Phase Transitions and Entanglement Entropy in Warped AdS_3/Warped CFT_2

Mahdis Ghodrati

Recent Trends in String Theory and Related Topics

Institute for Research in Fundamental Sciences (IPM)

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Based on

1. Phase transitions in BHT massive gravity, M. Ghodrati, A. Naseh, **Class.Quant.Grav. 34 (2017) no.7, 07500**
2. Revisiting conserved charges in higher curvature gravitational Theories, M. Ghodrati, K. Hajian, M.R. Setare, **Eur.Phys.J. C76 (2016) no.12, 701**
3. Phase transitions in warped AdS3 gravity S. Detournay, C. Zwikel, **JHEP 1505 (2015) 074**
4. Warped Conformal Field Theory S. Detournay, T. Hartman, D. Hofman, **Phys.Rev. D86 (2012) 124018**
5. Entanglement Entropy in Warped Conformal Field Theories. A. Castro, D. Hoffman, N. Iqbal, **JHEP 1602 (2016) 033**

Overview

- Review of 2d CFT
- Beyond AdS_3 /CFT_2
- Warped CFTs, Holographic WCFTs
- Phase Transitions in TMG theory
- Phase Transitions in NMG theory
- Entanglement entropy of WCFTs

2d CFT

- In 2d, scale invariance $\rightarrow$ conformal invariance.
- Any unitary theory with a discrete spectrum and invariant under translations, Lorentz transformations and scaling has an enlarged SL(2,R) × SL(2,R) global symmetry group and local symmetries given by two copies of centrally extended Virasoro algebra.
- Adding modular invariance gives Cardy's formula.

Modular Invariance

- The modular S-transformation implies $$\boxed{Z(\tau,\bar{\tau}) = Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}).}$$
- At $\Theta=0$, this relates the behavior of partition function at high and low T. In this regimes, the free energy for *generic* CFTs is universal and depends only on c. For holographic CFTs, this universal behavior extends to self dual temperature, $\beta_{sd} = 2\pi$.
- For large L_0 & $\bar{L}_0$ charges and fixed central charges, entropy is

$$\boxed{S_{\mathrm{CFT}} = 2\pi\sqrt{\frac{c_R}{6}L_0} + 2\pi\sqrt{\frac{c_L}{6}\bar{L}_0}\ .}$$

- This matches with Bekenstein-Hawking entropy of AdS_3 black hole.

Beyond AdS

- Flat spacetimes
- de Sitter spacetimes
- Kerr/CFT
- Lifshitz and Hyperscaling violating geometries (AdS/CMT)

Little is known about their dual field theories!

Warped Conformal Field Theory

- A 2d QFT with a chiral scaling symmetry that acts only on right movers $x^- \to \lambda x^-$, in contrast to CFTs which have a second independent scaling symmetry $x^+ \to \bar{\lambda} x^+$.
- They don't satisfy the Brown-Henneaux's Boundary condition.
- A 2d translational-invariant theory with a chiral scaling symmetry must have an extended local algebra.

CFT: The usual CFT with two copies of Virasoro algebra SL(2,R) × SL(2,R)

WCFT: One Virasoro algebra plus a U(1) Kac-Moody algebra → WCFT with SL(2,R) × U(1)

Local symmetries

- The local symmetries include two arbitrary functions worth of freedom in coordinate transformation

$$x^- \to f(x^-)\ , \qquad x^+ \to x^+ - g(x^-)$$

- These symmetries lead to a new type of modular transformation on the torus.
- Applied to finite T partition function, modular transformation relates thermodynamical quantities at low rotation to those at high rotations.

Asymptotic entropy

$$\boxed{S_{\mathrm{WCFT}} = -\frac{4\pi i P_0 P_0^{vac}}{k} + 4\pi\sqrt{-\left(L_0^{vac} - \frac{(P_0^{vac})^2}{k}\right)\left(L_0 - \frac{P_0^2}{k}\right)}}$$

- L_0 is the charge associated to SL(2,R).
- P_0 is the U(1) charge.
- c and k are the central extensions of Virasoro + Kac-Moody algebra.

Modular transformation → $$\boxed{Z(\beta,\theta) = e^{ik\frac{\beta^2}{4\theta}} Z\left(\frac{2\pi\beta}{\theta}, -\frac{4\pi^2}{\theta}\right)}$$

Holographic WCFTs

- The near horizon geometry of every extremal black hole in any dimension has the global SL(2,R)× U(1) symmetry.
- Examples: The near horizon geometry of the extremal 4d Kerr black hole at the fixed polar angle.
- The deformation of AdS_3 changes the asymptotic but preserves SL(2,R) × U(1) symmetry.
- Non-Einstein theories such as TMG and NMG.
- Asymptotic symmetry group (ASG) in TMG spacetime→ one Virasoro algebra, one U(1)current algebra extending the exact isometries → WCFT

Algebra

- The symmetry structure in a 2d Lorentzian theory with $SL(2,R)_R \times U(1)_L$ global invariance → right moving energy momentum tensor and a right moving U(1) Kac-Moody current.
- The commutators on the plane are:

$$
\begin{aligned}
i[T_\xi, T_\zeta] &= T_{\xi'\zeta-\zeta'\xi} + \frac{c}{48\pi}\int dx^- \left(\xi''\zeta' - \zeta''\xi'\right) \\
i[P_\chi, P_\psi] &= \frac{k}{8\pi}\int dx^- (\chi'\psi - \psi'\chi) \\
i[T_\xi, P_\chi] &= P_{-\chi'\xi}
\end{aligned}
$$

$$
T_\xi = -\frac{1}{2\pi}\int dx^- \, \xi(x^-) T(x^-) \quad P_\chi = -\frac{1}{2\pi}\int dx^- \, \chi(x^-) P(x^-)
$$

Algebra

- By a change of coordinate $x^- = e^{i\phi}$ and picking test function $\xi_n = (x^-)^n = e^{in\phi}$ the commutators on the cylinder are

$$\boxed{\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n-1)(n+1)\delta_{n+m} \\
[P_n, P_m] &= \frac{k}{2}n\delta_{n+m} \\
[L_n, P_m] &= -mP_{m+n}
\end{aligned}}$$

- Where $L_n = iT_{\xi_{n+1}}$ $\qquad P_n = P_{\chi n}$

The infinitesimal transformation

- The commutation relations imply the following infinitesimal transformations of energy momentum tensor and current

$$\boxed{\begin{aligned} \delta_\epsilon T(x^-) &= -\epsilon(x^-)\partial_- T(x^-) - 2\partial_-\epsilon(x^-)T(x^-) - \frac{c}{12}\partial_-^3\epsilon \\ \delta_\gamma T(x^-) &= -\partial_-\gamma(x^-)P(x^-) \\ \delta_\epsilon P(x^-) &= -\epsilon(x^-)\partial_- P(x^-) - \partial_-\epsilon(x^-)P(x^-) \\ \delta_\gamma P(x^-) &= \frac{k}{2}\partial_-\gamma(x^-) \end{aligned}}$$

- Where one defines $\boxed{\delta_{\epsilon+\gamma} = \delta_\epsilon + \delta_\gamma = -i[T_\epsilon, \cdot] - i[P_\gamma, \cdot]}$

Phase Transitions

- Hawking and Page → Thermal radiation in AdS beyond certain temperature leads to formation of a black hole. In the AdS/CFT picture it is dual to confining/deconfining phase transition at high T.

- The partition function is

$$\boxed{Z(\tau,\bar{\tau}) = \mathrm{Tr} e^{2\pi i\tau L_0} e^{2\pi i\bar{\tau}\bar{L}_0},}$$

$$z = \tfrac{1}{2\pi}(\phi + it)$$

$$Z(\tau) \sim \int \mathcal{D}g e^{-kS[g]}$$

$$\tau = \tfrac{1}{2\pi}(\Theta + i\beta)$$

$$Z(\tau,\bar{\tau}) \sim \sum_{g_c} e^{-kS[g_c]}, \quad k = \frac{c}{24} = \frac{\ell}{16G} \gg 0, \qquad Z(\tau,\bar{\tau}) = e^{-\beta G(\tau,\bar{\tau})}.$$

Warped AdS_3 black holes

- Squashed three sphere obtained by deforming AdS_3

$$\boxed{g_{W\,AdS} = g_{AdS_3} - 2H^2 \xi \otimes \xi}$$

- H^2 is a deformation parameter and ξ^{μ} is a constant norm Killing vector of SL(2,R) isometry group of AdS_3.
- The resulting geometry possesses an SL(2,R)× U(1) isometry group and for $\xi^2 = -1, 1, 0$, it is timelike, spacelike or Null Warped AdS_3.

Warped Solutions

- Spacelike $WAdS_3$ black hole ➔ $\xi^2 = 1$

$$\frac{ds^2}{\hat{l}^2} = dT^2 + \frac{d\hat{r}^2}{(\nu^2+3)(\hat{r}-\hat{r}_+)(\hat{r}-\hat{r}_-)} - \left(2\nu\hat{r} - \sqrt{\hat{r}_+\hat{r}_-(\nu^2+3)}\right) dT d\theta$$
$$+ \frac{\hat{r}}{4}\left(3(\nu^2-1)\hat{r} + (\nu^2+3)(\hat{r}_+ + \hat{r}_-) - 4\nu\sqrt{\hat{r}_+\hat{r}_-(\nu^2+3)}\right)$$

- Timelike $WAdS_3$ solution ➔ $\xi^2 = -1$

$$ds^2 = -dt^2 - 4\omega r dt d\phi + \frac{\ell^2 dr^2}{(2r^2(\omega^2\ell^2+1)+2\ell^2 r)} - \left(\frac{2r^2}{\ell^2}(\omega^2\ell^2-1) - 2r\right) d\phi^2.$$

Boundary conditions

- The boundary conditions are preserved by following set of diffeomorphisms:

$$\begin{aligned} \ell_n &= e^{in\theta}\partial_\theta - in\hat{r}e^{in\theta}\partial_{\hat{r}} \\ p_n &= e^{in\theta}\partial_T \ . \end{aligned}$$

- This generates a centerless Virasoro-Kac-Moody U(1) algebra

$$i[\ell_n\,\ell_m] = (n-m)\ell_{n+m}\ , \qquad i[\ell_n, p_m] = -m\ p_{n+m}\ , \qquad i[p_n, p_m] = 0\ .$$

- with the corresponding canonical charges

$$L_n := Q_{\ell_n}\ , \qquad P_n := Q_{p_n}$$

Topological Massive Gravity Theory

- Einstein-Hilbert action plus a gravitational Chern-Simons term

$$\boxed{S = \frac{1}{16\pi}\int d^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{16\pi}\frac{1}{2\mu}\int d^3x\sqrt{-g}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right)}$$

- μ is a Chern-Simons coupling and is positive
- The equation of motion is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0$$

- where

$$C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_\alpha\left(R_{\beta\nu} - \tfrac{1}{4}g_{\beta\nu}R\right)$$

Warped AdS$_3$ black holes in grand canonical ensemble

$$ds^2_{W\,BTZ} = ds^2_{BTZ} - 2H^2\xi\otimes\xi$$

$$ds^2_{BTZ} = \left(8M - \frac{r^2}{l^2}\right)dt^2 - \frac{r^2dr^2}{8Mr^2 - \frac{r^4}{l^2} - 16J^2} + 8J\,dt\,d\phi + r^2d\phi^2$$

$$\xi^\mu = \frac{1}{\sqrt{8}}\sqrt{\frac{l}{(M\,l - J)}}\,(-\partial_t + \partial_\phi)$$

$$\boxed{g_{\mu\nu} = \begin{pmatrix} -r^2 - \frac{H^2(-r^2-4J+8M)^2}{4(M-J)} + 8M & 0 & 4J - \frac{H^2(4J-r^2)(-r^2-4J+8M)}{4(M-J)} \\ 0 & \frac{1}{\frac{16J^2}{r^2}+r^2-8M} & 0 \\ 4J - \frac{H^2(4J-r^2)(-r^2-4J+8M)}{4(M-J)} & 0 & r^2 - \frac{H^2(4J-r^2)^2}{4(M-J)} \end{pmatrix}.}$$

Change of boundary conditions

- The change of coordinate is charge-dependent, so

$$\begin{aligned} g_{rr} &= \frac{l^2}{r^2} + O(r^{-4}), \ \ g_{++} = j_{++}r^4 + h_{++}r^2 + f_{++}, \ \ \ g_{+-} = (\frac{1}{2} + j_{+-})r^2 + O(1) \\ g_{--} &= f_{--}, \ \ \ g_{+r} = O(1/r), \ \ \ g_{-r} = O(1/r^3). \end{aligned}$$

$$\tilde{P}_0 := Q_{\partial_{x^+}} = \frac{P_0^2}{k}, \qquad \tilde{L}_0 := Q_{\partial_{x^-}} = L_0 - \frac{P_0^2}{k}.$$

$$\boxed{L_0^{\tilde{v}ac} = -\frac{c_R}{24}, \quad P_0^{\tilde{v}ac} = -\frac{c_L}{24},}$$

Vacuum Solutions

- Godel Spacetime

$$ds^2 = -dt^2 - 4\omega r dt d\phi + \frac{\ell^2 dr^2}{(2r^2(\omega^2\ell^2+1)+2\ell^2 r)} - \Big(\frac{2r^2}{\ell^2}(\omega^2\ell^2-1)-2r\Big)d\phi^2.$$

$$m^2 = -\frac{(19\omega^2\ell^2-2)}{2\ell^2}, \qquad \Lambda = -\frac{(11\omega^4\ell^4+28\omega^2\ell^2-4)}{2\ell^2(19\omega^2\ell^2-2)}.$$

Warped AdS_3 black holes in quadratic ensemble

$$\frac{ds^2}{l^2} = dt^2 + \frac{dr^2}{(\nu^2+3)(r-r_+)(r-r_-)} + \left(2\nu r - \sqrt{r_+ r_- (\nu^2+3)}\right) dt d\varphi$$
$$+ \frac{r}{4}\left[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_- (\nu^2+3)}\right] d\varphi^2.$$

$$m^2 = -\frac{(20\nu^2-3)}{2l^2}, \qquad \Lambda = -\frac{m^2(4\nu^4 - 48\nu^2 + 9)}{(20\nu^2-3)^2}.$$

- $\nu^2 > 1$ → spacelike streched
- $\nu^2 < 1$ → spacelike squashed
- $\nu^2 = 1$ → locally AdS_3 space (BTZ BH)

Calculating conserved charges

1. **ADT Formalism**
2. **The SL(2,R) reduction method**
3. **The Solution Phase Space Method**
   arXiv:1601.04403, arXiv:1512.05584

Gibbs Free Energy

- To study the stability in the grand canonical ensemble, we find the Gibbs free energy for black hole and vacuum as

$$\boxed{G(T,\Omega) = TS[g_c]} \qquad \boxed{G = M - TS - \Omega J.}$$

$$S_E[AdS(\tau)] = \frac{i\pi}{12\,l}(c\tau - \tilde{c}\bar{\tau}) \qquad c, \tilde{c} = \frac{3\,l}{2}\left(1 \pm \frac{1}{\mu\,l}\right) \qquad G_{AdS} = -\frac{1}{8\,l}\left(1 - \frac{\Omega_E}{\mu}\right).$$

$$ds^2_{BTZ}\left[-\frac{1}{\tau}\right] = ds^2_{AdS}[\tau] \qquad S_E[BTZ(\tau)] = -\frac{i\pi}{12\,l}\left(\frac{c}{\tau} - \frac{\tilde{c}}{\bar{\tau}}\right)$$

$$\boxed{\begin{aligned} G^{BTZ}(T,\Omega) &= -\frac{\pi^2 T^2}{2(1-\Omega^2)}\left(1 + \frac{\Omega}{\mu}\right) \\ G^{AdS}(T,\Omega) &= -\frac{1}{8}\left(1 - \frac{\Omega}{\mu}\right) \end{aligned}}$$

Hessian and free energy

- The hessian is defied as

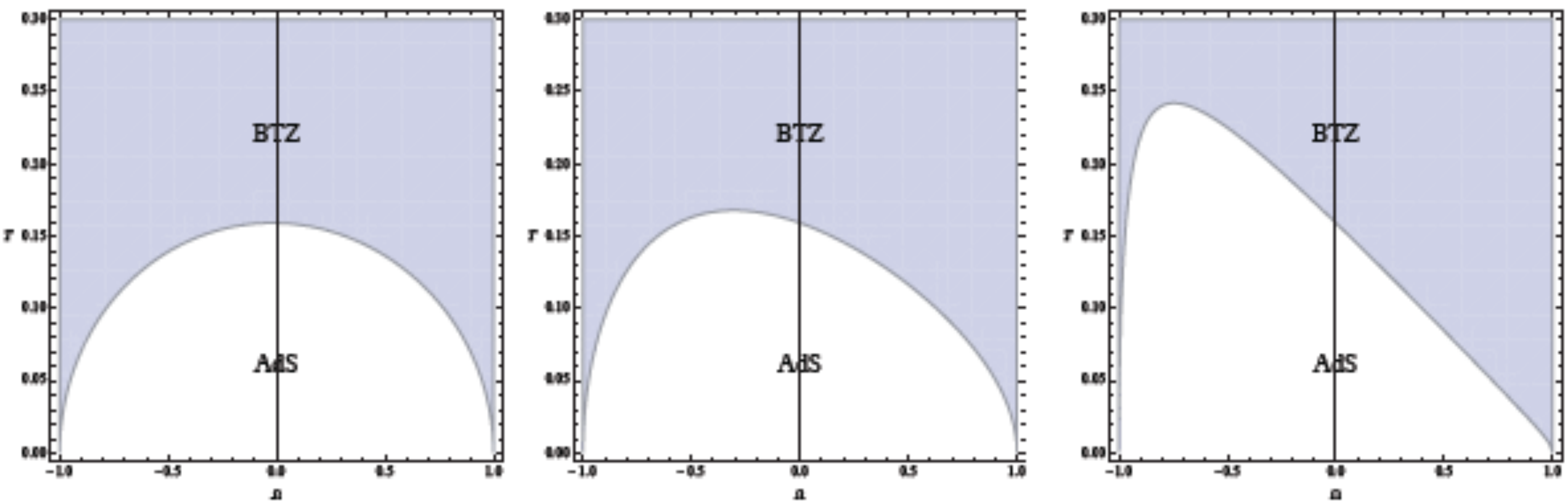
$$H = \begin{pmatrix} \frac{\partial^2 G}{\partial T^2} & \frac{\partial^2 G}{\partial T \partial \Omega} \\ \frac{\partial^2 G}{\partial \Omega \partial T} & \frac{\partial^2 G}{\partial \Omega^2} \end{pmatrix}$$

- Also the Gibbs free energy of warped solutions are

$$\boxed{\begin{aligned} G_{WAdS}(T,\Omega) &= -\frac{3-4H^2-\Omega}{24\sqrt{1-2H^2}} \\ G_{WBTZ}(T,\Omega) &= -\frac{\pi^2 T^2 \left(3-4H^2+\Omega\right)}{6\sqrt{1-2H^2}\left(1-\Omega^2\right)}. \end{aligned}}$$

Phase transitions of AdS_3 black hole

$$\Delta G = G_{AdS} - G_{BTZ}$$

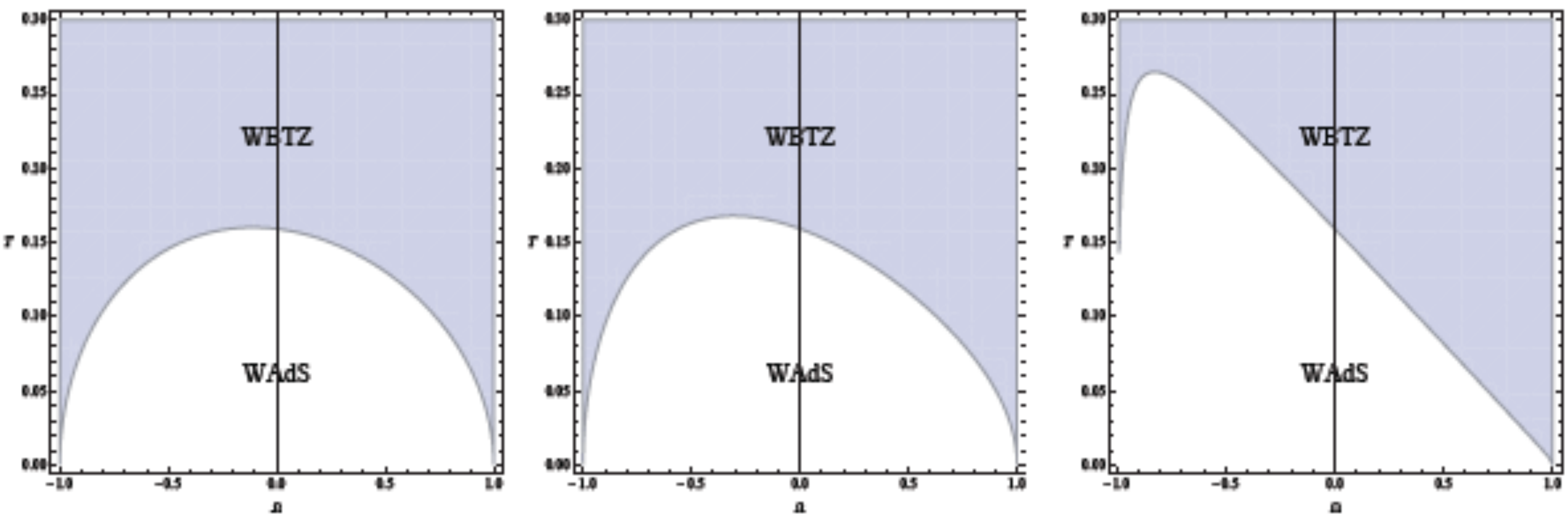


(a) GR limit, e.g. $\mu = 50$

(b) Stretched-squashed limit $\mu = 3$

(c) Local stability limit $\mu \lesssim 1$

Phase transitions of Warped AdS_3



(a) Flat space limit $H^2 = -\frac{3}{2}$ or $\mu = 36$ and $\Lambda \lesssim 0$

(b) Stretched-squashed limit $H^2 = 0$ or $\mu = 3$ and $\Lambda = -1$

(c) Local stability limit $H^2 \lesssim \frac{1}{2}$ or $\mu \gtrsim 0$ and $\Lambda \lesssim -2/3$

Bergshoeff-Hohm-Townsend theory

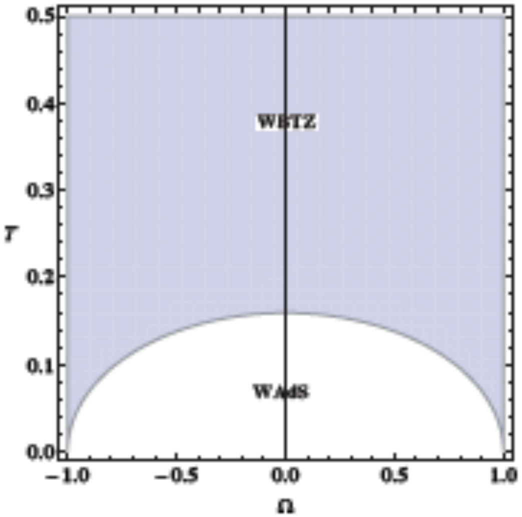
$$\boxed{S=\frac{1}{16\pi G_N}\int d^3x\sqrt{-g}\Big[R-2\Lambda+\frac{1}{m^2}\Big(R^{\mu\nu}R_{\mu\nu}-\frac{3}{8}R^2\Big)\Big]}$$

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}+\frac{1}{m^2}K_{\mu\nu}=0,$$

$$K_{\mu\nu}=\nabla^2R_{\mu\nu}-\frac{1}{4}\left(\nabla_\mu\nabla_\nu R+g_{\mu\nu}\nabla^2R\right)-4R^\sigma_\mu R_{\sigma\nu}+\frac{9}{4}RR_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\left(3R^{\alpha\beta}R_{\alpha\beta}-\frac{13}{8}R^2\right)$$

Conserved charges in grand canonical ensemble

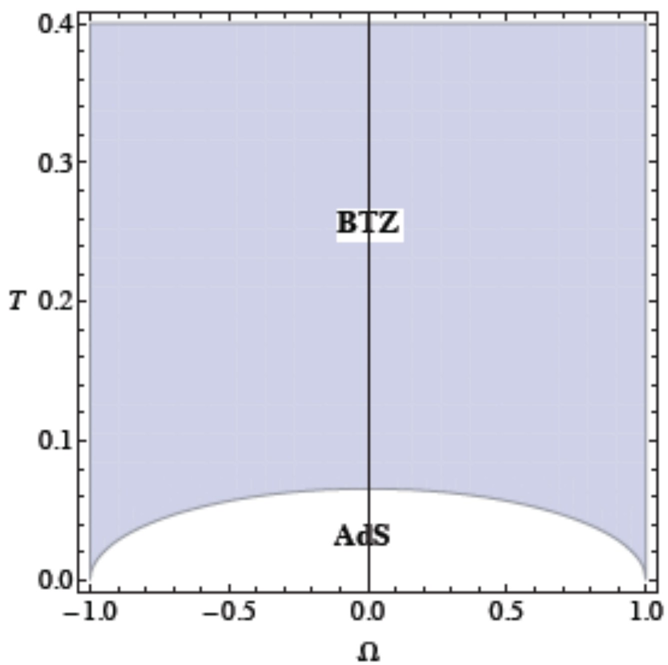
$$g_{\mu\nu}=\begin{pmatrix} -\frac{r^2}{l^2}-\frac{H^2(-r^2-4lJ+8l^2M)^2}{4l^3(lM-J)}+8M & 0 & 4J-\frac{H^2(4lJ-r^2)(-r^2-4lJ+8l^2M)}{4l^2(lM-J)} \\ 0 & \frac{1}{\frac{16J^2}{r^2}+\frac{r^2}{l^2}-8M} & 0 \\ 4J-\frac{H^2(4lJ-r^2)(-r^2-4lJ+8l^2M)}{4l^2(lM-J)} & 0 & r^2-\frac{H^2(4Jl-r^2)^2}{4l(lM-J)} \end{pmatrix}$$



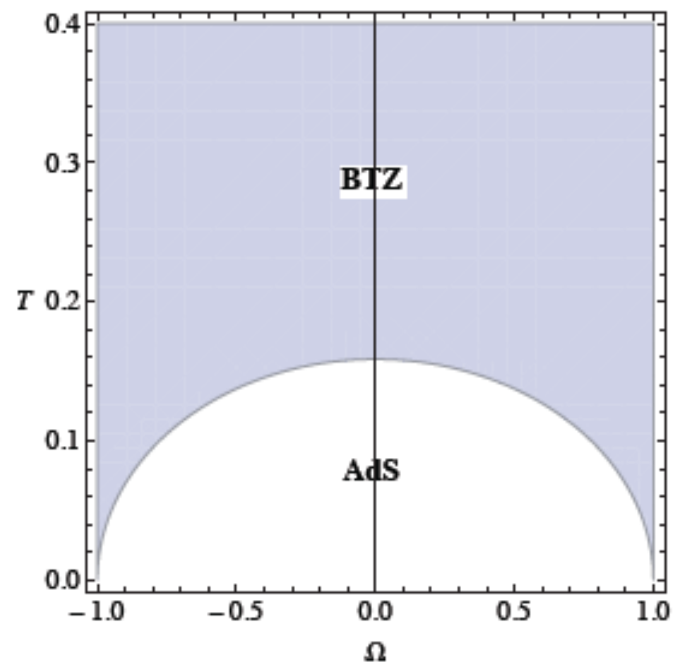
$$\mathbf{M}=\frac{16\left(1-2H^2\right)^{3/2}M}{GL\left(17-42H^2\right)},\qquad \mathbf{J}=\frac{16\left(1-2H^2\right)^{3/2}J}{G\left(17-42H^2\right)}.$$

$$\mathbf{S}=\frac{16\pi\left(1-2H^2\right)^{3/2}}{G\left(17-42H^2\right)}\sqrt{l^2M+\sqrt{l^4M^2-J^2l^2}}.$$

Hawking Page Phase diagrams of BTZ BH in grand canonical ensemble



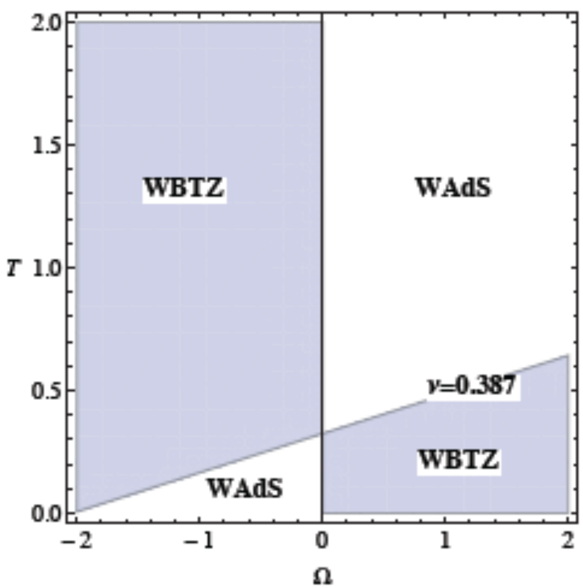
$m = 1.05.$



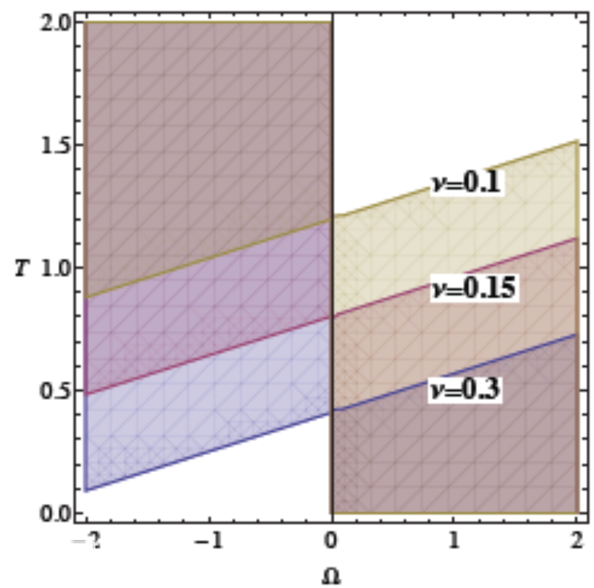
$m = 10.$

Warped BTZ BH in quadratic non-local ensemble

$$\frac{ds^2}{l^2} = dt^2 + \frac{dr^2}{(\nu^2+3)(r-r_+)(r-r_-)} + (2\nu r - \sqrt{r_+ r_-(\nu^2+3)})dtd\varphi$$
$$+\frac{r}{4}\left[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_-(\nu^2+3)}\right]d\varphi^2.$$

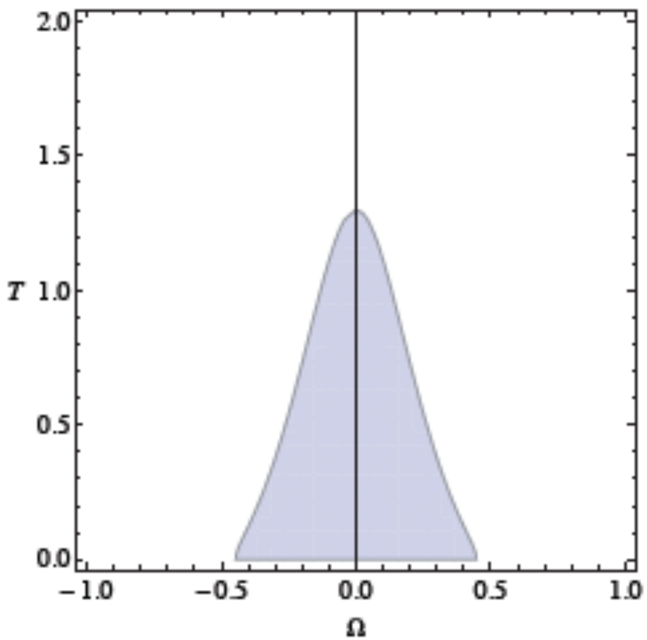


The phase diagram for $\nu = 0.387$.

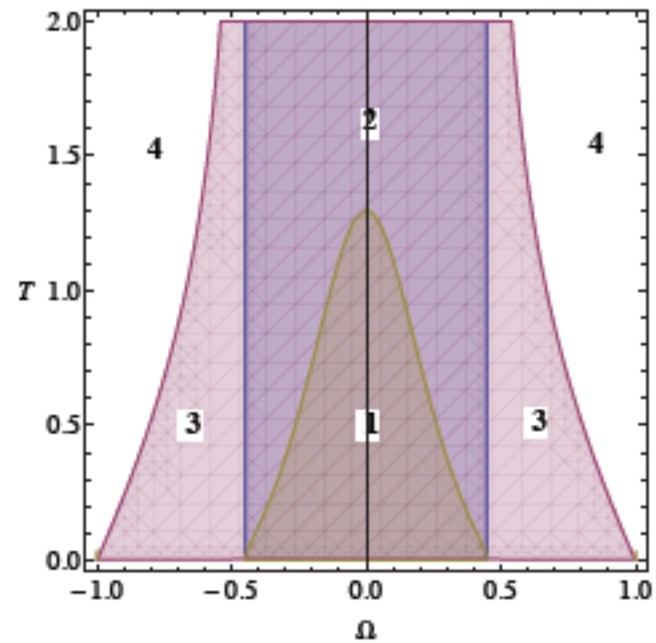


The phase diagram for different ν.

Phase diagram of hairy BH solution



The local stable region for $b = 20$.



The phase diagram for $b = 20$.

Entanglement Entropy (EE) of Warped CFTs

- The EE of a single interval in the vacuum of 2d CFT on the cylinder is

$$\boxed{S_{\mathrm{EE}} = \frac{c}{3}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right)}$$

- L is the length of circle and ℓ is the length of the interval.
- Using Warped CFT techniques, the analogous formula is

$$S_{\mathrm{EE}} = iP_0^{\mathrm{vac}}\ell\left(\frac{\bar{L}}{L} - \frac{\bar{\ell}}{\ell}\right) - 4L_0^{\mathrm{vac}}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right)$$

- Where L and $\bar{L}$ are identifications of the circle and ℓ and $\bar{\ell}$ are separations in space and time.
- The first term is independent of the Renyi replica index.

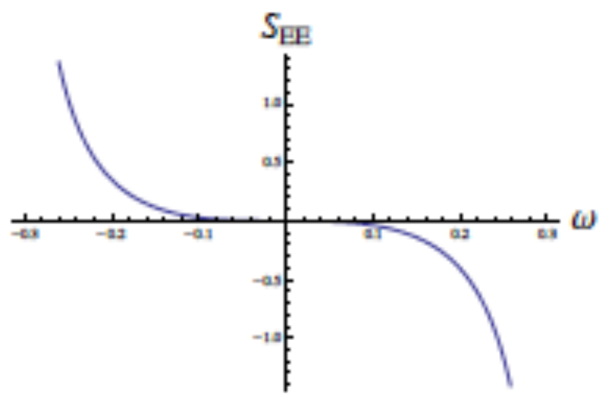
EE for the CFT_2 dual to AdS_3 in NMG

- For the NMG case, the Virasoro and Kac-Moody operators are

$$\tilde{P}_0^{(\mathrm{vac})} = \mathcal{M}^{(\mathrm{vac})}, \qquad \tilde{L}_0^{(\mathrm{vac})} = \frac{1}{k}\left(\mathcal{M}^{\mathrm{vac}}\right)^2 \qquad \mathcal{M}^{(\mathrm{vac})} = i\mathcal{M}_{\mathrm{God}} = -i\frac{4\ell^2\omega^2}{G(19\ell^2\omega^2-2)}.$$

- So the entanglement entropy is

$$S_{EE} = \frac{4\ell^2\omega^2}{G(19\ell^2\omega^2-2)}\left(\ell^*\left(\frac{\bar{L}}{L}-\frac{\bar{\ell}^*}{\ell^*}\right)+\frac{2\ell^2\omega}{1+\ell^2\omega^2}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell^*}{L}\right)\right).$$



Thanks for Your Attention!