The stress tensor of BMS invariant field theories

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Motivation

- Still open question: Extending gauge/gravity duality beyond the AdS/CFT correspondence.
- A related question: Holography of asymptotically flat spacetimes (AFS).
- Lessons of AdS/CFT: Asymptotic symmetry of asymptotically AdS spacetimes in (d+1)-dimensions is the same as conformal symmetry in d dimensions
- Asymptotic symmetry of AFS in three and four dimensions is infinite dimensional: BMS_3 and BMS_4 symmetry.
- One expects: The dual theory of AFS is BMS invariant.

Aspects of BMS Invariant Field Theories

- Non-trivial ASG for asymptotically Minkowski spacetimes at null infinity in three and four dimensions:
- BMS_3:

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [L_m, M_n] = (m-n)M_{m+n}, \quad [M_m, M_n] = 0. \tag{1}$$

[Ashtekar, Bicak, Schmidt 1996], [Barnich,Compere 2006]

- BMS_4:

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \quad [l_m, \bar{l}_n] = 0,$$
$$[l_l, T_{m,n}] = (\frac{l+1}{2} - m)T_{m+l,n}, \quad [\bar{l}_l, T_{m,n}] = (\frac{l+1}{2} - n)T_{m,n+l}.$$

[Bondi, van der Burg, Metzner; Sachs 1962], [Barnich,Troessaert 2010]

BMS Invariant Field Theories as contracted CFT

- $\mathcal{L}_n$ and $\bar{\mathcal{L}}_n$ are generators of ASG of asymptotically AdS_3:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \quad [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n}, \quad [\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0, \tag{2}$$

then

$$L_n = \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \quad M_n = \frac{1}{\ell}(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}) \tag{3}$$

ℓ is the radius of *AAdS*. In the limit $\ell \to \infty$, (2) results in (1). [Barnich,Compere 2006]

- What does $\ell \to \infty$ correspond to in the field theory side?
- Generators of two dimensional CFT on the cylinder:

$$\mathcal{L}_n = -e^{nw}\partial_w, \quad \bar{\mathcal{L}}_n = -e^{n\bar{w}}\partial_{\bar{w}} \quad (w = t + ix, \bar{w} = t - ix) \tag{4}$$

- Define $L_n = \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \quad M_n = \epsilon(\mathcal{L}_n + \bar{\mathcal{L}}_{-n})$.
- Use a spacetime contraction: $t \to \epsilon t, x \to x$.

BMS Invariant Field Theories as contracted CFT

- Final generators in the $\epsilon \to 0$ limit:

$$L_n = -e^{inx}(i\partial_x + nt\partial_t), \quad M_n = -e^{inx}\partial_t \tag{5}$$

- The resultant algebra in the $\epsilon \to 0$ limit is BMS_3 with central charges: $c_{LL} = C_1 = c - \bar{c}, \quad c_{LM} = C_2 = \epsilon(c + \bar{c})$
- In the level of algebra: $\ell \to \infty$ corresponds to contraction of time-coordinate.
- Our proposal: BMS invariant field theory is an ultra-relativistic field theory.
- Holographic dual of AFS in (d+1) is ultra relativistic field theories in one dimension lower.
  [A. Bagchi, R. F. (2012)]

A Cardy-like formula

- 3d asymptotically flat spacetimes $\longrightarrow$ states of field theory with BMS symmetry.
- The states are labelled by eigenvalues of L_0 and M_0:

$$L_0|h_L, h_M\rangle = h_L|h_L, h_M\rangle, \qquad M_0|h_L, h_M\rangle = h_M|h_L, h_M\rangle,$$

where

$$h_L = \lim_{\epsilon \to 0}(h - \bar{h}), \quad h_M = \lim_{\epsilon \to 0} \epsilon(h + \bar{h}) \tag{6}$$

- Modular invariance of CCFT partition function results in

$$S = \log d(h_L, h_M) = 2\pi \left(h_L \sqrt{\frac{C_M}{2h_M}} + C_L \sqrt{\frac{h_M}{2C_M}} \right) \tag{7}$$

[A. Bagchi, S. Detournay, R. F. , Joan Simon (2012)]

BMS_3 invariant field theory stress tensor: using contraction

- $\ell \to \infty$ in the gravity side corresponds to contraction in the field theory.
- Using contraction one can find the stress tensor of two dimensional BMS_3 invariant field theories.
- $T_{\mu\nu}$= stress tensor of original CFT,
  $\tilde{T}_{\mu\nu}$= contracted stress tensor
  $\pm$ = light cone coordinates :

$$\begin{aligned} \tilde{T}_{++} + \tilde{T}_{--} &= \lim_{\epsilon \to 0} \epsilon \left(T_{++} + T_{--} \right) \\ \tilde{T}_{++} - \tilde{T}_{--} &= \lim_{\epsilon \to 0} \left(T_{++} - T_{--} \right) \\ \tilde{T}_{+-} &= \lim_{\epsilon \to 0} \epsilon \, T_{+-}. \end{aligned} \tag{8}$$

[R.F. , Ali Naseh,2013]

- Are these correct?

BMS_3 invariant field theory stress tensor: using contraction

- According to dictionary, $\tilde{T}_{\mu\nu}$ is also the quasi-local stress tensor of bulk theory.
- Let us calculate it in the bulk side.
- First step is calculation of $T_{\mu\nu}$ in the context of AdS/CFT.
- The simple method is Brown and York's method for calculation of the quasi-local stress tensor of asymptotically AdS spacetimes:

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{\mu\nu}}, \tag{9}$$

- The gravitational action is

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} \left(R - \frac{2}{\ell^2} \right) - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{-\gamma}\, \mathcal{K} + \frac{1}{8\pi G} S_{ct}(\gamma_{\mu\nu}), \tag{10}$$

where

$$S_{ct} = -\frac{1}{\ell} \int_{\partial\mathcal{M}} d^2x \sqrt{-\gamma}. \tag{11}$$

BMS_3 invariant field theory stress tensor: using contraction

- We need a proper coordinate with well-defined flat-space limit.
- A choice is BMS gauge:

$$ds^2 = \left(-\frac{r^2}{\ell^2} + \mathcal{M}\right) du^2 - 2dudr + 2\mathcal{N}dud\phi + r^2 d\phi^2, \quad (12)$$

where

$$\mathcal{M}(u,\phi) = 2\left(\chi(x^+) + \bar{\chi}(x^-)\right), \qquad \mathcal{N}(u,\phi) = \ell\left(\chi(x^+) - \bar{\chi}(x^-)\right), \quad (13)$$

and $\chi, \bar{\chi}$ are arbitrary functions of $x^{\pm} = \frac{u}{\ell} \pm \phi$.

- The flat-space limit is well-defined:

$$ds^2 = Mdu^2 - 2dudr + 2Ndud\phi + r^2 d\phi^2. \quad (14)$$

where

$$M = \lim_{\frac{G}{\ell}\to 0} \mathcal{M} = \theta(\phi), \qquad N = \lim_{\frac{G}{\ell}\to 0} \mathcal{N} = \beta(\phi) + \frac{u}{2}\theta'(\phi), \quad (15)$$

BMS$_3$ invariant field theory stress tensor: using contraction

- The non-zero components of stress tensor at the boundary are

$$T_{uu} = \frac{\mathcal{M}}{16\pi G\ell}, \qquad T_{u\phi} = \frac{\mathcal{N}}{8\pi G\ell}, \qquad T_{\phi\phi} = \frac{\ell\mathcal{M}}{16\pi G}. \tag{16}$$

- In the light-cone coordinate:

$$\begin{aligned} T_{++} &= \frac{\ell}{8\pi G}\chi(x^+), \qquad T_{--} = \frac{\ell}{8\pi G}\bar{\chi}(x^-) \\ T_{+-} &= 0 \end{aligned} \tag{17}$$

- Using (8) we find

$$\tilde{T}_{uu} = \frac{M}{16\pi G^2}, \qquad \tilde{T}_{u\phi} = \frac{N}{8\pi G^2}, \qquad \tilde{T}_{\phi\phi} = \frac{M}{16\pi}. \tag{18}$$

- Comparison of (16) and (18) is instructive: In the gravity side one can scale the components of stress tensor by some appropriate powers of ℓ and then take the flat-space limit.

Conservation of BMS_3 stress tensor

- It is not difficult to check that stress tensor components (18) do not satisfy the conservation relation $\nabla_\mu \bar{T}^{\mu\nu} = 0$.
- There is a possibility of assuming non-symmetric stress tensors such that $\bar{T}_{u\phi}$ is non-zero and is given by (18) but $\bar{T}_{\phi u} = 0$.
- Then the standard conservation relation is satisfied.
- The fact that BMS invariant theories are not Poincare invariant makes this assumption reliable.

Conserved charges using Stress Tensor

- Using above stress tensor one can calculate the conserved charges:

$$Q_\xi = \int_\Sigma d\phi \sqrt{\sigma} v^\mu \xi^\nu \tilde{T}_{\mu\nu}, \tag{19}$$

- The geometry of spacetime which BMS invariant field theory lives on it, is the same as parent conformal field theory.
- This needs that we define an anisotropic conformal infinity.
- The final result is consistent with the known results.

BMS_3 invariant field theory stress tensor: A direct method

- It is possible to forget contraction and define a stress tensor for BMS invariant field theories directly.
- In this calculation we use the fact that BMS invariant field theories in d-dimensions are dual of asymptotically flat space times in d+1 dimensions.
- BMS invariant field theory lives on a spacetime which is given by anisotropic scaling of conformal infinity.
- Let us assume a two dimensional BMS_3 invariant field theory lives on

$$ds^2 = -du^2 + R^2 d\phi^2 \tag{20}$$

R is the radius of the cylinder.

BMS_3 invariant field theory stress tensor: A direct method

- Starting point is the formula which gives the conserved charges of symmetry generators ξ:

$$Q_\xi = R \int_0^{2\pi} d\phi J^u = R \int_0^{2\pi} d\phi T^{u\mu} \xi_\mu, \tag{21}$$

where J^μ is the symmetry current and $T^{\mu\nu}$ is the stress tensor.

- We just define a BMS invariant field theory by using its symmetry and details of the theory are not important.
- For a BMS_3 invariant theory that lives on the cylinder one can introduce a representation for the generators of BMS_3 algebra as

$$L_n = ie^{in\phi}\left(\partial_\phi + inu\partial_u\right), \qquad M_n = ie^{in\phi}\partial_u. \tag{22}$$

BMS_3 invariant field theory stress tensor: A direct method

- Thus we can write

$$Q_{M_n} = -iR \int_0^{2\pi} d\phi \, e^{in\phi} \, T^{uu},$$
$$Q_{L_n} = R \int_0^{2\pi} d\phi \, e^{in\phi} \left(nuT^{uu} + iR^2 T^{u\phi} \right). \tag{23}$$

- Using the orthogonality condition of Fourier modes, we can find T^{uu} and $T^{u\phi}$ from (23) as

$$T^{uu} = \frac{i}{2\pi R} \sum_n Q_{M_n} e^{-in\phi}$$
$$T^{u\phi} = \frac{-i}{2\pi R^3} \sum_n e^{-in\phi} \left(Q_{L_n} - iunQ_{M_n} \right) \tag{24}$$

- The other components must be determined by using the conservation and traceless-ness conditions.

BMS_3 invariant field theory stress tensor: A direct method

- we can find an interpretation for the charges Q_{M_n} and Q_{L_n} in the bulk side as the charges corresponding to the asymptotic symmetry generators.
- Using covariant phase space method in the bulk side we have

$$
\begin{aligned}
Q_{M_n} &= \frac{i}{16\pi G}\int_0^{2\pi} d\phi\, e^{in\phi}\theta(\phi) + \frac{i}{8G}\delta_n^0, \\
Q_{L_n} &= \frac{i}{8\pi G}\int_0^{2\pi} d\phi\, e^{in\phi}\chi(\phi).
\end{aligned} \tag{25}
$$

- The shift in the first line of (25) is necessary in order to the Poisson bracket of the charges produce the correct coefficient for the central term in the BMS_3 algebra.
- The interesting point here is that with this shift of charges we have $Q_{M_0} = Q_{L_0} = 0$ for the Minkowski metric.

BMS_3 invariant field theory stress tensor: A direct method

- Substituting (25) in (24) one can find components of the stress tensor as follows:

$$T^{uu} = -\frac{1}{16\pi GR}\left(1+\theta(\phi)\right),$$
$$T^{u\phi} = \frac{1}{8\pi GR^3}\left(\chi(\phi)+\frac{u}{2}\theta'(\phi)\right). \tag{26}$$

- This result is consistent with those of contraction method upto the shift in the T^{uu} component which is a result of shift in the definition of charges.
- Assuming non-symmetric stress tensors, we can calculate other components by using traceless-ness and conservation relation of stress tensor:[M. Asadi, O. Baghchesaraei, R. F. (2017)]

$$T_{uu} = -\frac{1}{16\pi GR}\left(1+\theta(\phi)\right),$$
$$T_{u\phi} = -\frac{1}{8\pi GR}\left(\chi(\phi)+\frac{u}{2}\theta'(\phi)\right),$$
$$T_{\phi\phi} = -\frac{R}{16\pi G}\left(1+\theta(\phi)\right), \; T_{\phi u} = 0. \tag{27}$$

Correlators of BMS_3 invariant field theory stress tensor

- We can use (27) and impose the invariance under the global part of BMS_3 to calculate the correlation function of stress tensor.
- The global part is generated by $\{L_0, L_{\pm 1}, M_0, M_{\pm 1}\}$.
- Components of the stress tensor are written in terms of two functions $\theta(\phi)$ and $\chi(\phi)$.
- An infinitesimal coordinate transformation generated by BMS_3 generators changes these functions to $\theta + \delta\theta$ and $\chi + \delta\chi$:

$$\delta_\xi \theta = Y\theta' + 2Y'\theta - 2Y''',$$
$$\delta_\xi \chi = \frac{1}{2}T\theta' + Y\chi' + 2Y'\chi + T'\theta - T'''. \qquad (28)$$

- We apply (28) in the gravity side to find the variation of the stress tensor in the boundary.

Correlators of BMS_3 invariant field theory stress tensor

- Imposing conditions

$$\delta_{M_n}\langle T_{ij}\rangle = 0, \qquad \delta_{L_n}\langle T_{ij}\rangle = 0, \qquad n = 0, \pm 1 \tag{29}$$

result in

$$\langle T_{ij}\rangle = 0, \tag{30}$$

- We want to determine the p-point functions by imposing

$$\delta_{M_n}\langle T_{ij}^1 \cdots T_{kl}^p\rangle = 0, \qquad \delta_{L_n}\langle T_{ij}^1 \cdots T_{kl}^p\rangle = 0, \qquad n = 0, \pm 1 \tag{31}$$

where $T_{ij}^l = T_{ij}(u_l, \phi_l)$.

- we find the following universal forms with to constants C_1 and C_2 related to the central charges of BMS_3 algebra.[M. Asadi, O. Baghchesaraei, R. F. (2017)]

$$\langle T_{uu}^1 T_{u\phi}^2 \cdots T_{u\phi}^p\rangle \propto C_2 \frac{e^{2i\sum_{k=1}^p \phi_k}}{\prod_{1\leq l<m\leq p}\left(e^{i\phi_l} - e^{i\phi_m}\right)^{\frac{4}{p-1}}}$$

$$\langle T_{u\phi}^1 T_{u\phi}^2 \cdots T_{u\phi}^p\rangle \propto \left(C_1 + \frac{C_2}{2}\sum_{k=1}^{p} u_k \partial_k\right) \frac{e^{2i\sum_{k=1}^p \phi_k}}{\prod_{1\leq l<m\leq p}\left(e^{i\phi_l} - e^{i\phi_m}\right)^{\frac{4}{p-1}}} \tag{32}$$

First step to BMS_4: Stress Tensor of Kerr Black Hole

- The previous method for construction of the tensor correlators can be used for the BMS_4 invariant field theories.
- They are three dimensional field theories which have the BMS_4 symmetry:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}, \\
[L_l, M_{m,n}] &= (\frac{l+1}{2} - m)M_{m+l,n}, \quad [\bar{L}_l, M_{m,n}] = (\frac{l+1}{2} - n)M_{m,n+l}.
\end{aligned} \tag{33}
$$

- BMS_4 is the asymptotic symmetry of the four dimensional asymptotically flat spacetimes. Thus we assume that BMS_4 invariant theories are dual to these spacetimes.

First step to BMS_4: Stress Tensor of Kerr Black Hole

- As the first step we consider Kerr black holes and propose a stress tensor for them.
- Similar to three dimensional case we can follow to methods. Take limit from Kerr-AdS or directly calculate stress tensor for Kerr.
- Take limit from the holographic calculations of Kerr-AdS black hole:

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \tag{34}$$

$$+\frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2, \tag{35}$$

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{\ell^2}\right) - 2MGr, \quad \Delta_\theta = 1 - \frac{a^2}{\ell^2}\cos^2\theta,$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{\ell^2}. \tag{36}$$

M is the mass of the black hole and $a = J/M$ where J is the

First step to BMS_4: Stress Tensor of Kerr Black Hole

- According to the AdS/CFT correspondence, the non-zero components of the Kerr-AdS stress tensor are

$$
\begin{aligned}
8\pi T_{tt} &= \frac{2M}{r\ell}, \\
8\pi T_{t\phi} &= -\frac{2aM}{r\ell\Xi}\sin^2\theta, \\
8\pi T_{\theta\theta} &= \frac{M\ell}{r\Delta_\theta}, \\
8\pi T_{\phi\phi} &= \frac{M\ell}{r\Xi^2}\sin^2\theta\left(\Xi + \frac{3a^2\sin^2\theta}{\ell^2}\right).
\end{aligned} \tag{37}
$$

- $\ell \to \infty$ is not well-defined! However applying proper powers of ℓ to the components of the stress tensor makes the flat limit well defined.

First step to BMS_4: Stress Tensor of Kerr Black Hole

- Our proposal for the Kerr stress tensor:

$$
\begin{aligned}
8\pi\tau_{tt} &= \frac{2M}{r\sqrt{G}}, \\
8\pi\tau_{t\varphi} &= -\frac{3aM}{r\sqrt{G}}\sin^2\theta, \\
8\pi\tau_{\theta\theta} &= \frac{M\sqrt{G}}{r}, \\
8\pi\tau_{\varphi\varphi} &= \frac{M\sqrt{G}}{r}\sin^2\theta.
\end{aligned} \tag{38}
$$

[O. Baghchesaraei, R. F. , Y. Izadi (2016)]

- The anisotropic conformal boundary is given by

$$
d\tilde{s}^2 = \frac{r^2}{G}\left[-dt^2 + Gd\theta^2 + G\sin^2\theta d\varphi^2\right]. \tag{39}
$$

- Using (38) and (38) in the Brown and York formula results in the correct charges of Kerr!

First step to BMS_4: Stress Tensor of Kerr Black Hole

- Why is (38) correct?
- Let us forget (38) and try to directly calculate the stress tensor of Kerr by using flat space holography.
- Consider BMS_4 field theory on a flat manifold with metric

$$d\tilde{s}^2 = \frac{r^2}{G}\left[-dt^2 + G d\theta^2 + G\sin^2\theta d\varphi^2\right] = \frac{r^2}{G}\left[-dt^2 + \frac{4G\,dx\,dy}{(1+xy)^2}\right]. \tag{40}$$

where x and y are defined by

$$x = e^{i\phi}\cot\frac{\theta}{2}, \qquad y = e^{-i\phi}\cot\frac{\theta}{2}. \tag{41}$$

and r is just a conformal factor.

First step to BMS_4: Stress Tensor of Kerr Black Hole

- A representation for the BMS_4 algebra:

$$
\begin{aligned}
L_n &= \frac{1}{2}\left(\frac{xy-1}{xy+1}-n\right)x^n t\partial_t - x^{n+1}\partial_x, \\
\bar{L}_n &= \frac{1}{2}(\frac{xy-1}{xy+1}-n)y^n t\partial_t - y^{n+1}\partial_y, \\
M_{m,n} &= \frac{2}{1+xy}x^m y^n \partial_t.
\end{aligned} \tag{42}
$$

- The starting point is definition of the conserved charges in the field theory,

$$
Q = \frac{r^3}{\sqrt{G}}\int d\theta\, d\phi\, \sin\theta\, J^t = \frac{r^3}{\sqrt{G}}\int d\theta\, d\phi\, \sin\theta\, T^{t\mu}\xi_\nu \tag{43}
$$

- Using (40) and (42) we have

$$
Q_{M_{m,n}} = \frac{2r^5}{G\sqrt{G}}\int d\theta\, d\phi\, \sin\theta \frac{x^m\, y^n}{1+xy} T^{tt}. \tag{44}
$$

First step to BMS_4: Stress Tensor of Kerr Black Hole

- Using

$$\sum_m x^{m-\frac{1}{2}} x'^{-m-\frac{1}{2}} = 2\pi i \delta(x - x'),$$
$$\sum_m y^{m-\frac{1}{2}} y'^{-m-\frac{1}{2}} = 2\pi i \delta(y - y'), \tag{45}$$

we can simplify the above equation and write

$$T^{tt} = \frac{G\sqrt{G}(1+xy)^3}{16\pi^3 i r^5} \sum_m \sum_n Q_{M_{m,n}} x^{-m-1} y^{-n-1}. \tag{46}$$

- The charges associated to $M_{m,n}$ for the Kerr black hole:

$$Q_{M_{m,n}} = \frac{M}{2\pi} \int d^2\Omega \frac{x^m y^n}{(1+xy)} \tag{47}$$

[G. Barnich and C. Troessaert,2011]

- If we substitute (47) in (46) and use (45) we will find a result which is in agreement with (38).
- There are similar checks for other components of stress tensor.[O. Baghchesaraei, R. F. , Y. Izadi (2016)]

Concluding remarks and future direction

- If we accept that the BMS invariant theories are ultra-relativistic theories then their stress tensor are non-symmetric.
- In the flat-space holography, the dual field theory lives on a spacetime given by anisotrophic scaling of conformal boundary.
- we expect that correlations of stress tensor for the three dimensional BMS_4 field theory is determined by a method similar to what we present in this talk. [Work in progress in collaboration with M. Asadi]

Thank you