# Entanglement in Lifshitz-type QFTs

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Recent Trends in String Theory and Related Topics
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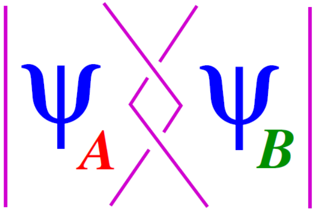
# Motivation

Entanglement entropy (EE) is an important concept which appears in condensed matter, quantum information and black hole physics.

EE is useful for studying

- Quantum phase transitions at $T = 0$
- Non-equilibrium processes, e.g., quantum quenches
- The connection between gauge theory and gravity
- $\cdots$

# Entanglement in QFTs



# (Pure) Entangled states

- Consider two quantum systems, i.e., $A$ and $B$

$$\mathcal{H}_A, \quad |i\rangle, \quad i = 1, \cdots, n, \quad \mathcal{H}_B, \quad |a\rangle, \quad a = 1, \cdots, m$$

- Construct $M$ using the tensor product of $A$ and $B$

$$\mathcal{H}_M = \mathcal{H}_A \otimes \mathcal{H}_B, \quad |i\rangle \otimes |a\rangle \equiv |i, a\rangle$$

- Separable states

$$|\chi\rangle_{\mathcal{H}_M} = |\psi\rangle_{\mathcal{H}_A} \otimes |\phi\rangle_{\mathcal{H}_B}$$

- (Pure) Entangled states

$$|\chi\rangle_{\mathcal{H}_M} \neq |\psi\rangle_{\mathcal{H}_A} \otimes |\phi\rangle_{\mathcal{H}_B}$$

# Example: Spin 1/2 Particles

- Separable states

$$\begin{aligned} |\Psi_1\rangle &= |\uparrow_A\rangle \otimes |\downarrow_B\rangle \\ |\Psi_2\rangle &= |\downarrow_A\rangle \otimes |\uparrow_B\rangle \end{aligned}$$

- Entangled states

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle \pm |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$$

**Challenge**

Entanglement Measures!

Entanglement entropy, Mutual information, $\cdots$

# Reduced Density Matrix

- Consider the density matrix for a pure system $M = A \cup B$

$$\rho_M = |\psi\rangle\langle\psi|$$

- Definition of reduced density matrix for $A$

$$\rho_A \equiv \mathrm{Tr}_B(\rho_M) = \sum_{i=1}^{dim[B]} \langle i_B|\rho_M|i_B\rangle$$

- For any $O_A \in A$

$$\langle O_A\rangle = \mathrm{Tr}_A(\rho_A O_A)$$

# Entanglement Entropy

- von-Neumann entropy for $\rho_A$

$$S_A \equiv -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right) = -\sum_{i=1}^{dim[A]} \langle i_A \big| \rho_A \log \rho_A \big| i_A \rangle$$

- Example:

$$\big|\Psi_1\rangle \text{ and } \big|\Psi_2\rangle, \qquad S_A = 0$$

$$\big|\Psi_3\rangle = \frac{1}{\sqrt{2}}\left(\big|\Psi_1\rangle \pm \big|\Psi_2\rangle\right), \qquad S_A = \log 2$$

# Entanglement Entropy

- Properties

1. EE corresponds to a non-linear operator in QM
2. When $A \cup B$ is pure $\quad S(A) = S(B)$
3. Subadditivity $\quad S(A) + S(B) \geq S(A \cup B)$

and $\cdots$

**Challenge**

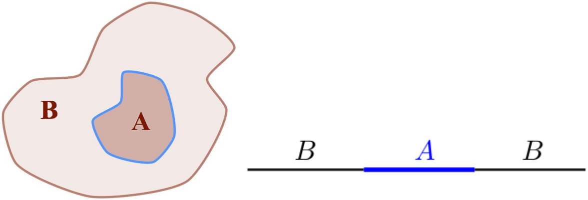
Generalization to QFT (Continuum Limit)?

# Different Notions of EE in QFT

- Different Hilbert space decompositions lead to different types of EE
  1. Field space entanglement entropy
     [Yamazaki '13, Mollabashi-Ryu-Takayanagi '14, MM-Mollabashi '15]
  2. Geometric (Entanglement) entropy
     [Bombelli-Koul-Lee-Sorkin '86, Srednicki '93, Callan-Wilczek '94]
  3. $\ldots$

# Geometric entropy

- Consider a local $d$-dimensional QFT on $\mathbb{R} \times \mathcal{M}^{(d-1)}$
- Divide $\mathcal{M}^{(d-1)}$ into two parts




- Locality implies $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$S_A = -\mathrm{Tr}_A \left(\rho_A \log \rho_A\right)$$

# Properties of EE in QFTs

- Infinite number of DOFs in QFT → Divergent EE
- In $d-1$ spatial dimensions we should have for any QFT

$$S_A \propto \frac{\mathcal{S}_{d-2}}{\epsilon^{d-2}} + \cdots + \frac{\mathcal{S}_1}{\epsilon} + \mathcal{S}_{\text{univ.}} \log \epsilon + \mathcal{S}_0$$

$\epsilon$ : Lattice constant (UV cutoff$^{-1}$)

$\mathcal{S}_{d-2}$: Area of the entangling surface

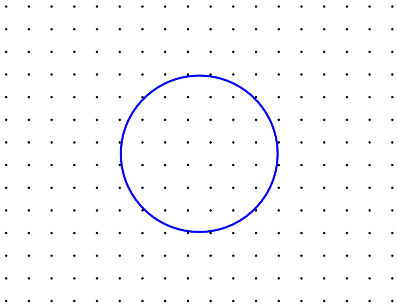
$\mathcal{S}_{\text{univ.}}$: Universal coefficient

$\mathcal{S}_0$: Finite part

# Area Law

$$S_A \propto \frac{\mathcal{S}_{d-2}}{\epsilon^{d-2}} + \cdots$$

- $\epsilon$ : Lattice constant (UV cutoff$^{-1}$)
- $\mathcal{S}_{d-2}$: Area of the entangling surface



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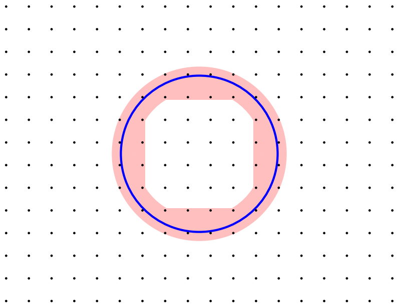
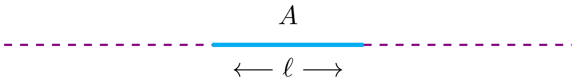


# Area Law

- Area law breaks down in specific cases:

1. Logarithmic divergence in $\text{CFT}_2$ [Calabrese-Cardy, '04]




2. Logarithmic divergence for fermions [Wolf '06, Gioev-Klich '06]

   Formation of Fermi surface

3. Volume law for non-local QFTs [Shiba-Takayanagi '13]

# How we compute EE?

- $S_A = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right)$

Taking the logarithm of $\rho_A$ is very complicated!

- Renyi Entropy and Replica Trick

$$S_A^{(n)} = \frac{1}{1-n}\mathrm{Tr}_A\left(\rho_A^n\right), \quad S_A = \lim_{n\to 1} S_A^{(n)}$$

- Different Methods for Computing EE

1. Heat Kernel Method
2. Correlator Method
3. Holographic Prescription

# Correlator Method

1. Having $\langle \mathcal{O}_A \rangle$ for any $\mathcal{O}$ restricted to A, one can fix $\rho_A$

$$\mathrm{Tr}\left(\rho_A \mathcal{O}_A\right) = \langle \mathcal{O}_A \rangle$$

2. In quadratic theories, $\mathrm{Tr}\left(\rho_A \mathcal{O}_A\right)$ can be written in terms of 2-point functions of the theory

3. The eigenvalues of $\rho_A$, i.e., $\varepsilon_k$, is given by

$$\varepsilon_k = 2 \coth^{-1}(2\nu_k)$$

where $\nu_k$'s are eigenvalues of $\sqrt{X.P}$

$$X_{ij} = \langle \phi_i \phi_j \rangle, \qquad P_{ij} = \langle \pi_i \pi_j \rangle$$

[I. Peschel 2003]

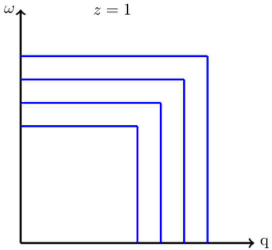
# Lifshitz-type QFTs

# Lifshitz Scaling at QCP

- Anisotropic scaling at a quantum critical point [Lifshitz, 1941]

  $t \to \lambda^z\, t, \qquad \vec{x} \to \lambda\, \vec{x}, \qquad [x] = -1, \qquad [t] = -z$

- Different Scaling Dimension $\longrightarrow$ Different RG Rate [Hertz, 1976]



# Free Scalar Theory

- Action

$$I = \frac{1}{2}\int dt d\vec{x} \left[\dot{\phi}^2 - \sum_{i=1}^{d}(\partial_i^z \phi)^2 - m^{2z}\phi^2\right]$$

- Mass Dimensions

$$[t] = -z, \quad [\vec{x}] = -1, \quad [m] = 1, \quad [\phi] = \frac{d-z}{2}$$

- Ground State Two Point Correlator

$$\langle \phi(0)\phi(r)\rangle \sim r^{-d+z}$$

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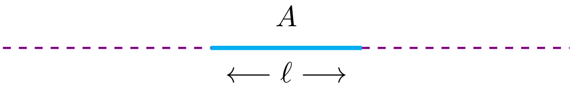
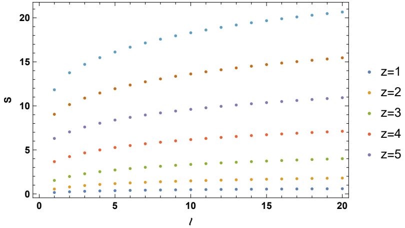
Correlations grow as the dynamical exponent increases

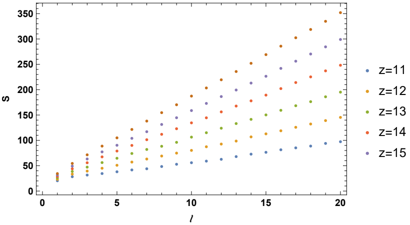
# Entanglement in Lifshitz Scalar Theory

# EE for Massless Scalar

- $1+1$-Dimensions

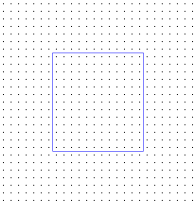


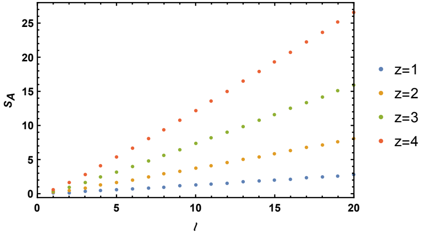



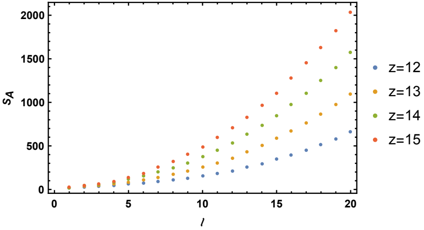



# EE for Massless Scalar

- 2 + 1-Dimensions (Square entangling region)

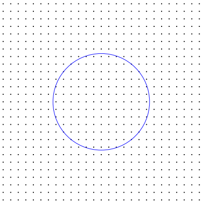


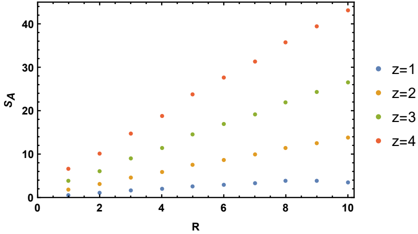


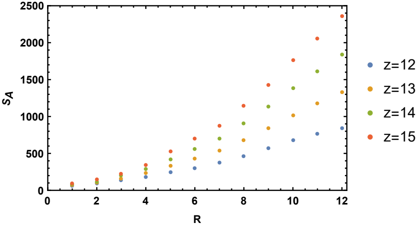



# EE for Massless Scalar

- $2+1$-Dimensions (Disk entangling region)





# Results

1. EE increases while the dynamical exponent is increased
2. For large values of $z$ and small enough entangling regions

   EE exhibits a volume law scaling!
3. For generic $z$, the leading term of EE is a function which interplay between area law and volume law scaling
   - The simplest choice

$$S_A \sim \left(\frac{\ell}{\epsilon}\right)^{d-\frac{1}{z}} + \cdots$$

# Results

- Kinetic Term in Hamiltonian $(\partial_i^z \phi)^2$
- Discretize Hamiltonian on a Lattice

$z = 1$ $\{\phi_{i+1}, \phi_i, \phi_{i-1}\} \in \mathcal{H}$

$z = 2$ $\{\phi_{i+2}, \phi_{i+1}, \phi_i, \phi_{i-1}, \phi_{i-2}\} \in \mathcal{H}$

.

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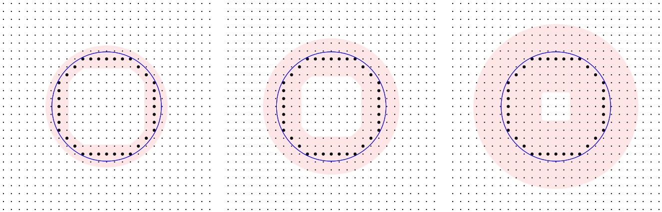
$z$ $\{\phi_{i+z}, \phi_{i+z-1}, \phi_{i+z-2}, \cdots, \phi_{i-z+2}, \phi_{i-z+1}, \phi_{i-z}\} \in \mathcal{H}$

# Results

- For larger values of $z$ the number of correlated points due to the kinetic term increases



$z = 1$    $z = 2$    $z = 4$

- The correlation between points inside and outside the entangling region increases

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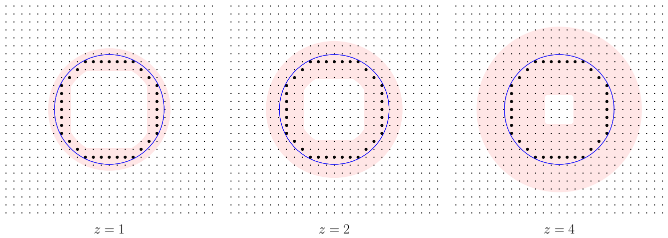



- The correlation between points inside and outside the entangling region increases

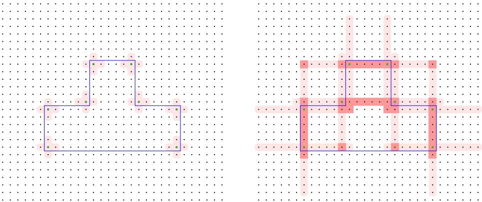
Non-local effects due to the nontrivial $z$

# Corner Contributions to EE

- For $z = 1$ corner contributions to EE are well-known to be local effects

$$S_A = \cdots + \mathcal{S}_{\text{corner}} \log \frac{\ell}{\epsilon} + \cdots$$

$$\mathcal{S}_{\text{corner}} = \sum_{\text{corners on } \partial \text{A}} a\left(\theta_i\right),$$



- Corner contributions are no more local effects for $z > 1$

# Conclusions

In a free scalar theory with Lifshitz scaling symmetry:

- EE is an increasing function of $z$
- For large values of $z$ and small enough entangling regions EE exhibits a volume law scaling
- Mutual information between subregions for a fixed configuration is an increasing function of $z$
- The tripartite information is positive (Holographic dual!)
- For large values of $z$ the corner contributions are no more additive

# Further Studies

- Considering other entanglement measures which are more suitable for mixed states, e.g., logarithmic negativity
- Study the time evolution of EE after a quantum quench
- Investigating the effects due to a non-zero mass on the entanglement spectrum
- . . .