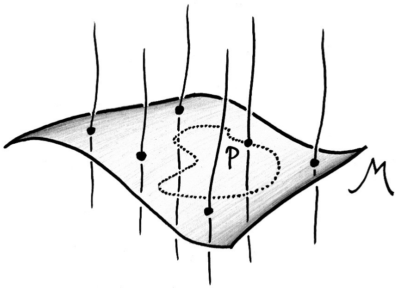

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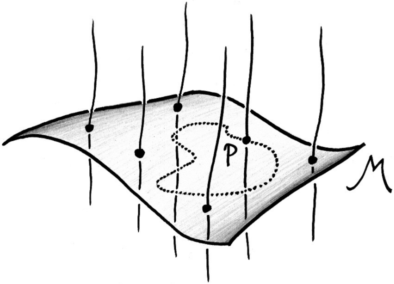

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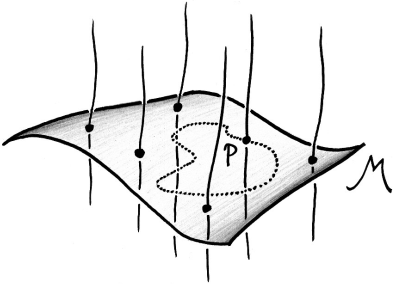

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Berry phases & groups

Group G

Berry phases & groups

Group G, algebra $\mathfrak{g}$

BERRY PHASES & GROUPS

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Berry phases & groups

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Berry phases & groups

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Berry phases & groups

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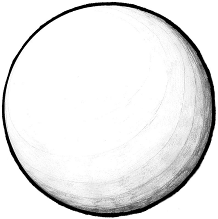

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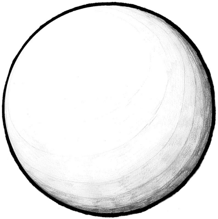

Let $f(t) \in \mathrm{SU}(2)$ with closed projection on S^2

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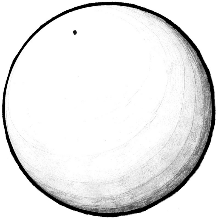

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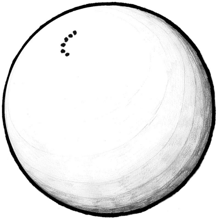

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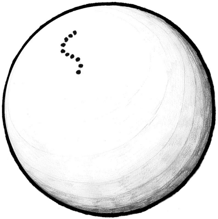

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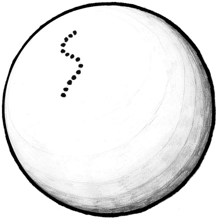

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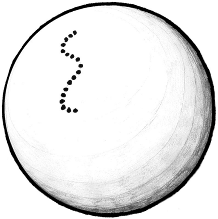

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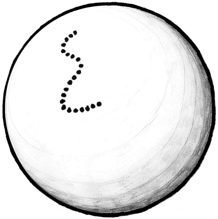

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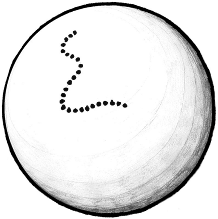

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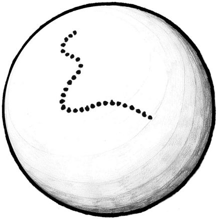

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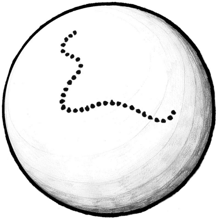

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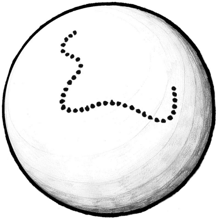

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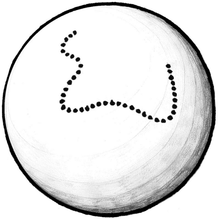

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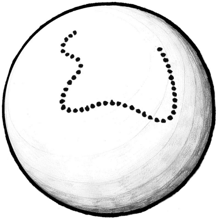

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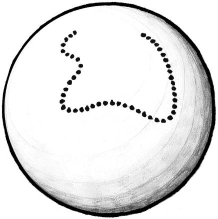

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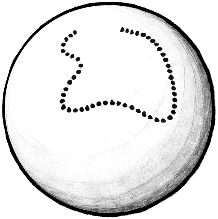

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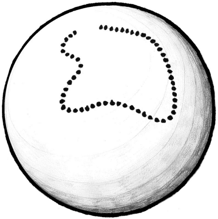

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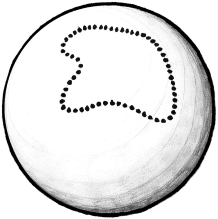

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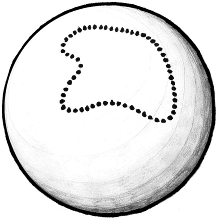

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Berry phases & groups

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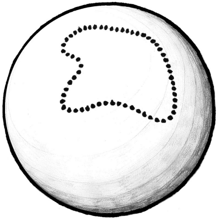

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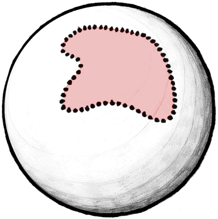

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A. Maurer-Cartan form of Diff S^1

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DIFF S^1

$\mathrm{Diff}\, S^1 =$ group of **circle diffeos**

Diff S^1

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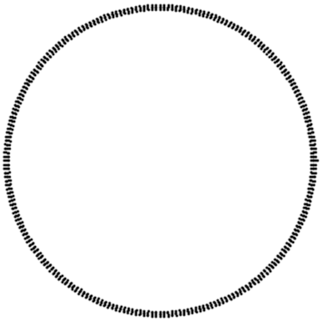

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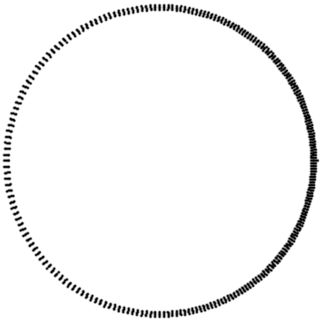

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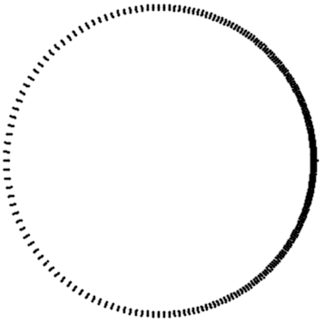

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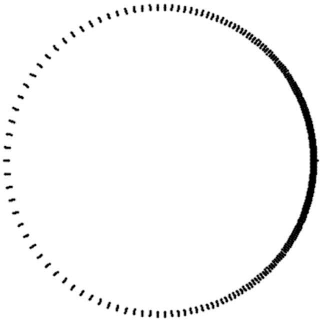

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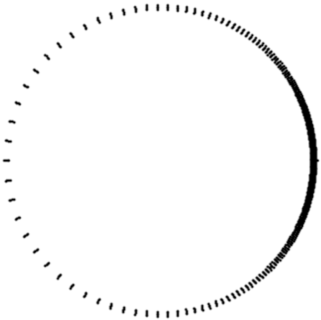

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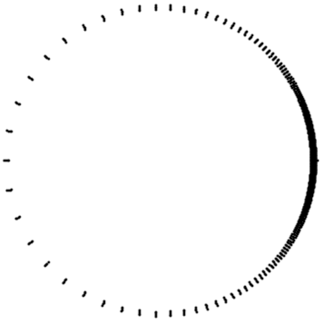

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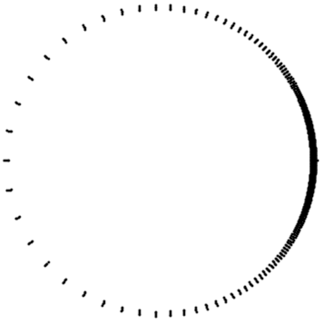

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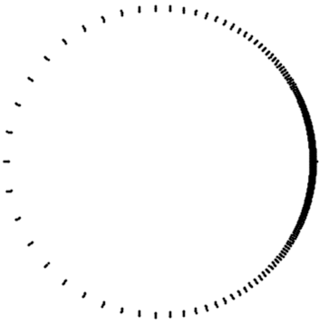

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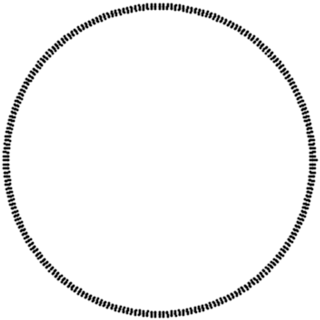

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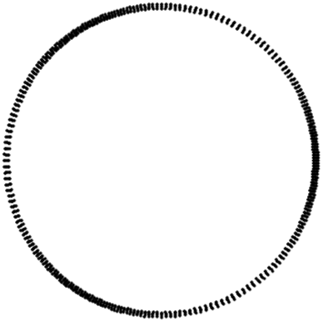

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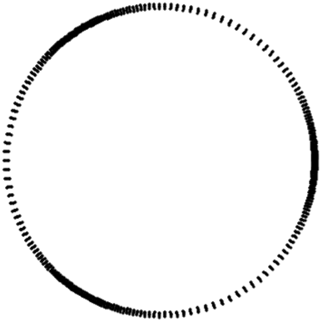

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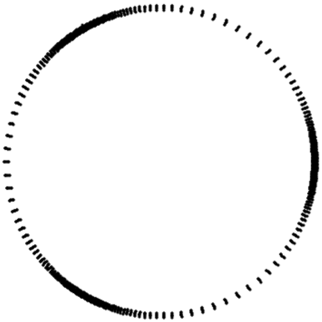

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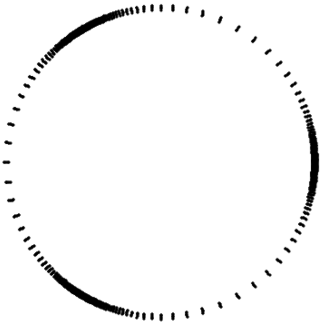

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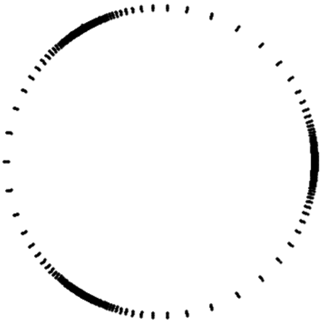

DIFF S^1

Maurer-Cartan form ?

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DIFF S^1

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DIFF S^1

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- Maurer-Cartan maps $\dot{f}$ on $X(\varphi)\partial_\varphi$

DIFF S^1

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[Alekseev, Shatashvili 1989]

3. Virasoro Berry phases

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A. General derivation

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C. Berry phases in AdS_3

General case

$$\text{Berry}: \quad B_\phi[f(t)] = i\int_f \langle\phi|\mathfrak{u}[\Theta_f]|\phi\rangle - i\log\langle\phi|\mathcal{U}[f(0)^{-1}f(T)]|\phi\rangle$$

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Apply this to Virasoro !

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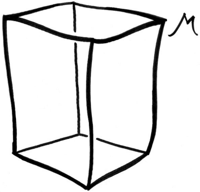

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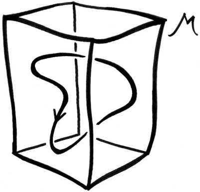

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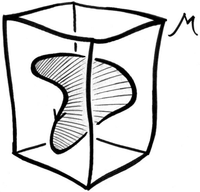

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Berry phases in AdS_3

AdS_3 space-time

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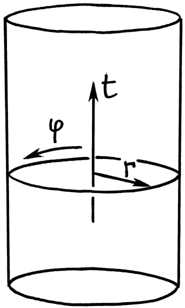
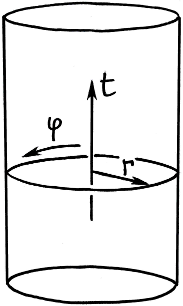


BERRY PHASES IN AdS_3

AdS_3 space-time

- Infinity = time-like cylinder



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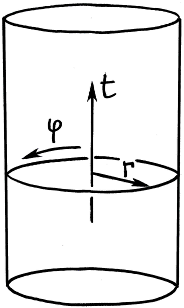

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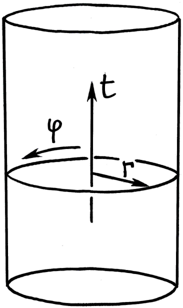

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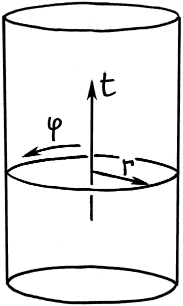

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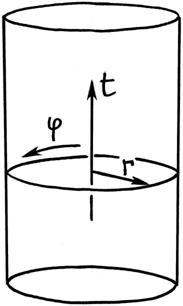

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On-shell AdS$_3$ metrics :

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- Interpretation of Berry phase ?

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Circular path with $g(\varphi) = \bar{g}(\varphi) =$ superboost

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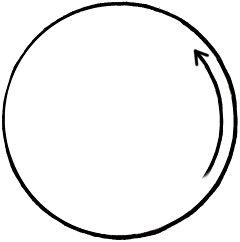

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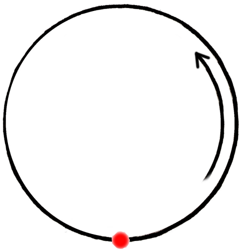

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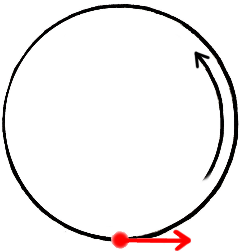

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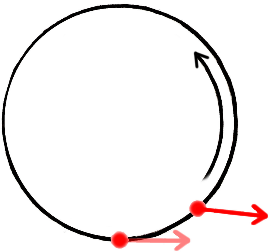

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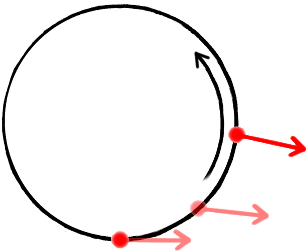

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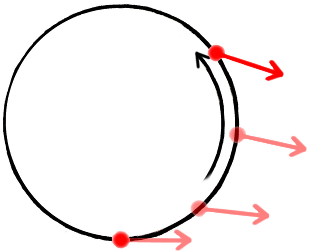

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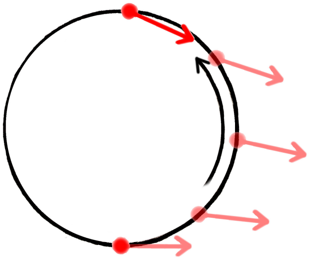

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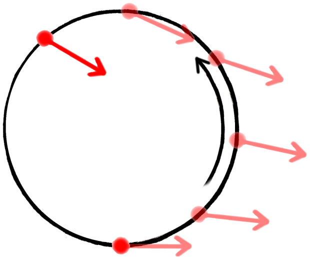

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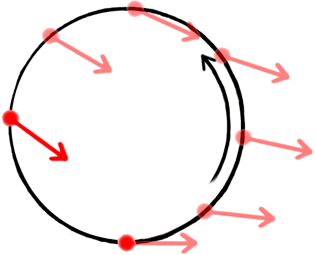

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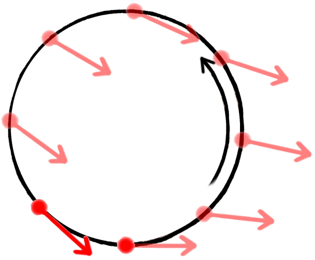

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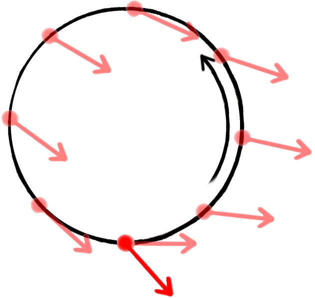

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  [with T. Neupert & F. Schindler]

Thank you for listening !