Construction of Universal Metrics

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Workshop on Recent Trends in String Theory and Related Topics, IPM, 8-11 May 2017, Tehran-Iran

May 6, 2017

Outline

Prelude

1. I would like to thank Prof Jabbari for his kind invitation to IPM. Frankly speaking I hesitated to travel far away. I mentioned about the invitation to my wife, Unsal. She said, without hesitation, we shall go to Tehran. That was the decision, we are now here.

2.The Campus of IPM remind me ICTP (Abdus Salam Institute of Theoretical Physics at Trieste). I was young and senior associate of ICTP. Almost all summers I used to visit ICTP. I am sure that ICTP contributed too much to my research carrier. I hope that IPM does similar job in IRAN and neighbor countries.

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4. I am a half physicist and half mathematician. In mathematics one of research area is Integrable Surfaces in soliton Theory. There are certain functionals defined on 2-surfaces where the Lagrange function is a function of Gauss and Mean curvatures. The critical points of these functionals are important and they are related to many different fields. I am interested in critical points of the most general functionals. i.e., critical points solving Euler-Lagrange equations of any functional. We call such critical points as "Universal" critical points. Plane and Sphere are examples of such universal critical points. A special torus is also universal.

5. Is it possible to have universal critical points in GR? This means that, are there Riemannian metrics in arbitrary dimensions solve field equations of any generic gravity theory?. Our aim is to give an answer to this question and present these metrics explicitly.

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Collaborators

This is a series of works with my colleagues :

Ibrahim Güllü,

Tahsin Çağrı Şişman,

and

Bayram Tekin

Publications

AdS-wave solutions of quadratic curvature gravities:

- I. Gullu, M. Gurses,Tahsin Cagri Sisman and B. Tekin, "AdS Waves as Exact Solutions to Quadratic Gravity," Phys. Rev. D **83**, 084015 (2011).
- M. Gurses, Tahsin Cagri Sisman and B. Tekin, "New Exact Solutions of Quadratic Curvature Gravity," Phys. Rev. D **86**, 024009 (2012).

Kerr-Schild–Kundt class metrics:

- M. Gurses, S. Hervik, Tahsin Cagri Sisman and B. Tekin, "Anti–de Sitter–Wave Solutions of Higher Derivative Theories," Phys. Rev. Lett. **111**, 101101 (2013).
- M. Gurses, Tahsin Cagri Sisman and B. Tekin, "AdS-plane wave and *pp*-wave solutions of generic gravity theories," Phys. Rev. D **90**, no. 12, 124005 (2014).
- M. Gurses,Tahsin Cagri Sisman and B. Tekin, "Kerr-Schild–Kundt Metrics are Universal,"Class. Quantum Grav, **34**, no.7, 075003, (2017).

Publications

KSK Metrics in three dimesions

- M. Gurses, Tahsin Cagri Sisman and B. Tekin, "Gravity waves in three dimensions," Phys. Rev. D **92**, 084016 (2015).

Solution generation technique for (A)dS-waves:

- M. Gurses, Tahsin Cagri Sisman. and B. Tekin, "From Smooth Curves to Universal Metrics," Phys. Rev. **D94**, 044041 (2016).

Proof of Universality of the KSK Metrics:

- M. Gurses, S. Hervik, Tahsin Cagri Sisman. and B. Tekin, "Anti–de Sitter–Wave Solutions of Higher Derivative Theories," Phys. Rev. Lett. **111**, 101101 (2013).
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Why higher curvature/derivative theory?

- **Problem:** Einstein's gravity

$$I = \frac{1}{\kappa}\int d^4x\sqrt{-g}\,R,$$

is *not* renormalizable ('t Hooft and Veltman, 1974; Deser and van Nieuwenhuizen, 1974). $(\kappa \equiv 16\pi G)$

- **Effective field theory perspective:** High energies $\Rightarrow$ Higher curvature and derivative terms;

$$I = \int d^Dx\sqrt{-g}\left\{\frac{1}{\kappa}(R-2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) + \sum_{n=3}^{\infty} a_n(\text{Riem, Ric, R, }\nabla\text{Riem, }\ldots)^n\right\}.$$

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A basic question

- What can we say about the solutions of

$$I = \int d^D x \sqrt{-g}\, \mathscr{L}(\text{Riem}, \nabla\text{Riem}, \ldots, \nabla^n\text{Riem}, \ldots),$$

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Derivative order of the field equations =2N+2

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What we know: plane wave metric solves the generic theory

- plane-wave solution for Einstein's gravity

$$I=\frac{1}{\kappa}\int d^Dx\sqrt{-g}\,R,$$

is a solution of the generic higher derivative gravity theory (Guven, 1987; Amati and Klimcik, 1989; Horowitz and Steif, 1990)

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What is plane-wave?

- *plane-wave* metric is a special case of pp-wave metric
- pp-wave metric in Kerr-Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + 2V\lambda_\mu\lambda_\nu,$$

where $\eta_{\mu\nu}$ is the Minkowski metric and

$$\lambda^\mu\lambda_\mu = 0, \qquad \partial_\mu\lambda_\nu = 0, \qquad \lambda^\mu\partial_\mu V = 0.$$

For plane waves $V(u, x^i)$ is quadratic in x^i's.

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Kerr-Schild–Kundt (KSK) metrics

- Kerr-Schild metrics

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$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}, \qquad \xi_\mu\lambda^\mu = 0, \qquad \lambda^\mu\partial_\mu V = 0.$$

- With $\nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}$ and $\xi_\mu\lambda^\mu = 0$, λ_μ is divergenceless, shear-free, and non-twisting $\Rightarrow$ Kundt spacetime

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Properties of KSK metrics

- Type-N Weyl

$$C_{\mu\alpha\nu\beta} = \lambda_\mu \lambda_\nu \Omega_{\alpha\beta} + \lambda_\alpha \lambda_\beta \Omega_{\mu\nu} - \lambda_\mu \lambda_\beta \Omega_{\alpha\nu} - \lambda_\alpha \lambda_\nu \Omega_{\mu\beta},$$

where $\lambda^\alpha \Omega_{\alpha\beta} = 0$, $\Omega^\alpha_\alpha = 0$.

- Type-N traceless-Ricci

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{D} g_{\mu\nu} R = -\left(\Box + \frac{2}{\ell^2}\right)(\lambda_\mu \lambda_\nu V) = \rho\, \lambda_\mu \lambda_\nu,$$

where $\Box \equiv \nabla^\mu \nabla_\mu$ and ρ is given by

$$\rho = -\left(\Box + 2\xi^\mu \partial_\mu + \frac{1}{2}\xi^\mu \xi_\mu - \frac{2(d-2)}{\ell^2}\right) V \equiv -\mathscr{Q} V, \quad (1)$$

where we defined the operator $\mathscr{Q}$.

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where we defined the operator $\mathcal{Q}$.

The Main Theorem

Let (M, g) be the space-time with the Kerr-Schild type of metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu,$$

satisfying the properties

$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}, \qquad \xi_\mu\lambda^\mu = 0, \qquad \lambda^\mu\partial_\mu V = 0,$$

where $\bar{g}_{\mu\nu}$ is the metric of a space of constant curvature (AdS or dS) then any second rank symmetric tensor constructed from the Riemann tensor and its covariant derivatives can be written as a linear combination of $g_{\mu\nu}$, $S_{\mu\nu}$, and higher derivatives of $S_{\mu\nu}$ in the form $\Box^n S_{\mu\nu}$ where $\Box$ represents the d'Alembertian with respect to $g_{\mu\nu}$, that is

$$E_{\mu\nu} = e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu}.$$

The Main Theorem

A stronger version of this theorem for Einstein spaces

$$R_{\alpha\beta} = \lambda g_{\alpha\beta}$$

was stated in:

A A Coley, G W Gibbons, S Hervik, and C N Pope, *Metrics with vanishing quantum corrections*, Classical and Quantum Gravity, **25**, 145015 (2008).

Our main theorem covers the non-Einsteinian case:

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Field Equations of Generic Gravity

Let the most general gravity theory be a $(2N+2)$-derivative theory. As examples, for Einstein's gravity (and Einstein–Gauss-Bonnet gravity) $N=0$, for quadratic and f(Riemann) theories $N=1$. We have shown that the equations of the most general $(2N+2)$-derivative gravity theory reduce, when evaluated for these metrics, to

$$\sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0, \quad (2)$$

where a_ns are constants which are functions of the parameters of the theory. After some algebraic manipulations (2) reduces to

$$\prod_{n=1}^{N} \left(\mathcal{Q} - m_n^2\right) \mathcal{Q}\, V = 0, \quad (3)$$

whose generic solution is

$$V = V_E + \sum_{n=1}^{N} V_n \quad (4)$$

where the Einsteinian part (V_E) and the other (massive) parts satisfy the following equations

Field Equations of Generic Higher Derivative Gravity

$$\mathcal{Q} V_E = 0, \qquad \left(\mathcal{Q} - m_n^2\right) V_n = 0, \tag{5}$$

for all $n = 1, 2, \cdots, N$, provided that all m_n's are different and none is zero. If any two or more m_n's coincide or equal to zero, then the second equation in (5) changes in the following way: let r be the number (multiplicity) of m_n's that are equal to m_r, then the corresponding V_r satisfies an irreducibly higher derivative equation

$$(\mathcal{Q} - m_r^2)^r \, V_r = 0, \tag{6}$$

with new branches so called log-solutions appear:

$$V = V_E + V_r + \sum_{n=0}^{N-r} V_n \tag{7}$$

and V_r contains $\log^{r-1}$ terms. Maximal criticality is $r = N+1$. Note that derivative order of the theory is twice of the maximal criticality of the theory

A Remark: Linearized Equations

KSK Metrics solve also the linearized field equations of any generic theory of Gravity.

For a gravity theory of rank $N+1$ the linearized equations are (see T. Nutma, Polycritical Gravities,PRD. **85**, 124040 (2012))

$$\prod_{n=1}^{N} (\bar{\Box} - 2\Lambda - m_n^2)\, \bar{\Box}\, h_{\mu\nu} = 0, \tag{8}$$

with

$$\bar{\nabla}_\mu\, h^{\mu\nu} = 0. \tag{9}$$
$$\bar{g}^{\mu\nu}\, h_{\mu\nu} = 0 \tag{10}$$

For KSK Metics $h_{\mu\nu} = 2V\lambda_\mu\, \lambda_\nu$

The Proof: Basic Idea

- Weyl tensor and traceless-Ricci tensor have two free index λ.
- Any contraction of the λ vector with any other tensorial form always yields a free-index λ vector or zero.
- To have a nonzero two-tensor from the contractions of the Riemann tensor and its derivatives, one should start with a tensorial form involving 2 λ's.
- So, one has the possibilities $R_{\alpha\beta\rho\sigma}$ yielding $R_{\mu\nu}$ and $\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_n}R_{\alpha\beta\rho\sigma}$ yielding the $\Box^{n/2}S_{\mu\nu}$ and lower orders.

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Field Equations for Einstein's Gravity and Quadratic Curvature Gravity

For KSK class of metrics,

- the field equations of cosmological Einstein's gravity

$$I = \frac{1}{\kappa} \int d^D x \sqrt{-g} \, (R - 2\Lambda_0),$$

reduces to

$$S_{\mu\nu} = \left(\Box + \frac{2}{\ell^2} \right) (V \lambda_\mu \lambda_\nu) = \left(\bar{\Box} + \frac{2}{\ell^2} \right) (V \lambda_\mu \lambda_\nu) = 0.$$

- the field equations of quadratic curvature gravity

$$I = \int d^D x \sqrt{-g} \left\{ \frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \right\},$$

reduces to

$$(\Box - a) S_{\mu\nu} = \left(\Box + \frac{2}{\ell^2} - M^2 \right) S_{\mu\nu} = \left(\bar{\Box} + \frac{2}{\ell^2} - M^2 \right) S_{\mu\nu} = 0,$$

where $M^2 = -\frac{1}{\beta} \left(\frac{1}{\kappa} + \frac{4\Lambda D}{D-2} \alpha + \frac{4\Lambda}{D-1} \beta + \frac{4\Lambda (D-3)(D-4)}{(D-1)(D-2)} \gamma \right) + \frac{2}{\ell^2}$.

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where $M^2=-\frac{1}{\beta}\left(\frac{1}{\kappa}+\frac{4\Lambda D}{D-2}\alpha+\frac{4\Lambda}{D-1}\beta+\frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)}\gamma\right)+\frac{2}{\ell^2}$.

AdS-waves solve the generic theory

- *AdS-waves are members of KSK class* $\Rightarrow$ *AdS-waves are solutions to any higher derivative theory*

- AdS-wave solutions of cosmological Einstein's gravity

$$I = \frac{1}{\kappa}\int d^D x\sqrt{-g}\,(R - 2\Lambda_0),$$

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$$I = \int d^D x \sqrt{-g}\left\{\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)\right\},$$

are solutions of the generic higher derivative theory

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AdS-waves solve the generic theory

- *AdS-waves are members of KSK class* $\Rightarrow$ *AdS-waves are solutions to any higher derivative theory*

- AdS-wave solutions of cosmological Einstein's gravity

$$I = \frac{1}{\kappa} \int d^D x \sqrt{-g} \left(R - 2\Lambda_0 \right),$$

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AdS-plane and AdS-spherical Waves

Metric forms: *(for simplicity, let us have 4D)*

- AdS-plane wave or Siklos metric form:

$$ds^2 = \frac{\ell^2}{z^2}\left(2dudv + dy^2 + dz^2\right) + 2V(u,y,z)\,\mathrm{d}u^2.$$

- AdS-spherical wave:

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(\frac{4dudv}{(u+v)^2} + d\theta^2 + \sin^2\theta\, d\phi^2\right) + 2V(u,\theta,\phi)du^2.$$

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4D AdS-plane wave solution

- Einstein's gravity solution is (Kaigorodov, 1963; see also Chamblin and Gibbons, 2000)

$$V(u,y,z) = \frac{1}{\sqrt{z}}\left[c_1(u) I_\nu(az) + c_2(u) K_\nu(az)\right]\sin(ay + c_3(u)),$$

where $\nu = \frac{3}{2}$ and a is an arbitrary constant; and for $a = 0$,

$$V(u,z) = c(u)\, z.$$

- Quadratic curvature gravity solution found as (Alishahiha and Fareghbal, 2011; Gullu, Gurses, Sisman and Tekin, 2011)

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4D AdS-spherical wave solution

- Einstein's gravity solution (Gurses, Sisman and Tekin, 2012)

$$V(u,\theta,\phi)=\left[c_1(u)\left(\tan\frac{\theta}{2}\right)^a\sec\theta\,(a+\sec\theta)\right.$$
$$\left.+c_2(u)\left(\tan\frac{\theta}{2}\right)^{-a}\sec\theta\,(a-\sec\theta)\right]$$
$$\times\left[c_3(u)\cos(a\phi)+c_4(u)\sin(a\phi)\right],$$

where a is an arbitrary constant; and for $a=0$,

$$V(u,\theta,\phi)=\frac{1}{\cos^2\theta}\left[1+c_2(u)\left(\cos\theta+\log\left[\tan\left(\frac{\theta}{2}\right)\right]\right)\right](c_3(u)+c_4(u)\phi).$$

Null vector field of KSK class

- Remember $\nabla_\mu \lambda_\nu = \xi_{(\mu} \lambda_{\nu)}$ and consider AdS and dS metrics as

$$d\bar{s}^2 = \frac{\ell^2}{z^2} \left(-dt^2 + \sum_{m=1}^{d-2} (dx^m)^2 + dz^2 \right),$$

and

$$d\bar{s}^2 = \frac{\ell^2}{t^2} \left(-dt^2 + \sum_{m=1}^{d-1} (dx^m)^2 \right).$$

- Consider $\partial_\nu \lambda_\mu$ in these coordinates

$$\partial_\nu \lambda_\mu = a\, \eta_{\mu\nu} + \lambda_\mu \left(\frac{1}{2} \xi_\nu - \zeta_\nu \right) + \lambda_\nu \left(\frac{1}{2} \xi_\mu - \zeta_\mu \right),$$

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Questions

- KSK metrics

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu,$$

where $\bar{g}_{\mu\nu}$ is the (A)dS metric and

$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}, \qquad \xi_\mu\lambda^\mu = 0, \qquad \lambda^\mu\partial_\mu V = 0.$$

- How we compute the vectors λ_μ and ξ_μ ?
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A solution generation technique

- The crucial point is to find the λ and ξ vectors in the KSK metrics. For this purpose we write $\partial_\nu \lambda_\mu$ in the AdS and dS backgrounds

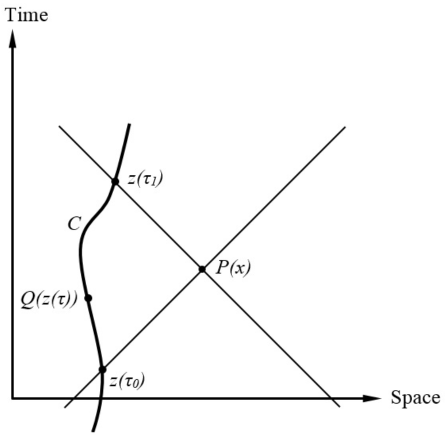
$$\partial_\nu \lambda_\mu = a\, \eta_{\mu\nu} + \lambda_\mu \left(\frac{1}{2}\xi_\nu - \zeta_\nu\right) + \lambda_\nu \left(\frac{1}{2}\xi_\mu - \zeta_\mu\right),$$

where $a = \frac{\lambda_z}{z}$, $\zeta_\nu = \frac{1}{z}\delta^z_\nu$ for AdS and $a = \frac{\lambda_t}{t}$, $\zeta_\nu = \frac{1}{t}\delta^t_\nu$ for dS.

An unexpected way of constructing the vectors λ and ξ

- KSK metrics in D-dimensions can be generated from curves in one less dimensions. (From Smooth Curves to Universal Metrics, PRD , Gurses, Sisman and Tekin, 2016)

Curves in D-Dimensional Minkowski



Curve C is parametrized by the parameter τ, $z^\mu = z^\mu(\tau)$ so that $\eta_{\mu\nu}\, \dot{z}^\mu(\tau)\, \dot{z}^\nu(\tau) = \varepsilon$ where $\varepsilon = \pm 1, 0$.

- The distance between the points $P(x^\mu)$ and $Q(z^\mu)$

$$\Omega^2 = \eta_{\mu\nu}\,(x^\mu - z^\mu(\tau))\,(x^\nu - z^\nu(\tau)).$$

Null vector field out of the curves of Minkowski Geometry

- Differentiating $\Omega(\tau_0) = 0$ with respect to x_μ yields a null vector: (Bonnor-Vaidya, 1972)

$$\ell_\mu \equiv \partial_\mu \tau_0 = \frac{x_\mu - z_\mu(\tau_0)}{R},$$

where $R \equiv \dot{z}^\alpha(\tau_0)\left(x_\alpha - z_\alpha(\tau_0)\right)$ with $\dot{z}^\alpha(\tau_0) \equiv \partial_{\tau_0} z^\alpha(\tau_0)$.

- One more derivative:

$$\partial_\nu \ell_\mu = \frac{1}{R}\left(\eta_{\mu\nu} - \dot{z}_\mu \ell_\nu - \dot{z}_\nu \ell_\mu - (A - \varepsilon)\,\ell_\mu \ell_\nu\right),$$

with $A \equiv \ddot{z}^\mu \left(x_\mu - z_\mu\right)$ and $\varepsilon \equiv \dot{z}^\mu \dot{z}_\mu$.

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The prescription

- Observing the similarity:
  Null vector ℓ_μ from any smooth curve in M_D

$$\partial_\nu \ell_\mu = \frac{1}{R}\left(\eta_{\mu\nu} - \dot{z}_\mu \ell_\nu - \dot{z}_\nu \ell_\mu - (A-\varepsilon)\,\ell_\mu \ell_\nu\right),$$

  and null vector λ_μ of the KSK Metric

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  suggests the prescription:

  1. Take the vectors ℓ_μ and λ_μ to be equal and derive the corresponding vector ξ_μ
  2. Set $\lambda^\mu \xi_\mu = 0$, and obtain the constraint on $z^\mu(\tau)$.

- 2nd step constrains $z^\mu(\tau)$ curves to live in one less dimension.

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The prescription

Let us execute this procedure, when the seed metric is AdS

$$d\bar{s}^2 = \frac{\ell^2}{z^2}\left(-dt^2 + \sum_{m=1}^{d-2}(dx^m)^2 + dz^2\right), \tag{11}$$

equating λ vector to ℓ vector one finds

$$\xi_\mu = -\frac{2}{R}\left(\dot{z}_\mu + \frac{1}{2}(A-\varepsilon)\lambda_\mu\right) + \frac{2}{z}\delta^z_\mu. \tag{12}$$

To satisfy $\lambda^\mu \xi_\mu = 0$, we must have $\lambda_z = \frac{z}{R}$ and $z_z = 0$. Hence, all these curves live in a $(d-1)$-dimensional Minkowski spacetime. In the AdS case curves are timelike, null and spacelike.

The prescription

When the seed metric is the dS metric:

$$d\bar{s}^2 = \frac{\ell^2}{t^2}\left(-dt^2 + \sum_{m=1}^{d-1}(dx^m)^2\right), \tag{13}$$

we find

$$\xi_\mu = -\frac{2}{R}\left(\dot{z}_\mu + \frac{1}{2}(A-\varepsilon)\lambda_\mu\right) + \frac{2}{t}\delta^t_\mu. \tag{14}$$

To satisfy $\lambda^\mu \xi_\mu = 0$, we must have $\lambda_t = \frac{t}{R}$ and $z_t = 0$. Hence, the curve C in this case lives in a $(d-1)$-dimensional Euclidean space where we can have only spacelike curves.

KSK Metrics in Robinson Trautman Coordinates

- The KSK Metrics in the coordinates (t, x, y, z) gives a very complicated expression for the operator $\mathscr{Q}$.
- With this form, it is highly difficult to solve the equations for the metric function V.
- In addition, we must also satisfy $\lambda^{\mu}\partial_{\mu}V = 0$. For this purpose, there is a need to introduce new coordinates where both the metric and the operator $\mathscr{Q}$ take simpler forms.
- Two of the new coordinates are (the natural coordinates) τ and R. They are defined through $\Omega(\tau) = 0$ and $R = \dot{z}^{\alpha}(\tau)\left(x_{\alpha} - z_{\alpha}(\tau)\right)$.

KSK Metrics in Robinson Trautman Coordinates

The coordinate transformation can be given as (Newman-Unti,JMP, 1963)

$$x^{\mu} = R\lambda^{\mu}(\tau, \text{angular coordinates}) + z^{\mu}(\tau), \qquad \mu = 0,1,2,3. \tag{15}$$

Here, the null vector λ^{μ} does not depend on the new coordinate R. In these new coordinates, we have

$$\partial_R V = \frac{\partial x^{\mu}}{\partial R}\partial_{\mu} V = \lambda^{\mu}\partial_{\mu} V = 0. \tag{16}$$

Hence, the metric function is independent of the new coordinate R. Furthermore, in the new coordinates, $\lambda_{\mu} dx^{\mu} = d\tau$. Hence, we have

$$ds^2 = d\bar{s}^2 + 2V(\tau, \text{angular coordinates})\, d\tau^2, \tag{17}$$

where $d\bar{s}^2$ is the background line element. The new metric is found as (letting $r = R$)

$$ds^2 = \frac{1}{f^2}[H d\tau^2 + 2d\tau dr + \frac{r^2}{2P^2}\, d\zeta\, d\bar{\zeta}] + 2V(\tau, \zeta, \bar{\zeta})\, d\tau^2 \tag{18}$$

KSK Metrics in AdS Background $D = 4$

For AdS Background:

$$H = \varepsilon - 2r \frac{P_{,\tau}}{P}, \quad (19)$$

$$f = r \frac{k}{P} \left(1 - \zeta \bar{\zeta}/4\right) \quad (20)$$

$$\varepsilon = -(\dot{z}^0(\tau))^2 + (\dot{z}^1(\tau))^2 + (\dot{z}^2(\tau))^2 + (\dot{z}^3(\tau))^2 \quad (21)$$

where

$$P = \dot{z}^0(\tau) - \dot{z}^3(\tau) - \frac{1}{2}(\dot{z}^2(\tau) + i\dot{z}^1(\tau))\,\zeta - \frac{1}{2}(\dot{z}^2(\tau) - i\dot{z}^1(\tau))\bar{\zeta} + \frac{1}{4}(\dot{z}^0(\tau) + \dot{z}^3(\tau))\,\zeta\,\bar{\zeta} \quad (22)$$

and $z^3(\tau) = 0$.

KSK Metrics in AdS Background in $D = 4$

The Ricci tensor takes the form

$$R_{\mu\nu} = \rho \lambda_\mu \lambda_\nu + \lambda g_{\mu\nu} \tag{23}$$

where $\lambda = -3k^2$ and $\lambda_\mu = \delta^0_\mu$. Here $z^\mu = (z^0(\tau), z^1(\tau), z^2(\tau), z^3(\tau) = 0)$ is the parametrization of an arbitrary curve C satisfying with $\varepsilon = -1, 0, 1$.

Letting $\zeta = 2\tan(\theta/2)\, e^{i\phi}$ we obtain

$$ds^2 = \frac{1}{k^2 \cos^2(\theta)} [Q_1^2 \frac{H d\tau^2 + 2 d\tau dr}{r^2} + \frac{1}{2}(d\theta^2 + \sin^2(\theta) d\phi^2)] + 2V(\tau, \theta, \phi)\, d\tau^2 \tag{24}$$

where

$$Q_1 = \dot{z}^{(0)} - \dot{z}^{(2)} \sin\theta \cos\phi - \dot{z}^{(1)} \sin(\theta)\sin(\phi) \tag{25}$$

and $H = \varepsilon - 2r\, \frac{Q_{1,\tau}}{Q_1}$

KSK Metrics in For dS Background in $D = 4$

$$H = 1 - 2r\frac{P_{,\tau}}{P}, \quad (26)$$

$$f = r\frac{k}{P}\left(1 + \zeta\bar{\zeta}/4\right) \quad (27)$$

$$1 = (\dot{z}^1(\tau))^2 + (\dot{z}^2(\tau))^2 + (\dot{z}^3(\tau))^2 \quad (28)$$

where

$$P = P = \dot{z}^0(\tau) - \dot{z}^3(\tau) - \frac{1}{2}(\dot{z}^2(\tau) + i\dot{z}^1(\tau))\,\zeta - \frac{1}{2}(\dot{z}^2(\tau) - i\dot{z}^1(\tau))\bar{\zeta} + \frac{1}{4}(\dot{z}^0(\tau) + \dot{z}^3(\tau))\,\zeta\,\bar{\zeta} \quad (29)$$

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$$R_{\mu\nu} = \rho\lambda_\mu\,\lambda_\nu + \lambda\,g_{\mu\nu} \quad (30)$$

where $\lambda = 3k^2$ and $\lambda_\mu = \delta^0_\mu$. Here $z^\mu = (z^0(\tau) = 0, z^1(\tau), z^2(\tau), z^3(\tau))$ is the parametrization of an arbitrary curve C satisfying.

KSK Metrics in For dS Background in $D = 4$

Letting $\zeta = 2\tanh(\theta/2)\, e^{i\phi}$ we obtain

$$ds^2 = \frac{1}{k^2\cosh^2(\theta)}\,[Q_2^2\,\frac{Hd\tau^2 + 2d\tau dr}{r^2} + \frac{1}{2}\,(d\theta^2 + \sinh^2(\theta)d\phi^2)] + 2V(\tau,\theta,\phi)\,d\tau^2 \tag{31}$$

where

$$Q_2 = \dot{z}^{(3)} - \dot{z}^{(2)}\sinh\theta\cos\phi - \dot{z}^{(1)}\sinh(\theta)\sin(\phi) \tag{32}$$

and $H = 1 - 2r\,\frac{Q_{2,\tau}}{Q_2}$

Summary

- We have shown that the KSK metrics are universal.
- We have shown that KSK Metrics in D-dimensions can be constructed from the smooth metrics in (D-1) dimensions.
- We have given all KSK metrics in RT coordinates

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Concluding Remarks

- It is straightforward to write KSK Metrics in AdS and dS backgrounds in an arbitrary dimensions.
- What is left to determine? Solving the Klein-Gordon type of equations satisfied by the metric function V

$$\mathcal{Q} V_E = 0, \qquad \left(\mathcal{Q} - m_n^2\right) V_n = 0, \tag{33}$$

  in the backgrounds AdS and dS
- All known examples correspond to straight curves (curvature of the curve vanishes). AdS plane wave metric correspond to straight null line ($\varepsilon = 0$) and AdS spherical wave metric correspond to straight time like line ($\varepsilon = 1$). Curves with acceleration has not been studied yet.
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