Homogeneous Solutions of Minimal Massive 3D Gravity

Nihat Sadik Deger

Bogazici University

IPM, 8 May 2017

Jumageldi Charyyev, N.S.D.
arXiv:1703.06871

- The MMG model
- Method
- Some examples
- Conclusion

Model: Minimal Massive 3D Gravity

E. Bergshoeff, O. Hohm, W. Merbis, A.J. Routh and P.K. Townsend, *Minimal Massive 3D Gravity*, Class.Quant.Grav. 31 (2014) 145008, arXiv:1404.2867.

- Three dimensions is a useful laboratory.

- AdS_3/CFT_2 is better understood.

The theory is defined by the field equation

$$G_{\mu\nu} + ag_{\mu\nu} + bC_{\mu\nu} + cJ_{\mu\nu} = 0\,,$$

where $G_{\mu\nu}$ is the Einstein tensor and the Cotton tensor $C_{\mu\nu}$, which is symmetric, traceless and covariantly conserved, is related to the Schouten tensor $S_{\sigma\nu}$ as

$$C^{\mu}{}_{\nu} \equiv \frac{1}{\sqrt{-g}}\varepsilon^{\mu\rho\sigma}\nabla_{\rho}S_{\sigma\nu},\ \ S_{\sigma\nu} \equiv R_{\sigma\nu} - \frac{1}{4}Rg_{\sigma\nu}\,,$$

with $\varepsilon_{012} = +1$.

The J-tensor is given as

$$J^{\mu\nu} \equiv R^{\mu\rho} R^{\nu}{}_{\rho} - \frac{3}{4} R^{\mu\nu} R - \frac{1}{2} g^{\mu\nu} (R^{\rho\sigma} R_{\rho\sigma} - \frac{5}{8} R^2) \, .$$

It is not covariantly conserved, but instead one finds:

$$\sqrt{-g} \nabla_\mu J^{\mu\nu} = \varepsilon^{\nu\rho\sigma} S_\rho{}^\tau C_{\sigma\tau} \, ,$$

which is not automatically zero. It follows that the MMG field equation cannot be derived from an action that contains only the metric field. However, for any solution of the field equation one can show that the right hand side of the above vanishes, which establishes the consistency of the model in a novel way.

The coefficients a, b and c in terms of physical parameters are

$$a = \frac{\bar{\Lambda}_0}{\bar{\sigma}},\ b = \frac{1}{\mu\bar{\sigma}},\ c = \frac{\gamma}{\mu^2\bar{\sigma}}.$$

When $\gamma = 0$ (i.e., $c = 0$) the model reduces to the (cosmological) TMG model:

S. Deser, R. Jackiw and S. Templeton, *Topologically Massive Gauge Theories*, Ann.Phys. 140 (1982) 372-411.

There are two special points in the parameter space of the MMG theory. The first is called the **chiral point** at which one of the boundary central charges vanish. It is given by:

$$
\begin{aligned}
\bar{\sigma} + \frac{\gamma}{2}(\bar{\sigma}^2 - \frac{\gamma\bar{\Lambda}_0}{\mu^2}) \pm \sqrt{\bar{\sigma}^2 - \frac{\gamma\bar{\Lambda}_0}{\mu^2}} &= 0 \text{ , or} \\
1 + \frac{c}{2b^2}(1 - ac) \pm \sqrt{1 - ac} &= 0.
\end{aligned}
$$

The second one is called the **merger point** where two possible values of the effective cosmological constant of maximally symmetric vacua coincide.

$$
G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \Rightarrow \gamma\Lambda^2 - 4\mu^2\bar{\sigma}\Lambda + 4\mu^2\bar{\Lambda}_0 = 0
$$

Roots are the same if

$$
\bar{\Lambda}_0 = \frac{\mu^2\bar{\sigma}^2}{\gamma} \quad \text{or} \quad ac = 1.
$$

It is 'minimal' since there is only one propagating spin-2 mode in the bulk like in TMG. But unlike TMG, it avoids the bulk-boundary unitarity clash since it is possible to have both central charges of the dual CFT and energy of the bulk graviton positive. The MMG parameters can be expressed in terms of those of TMG as:

$$\begin{aligned} \bar{\sigma} &= 1 + \alpha + \frac{\alpha^2 \Lambda_0}{2\mu^2(1+\alpha)^2} , \\ \gamma &= -\frac{\alpha}{(1+\alpha)^2} , \\ \bar{\Lambda}_0 &= \Lambda_0 \left(1 + \alpha - \frac{\alpha^3 \Lambda_0}{4\mu^2(1+\alpha)^2} \right) , \end{aligned}$$

where α is a dimensionless parameter such that one gets TMG in the $\alpha \to 0$ limit. For bulk and boundary unitarity we need:

$$-1 < \alpha < 0 \, , \; \Lambda_0 < \frac{4\mu^2(1+\alpha)^3}{\alpha^3} \, .$$

A.S. Arvanitakis and P.K. Townsend, *Minimal Massive 3D Gravity Unitarity Redux*, Class.Quant.Grav. 32 (2015) 085003, arXiv:1411.1970.

Homogeneous Solutions

A homogeneous spacetime M has the form of a quotient G/H, where G is its group of isometries, which is a Lie group, and H is a closed subgroup of G.

We are interested in the simply transitive case, that is we take H to be just the identity so that the stability group of any point is trivial.

In this case M and G can be identified and considering left action of such a Lie group on itself one can construct its left-invariant metric up to automorphisms using left-invariant 1-forms.

This method was successfully used for the cosmological TMG model in:

G. Moutsopoulos, *Homogeneous anisotropic solutions of topologically massive gravity with cosmological constant and their homogeneous deformations*, Class.Quant.Grav. 30 (2013) 125014, arXiv:1211.2581.

Homogeneous solutions of TMG with cosmological constant zero were studied with a similar approach earlier in:

M.E. Ortiz, *Homogeneous Solutions to Topologically Massive Gravity*, Ann.Phys. 200 (1990) 345.

Method

• First a Lie algebra basis is fixed for each three-dimensional Lie algebra $\mathfrak{g}$ which induces left-invariant Maurer-Cartan 1-forms.

• A left-invariant metric for the Lie group at the identity is identified by a non-degenerate metric on the Lie algebra up to automorphism group of this Lie algebra $[e_i, e_j] = C^k_{ij} e_k$. The metric $g_{ij} =< e_i, e_j >$ is put into a simple form using automorphisms which are invertible linear transformations $A : \mathfrak{g} \to \mathfrak{g}$ such that $A([e_i, e_j]) = [A(e_i), A(e_j)]$ for all basis vectors $\{e_i\}$.

The metric is expressed in terms of left-invariant 1-forms with constant coefficients, which implies that all curvature calculations, and hence the MMG field equation, become algebraic.

Instead of fixing the Lie algebra basis, alternatively an orthonormal frame may be chosen.

The classification of three-dimensional Lie algebras was done by Bianchi:

L. Bianchi, *Sugli spazii a tre dimensioni che ammettono un gruppo continuo di movimenti*, Soc.Ital.Sci.Mem. di Mat. 11 (1898) 267.

Besides the abelian $\mathbb{R}^3$ and the two familiar algebras $\mathfrak{sl}_2$ and $\mathfrak{su}_2$, we also have the Lie algebras $\mathfrak{a}_\infty$ and $\mathfrak{a}_0$, and two continuous families of Lie algebras: $\mathfrak{iso}(1,1;\theta)$ and $\mathfrak{iso}(2;\theta)$ where parameter θ varies in $(0, \frac{\pi}{2}]$. In the first, θ values $\{0, \frac{\pi}{4}\}$ are special and should be considered separately which in total leads to 9 Bianchi classes. One can systematically go through this list and search for simply transitive homogeneous solutions of a three dimensional theory.

Instead of solving algebraic equations for the constants in the metric $\{u, v, w, ...\}$ in terms of the parameters $\{a, b, c\}$ of the MMG theory it is more convenient and illuminating to display the parameters of theory as functions of the metric parameters. This amounts to solving a system of linear equations of the form

$$A \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = V$$

for $\{a, b, c\}$ where A is a matrix is of the dimension $k \times 3$ and V is a $k \times 1$ vector with $k = 3, 4$ or 6. The number k is determined by the number of independent components of the field equation.

The rank of the matrix A can be at most three. The cases when the rank of A is less than three should be considered separately since new solutions may arise.
We focus on Lorentzian metrics with mostly plus signature.

Solutions on $SL(2,\mathbb{R})$

For the Lie algebra $\mathfrak{sl}_2$ a basis $\{\tau_0, \tau_1, \tau_2\}$ can be fixed as

$$[\tau_0, \tau_1] = \tau_2,\ [\tau_2, \tau_1] = \tau_0,\ [\tau_2, \tau_0] = \tau_1.$$

Let θ^a be a dual basis of τ_a.

Elements of $SL(2,\mathbb{R})$ can be parametrized by a group representative as:

$$\mathcal{V} = e^{t(\tau_0+\tau_2)} e^{\sigma\tau_1} e^{\zeta\tau_2}.$$

It follows that the Maurer-Cartan one-forms are

$$\begin{aligned} \mathcal{V}^{-1} d\mathcal{V} &= (e^{\sigma}\cosh\zeta dt - \sinh\zeta d\sigma)\tau_0 \\ &+ (\cosh\zeta d\sigma - e^{\sigma}\sinh\zeta dt)\tau_1 + (d\zeta + e^{\sigma} dt)\tau_2. \end{aligned}$$

There are 4 classes of left-invariant metrics on $SL(2,\mathbb{R})$:

- 111-type metric is of the form

$$g = u\theta^0\theta^0 + v\theta^1\theta^1 + w\theta^2\theta^2\,,$$

where $uw < 0$ and $v > 0$ for Lorentzian, mostly plus signature. The case $-u = v = w$ corresponds to the round AdS_3.

The general line element is

$$\begin{aligned} ds^2 &= e^{2\sigma}(u\cosh^2\zeta + v\sinh^2\zeta)dt^2 + (u\sinh^2\zeta + v\cosh^2\zeta)d\sigma^2 \\ &- 2(u+v)e^{\sigma}\cosh\zeta\sinh\zeta\, dt d\sigma + w(d\zeta + e^{\sigma}dt)^2\,. \end{aligned}$$

The coefficients a, b, c and the scalar curvature R in terms of u, v, and w are

$$\begin{aligned}
a &= \frac{1}{Q}\cdot\frac{[(u+v+w)^2-4vw]^3}{8uvw}\,,\\
b &= -\frac{1}{Q}\cdot 8\sqrt{-uvw}[u^2-(v-w)^2](u+v+w)\,,\\
c &= \frac{1}{Q}\cdot 8uvw[(u+v+w)^2-4vw]\,,\\
R &= -\frac{(u+v+w)^2-4vw}{2uvw} = -\frac{c\,Q}{16(uvw)^2} = -\frac{(a\,Q)^{1/3}}{(uvw)^{2/3}}\,,
\end{aligned}$$

where

$$Q = [(u+v+w)^2-4vw]^2 + 8[u^2-(v+w)^2][u^2-(v-w)^2]\,.$$

This solution in general represents a triaxially deformed AdS spacetime. Note that when $c = 0$, i.e. for TMG, $R = 0$. In this case, the cosmological constant a vanishes as well.

The matrix A is a 3×3 matrix with the determinant

$$detA = -Q \cdot \frac{(u+v)(v-w)(u+w)}{8(-uvw)^{5/2}}.$$

Thus the cases $Q = 0$, $u = -v$ (or equivalently $u = -w$) and $v = w$ should be considered separately. The case $Q = 0$ does not give rise to any solution.

u=-v: In this case the spacetime metric becomes:

$$ds^2 = v[-e^{2\sigma}dt^2 + d\sigma^2] + w(d\zeta + e^{\sigma}dt)^2 \equiv g_{(2)} + w(d\zeta + \chi)^2\,,$$

where $w > 0$. This spacetime is called the spacelike warped AdS.

$$\begin{aligned} a &= \frac{16v^2(w-4v) + c(4v-7w)(4v-3w)}{192v^4}, \\ b &= \frac{8v^2 + c(4v-3w)}{12\sqrt{v^2 w}} \end{aligned}$$

This solution was found before in:

A.S. Arvanitakis, A.J. Routh and P.K. Townsend, *Matter coupling in 3D 'minimal massive gravity*, Class.Quant.Grav. 31 (2014) 235012, arXiv:1407.1264.

- 12-type metric

$$g = v(-\theta^0\theta^0 + \theta^1\theta^1) + w\theta^2\theta^2 + z(\theta^0 + \theta^1)^2\,,$$

with $z \neq 0$, $v > 0$ and $w > 0$. Here, z can be scaled to ± 1. Notice that it is a z-deformation of the spacelike warped AdS metric.

$$\begin{aligned} a &= \frac{1}{Q}\cdot\frac{(w-4v)^3}{8v^2}\,,\\ b &= \frac{1}{Q}\cdot 8(2v-w)\sqrt{v^2 w}\,,\\ c &= \frac{1}{Q}\cdot 8v^2(w-4v)\,,\\ R &= \frac{w-4v}{2v^2}\,,\end{aligned}$$

where $Q = -(w-4v)^2 + 8(4v^2 - w^2)$. Note that in the TMG limit, i.e. $c = 0$, both the scalar curvature and cosmological constant a vanish.

After some coordinate transformations we obtain

$$ds^2 = d\rho^2 + 2dydx + (\frac{R}{2} + \frac{3k^2}{4l^2})x^2dy^2 + \frac{2k}{l}xd\rho dy + \frac{z}{l^4}e^{-\frac{2}{kl}\rho}dy^2\,,$$

where $v = l^2$ and $w = k^2l^2$ with $k > 0$. This solution is Kundt type and corresponds to a special case found in:

N.S.D. and O. Sarioglu, *Kundt solutions of Minimal Massive 3D Gravity*, Phys.Rev. D92 (2015) 104015, arXiv:1505.03387.

Only $Q = 0$ and $v = w$ cases should be considered separately, and the first does not provide any solution.

v=w: Here the coefficients a and b are equal to

$$a = -\frac{16w + c}{64w^2}, \;\; b = \frac{8w + c}{12\sqrt{w}}.$$

After a coordinate change the metric becomes

$$ds^2 = d\rho^2 + 2e^{-\frac{\rho}{l}} dy d\theta + \frac{z}{l^4} e^{-\frac{2\rho}{l}} dy^2 \,,$$

which corresponds to the null warped AdS (Schrödinger) spacetime that was obtained before.

Note that in all the above examples when $b = 0$ we get the merger point condition $ac = 1$.

- Type-3 metric

$$g = v(-\theta^0\theta^0 + \theta^1\theta^1 + \theta^2\theta^2) + z(\theta^0\theta^2 + \theta^1\theta^2)\,,$$

with $z \neq 0$ and $v > 0$. Note that it is a z-deformation of the round AdS. It is a solution of MMG with

$$a = -\frac{9}{40v},\; b = \frac{8\sqrt{v}}{15},\; c = -\frac{8v}{5},\; R = -\frac{3}{2v}.$$

The z-deformation has no affect on the scalar curvature which is the same as round AdS_3. Moreover, the solution is attained at the chiral point with the plus sign. After some coordinate transformations we obtain

$$ds^2 = d\rho^2 + 2dydx + (\frac{2x}{l} + \frac{z}{l^3}e^{-\rho/l})dyd\rho + \frac{z}{l^4}e^{-\rho/l}xdy^2\,,$$

where $v = l^2$ which is a Kundt solution found before.

Solutions on A_0

The Lie algebra $\mathfrak{a}_0$ of A_0, spanned by r, x, and y, has non-vanishing brackets

$$[r, x] = x,\ [r, y] = x + y.$$

We denote the dual basis as $\{\tilde{r}, \tilde{x}, \tilde{y}\}$. The Maurer-Cartan one-forms are

$$\mathcal{V}^{-1} d\mathcal{V} = (e^{-\alpha} d\xi - \alpha e^{-\alpha} d\rho)x + (e^{-\alpha} d\rho)y + (d\alpha)r.$$

There are 4-types of metrics.

- B_2-type metric

$$B_2 = z\tilde{r}^2 \pm 2\tilde{x}\tilde{y}\,,$$

with $z > 0$. The MMG field equation is satisfied if

$$a = -\frac{4z + c}{4z^2}, b = \mp\frac{2z + c}{2\sqrt{z}}, R = -\frac{6}{z}.$$

Note that the solution is attained at the chiral point. After some coordinate transformations the metric becomes

$$ds^2 = \mp\frac{l^2}{w^2}[2\log(w)(dx^+)^2 + 2dx^+dx^- \mp dw^2]\,,$$

which is the logarithmic pp-wave solution found in:

M. Alishahiha, M.M. Qaemmaqami, A. Naseh and A. Shirzad, *On 3D Minimal Massive Gravity*, JHEP 1412 (2014) 033, arXiv:1409.6146.

Solutions on $ISO(1,1;\theta)$

The basis $\{l, m_1, m_2\}$ of $\mathfrak{iso}(1,1;\theta)$ has the brackets

$$[l, m_1] = 2\cos\theta m_1 + 2\sin\theta m_2,$$

$$[l, m_2] = 2\cos\theta m_2 + 2\sin\theta m_1\,,$$

and the dual basis is $\{\tilde{l}, \tilde{m}_1, \tilde{m}_2\}$. The Maurer-Cartan one-forms are

$$\mathcal{V}^{-1}d\mathcal{V} = e^{-2\rho\cos\theta}[\cosh(2\rho\sin\theta)dx - \sinh(2\rho\sin\theta)dy]m_1$$
$$+(d\rho)l + e^{-2\rho\cos\theta}[\cosh(2\rho\sin\theta)dy - \sinh(2\rho\sin\theta)dx]m_2$$

with $\theta \in [0, \pi/2]$. There are two types of metrics.

- B_1-type metric

$$B_1 = \delta\tilde{l}\tilde{l} + u(\tilde{m}_1 + \tilde{m}_2)^2 + v(\tilde{m}_1 - \tilde{m}_2)^2 + 2w(\tilde{m}_1\tilde{m}_1 - \tilde{m}_2\tilde{m}_2)$$

with $w^2 > uv$ and $\delta > 0$. Two of the parameters (u, v, w) can be set to ± 1 whenever they are non-zero. The curvature scalar is

$$R = -\frac{8[3(uv - w^2)\cos^2\theta + uv\sin^2\theta]}{\delta(uv - w^2)}.$$

There is no general solution for a, b, c. However, the cases $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$, and $uv = 0$ should be considered separately.

$\theta = \frac{\pi}{4}$: When $w = 0$, the Cotton tensor vanishes identically and we are at the merger point with $c = -\delta/4 = 1/a = 4/R$.

$$ds^2 = \delta d\rho^2 + u e^{-4\sqrt{2}\rho}(dx + dy)^2 + v(dx - dy)^2\,,$$

The metric becomes $(A)dS_2 \times S^1$ that was found before, which is clearly not a solution of TMG since c cannot be zero. It is not a solution away from the merger point.

$\theta = \frac{\pi}{2}$: In this case we have

$$\begin{aligned} a &= \frac{2u^2v^2}{\delta(uv + 8w^2)(uv - w^2)}, \quad b = -\frac{2w\sqrt{\delta(w^2 - uv)}}{uv + 8w^2}, \\ c &= \frac{\delta(uv - w^2)}{2(uv + 8w^2)}. \end{aligned}$$

After some coordinate transformations its spacetime metric becomes:

$$ds^2 = \frac{\delta}{4}\frac{dr^2}{r^2} + \frac{d\alpha^2}{r^2} - r^2 dt^2 + 2w d\alpha dt\,,$$

where we set $u = 1$, $v = -1$.
Notice that the following constant rescalings

$$r \to \lambda r,\ \alpha \to \lambda\alpha,\ t \to \lambda^{-1} t$$

leave the metric invariant. When $w = 0$ (which sets $b = 0$ and the merger point condition is satisfied), this corresponds to the static Lifshitz spacetime with the dynamical exponent $z = -1$, and for $w \neq 0$ it is a stationary Lifshitz metric. Note that the rotation parameter w is non-zero only when there is a contribution from the Cotton tensor.

- B_2-type metric

$$B_2 = \delta \tilde{l}\tilde{l} + \tilde{l}\tilde{m}_1 + u(\tilde{m}_1 + \tilde{m}_2)^2 + v(\tilde{m}_1 - \tilde{m}_2)^2 + 2w(\tilde{m}_1\tilde{m}_1 - \tilde{m}_2\tilde{m}_2)$$

with $w^2 = uv > 0$ and $u + v \neq 2w$. After some coordinate transformations the line element becomes

$$\begin{aligned} ds^2 &= \frac{\delta}{4}\frac{dr^2}{r^2} + ur^{-2n}d\alpha^2 + vr^{-2m}dt^2 + 2wr^{-(n+m)}d\alpha dt \\ &+ \frac{1}{4}r^{-(n+1)}d\alpha dr + \frac{1}{4}r^{-(m+1)}dtdr \,, \end{aligned}$$

where $n = (\cos\theta + \sin\theta)$ and $m = (\cos\theta - \sin\theta)$. Note that the metric is invariant under the scalings

$$r \to \lambda r,\ \alpha \to \lambda^n \alpha,\ t \to \lambda^m t \,.$$

Hence, the solution possesses a generalized (anisotropic) Lifshitz symmetry. For $\theta = \pi/2$, it becomes a stationary Lifshitz solution with dynamical exponent $z = -1$ similar to the one we found above.

Summary

Homogeneous Solutions					
Groups	**Metric**	**MMG**	**TMG**	**Description**	**Type**
$SL(2;\mathbb{R})$	111-type		✓ ($R=0$)	triaxially deformed AdS	$I_{\mathbb{R}}$
			✓	spacelike warped AdS	D
			✓	timelike warped AdS	D
	12-type		✓ ($R=0$)	Kundt	II
			✓	null warped AdS	N
	3-type	chiral pt.	X	Kundt	III
	$1z\bar{z}$-type		✓ ($R=0$)	generic	$I_{\mathbb{C}}$
$SU(2)$			✓ ($R=0$)	triaxially deformed sphere	$I_{\mathbb{R}}$
			✓	stretched/squashed sphere	D
A_∞			✓	warped flat	D
A_0	B_2-type	chiral pt.	✓	logarithmic pp-wave	N
	B_4-type		X	generic	II
$ISO(2;\theta)$	B_1-type	$\theta = \pi/2$	X	generic	$I_{\mathbb{R}}$
	B_2-type	$\theta \neq 0$	X	generic	II
$ISO(1,1;\theta)$	B_1-type	$\theta = \pi/4$	✓	warped AdS	D
		$\theta = \pi/2$	X	stationary Lifshitz	$I_{\mathbb{R}}$
			✓	pp-wave	N
			✓	pp-wave	N
	B_2-type	$\theta \neq 0$	X	generalized Lifshitz	II
		$\theta = \pi/4$	✓	warped flat	D

In the last column we give a classification of our solutions with respect to the Segre-Petrov type of their traceless Einstein tensor

$$P^a{}_b \equiv R^a{}_b - \frac{1}{3} R \delta^a{}_b \,, \tag{0.1}$$

as was proposed in:

D.D.K. Chow, C.N. Pope and E. Sezgin, *Classification of solutions in topologically massive gravity*, Class.Quant.Grav. 27 (2010) 105001, arXiv:0906.3559.

From the Table we see that homogeneous solutions of MMG in comparison to TMG can be grouped into three as follows:

Group 1: Solutions which are type N or D in the Segre-Petrov classification can be obtained from TMG solutions with a redefinition of constants. Corresponding solutions have the same curvature. This was shown in general in:

E. Altas and B. Tekin, *On Exact Solutions and the Consistency of 3D Minimal Massive Gravity*, Phys.Rev. D92 (2015) 025033, arXiv:1503.04726.

For Type D solutions we find

$$
\begin{aligned}
a_{\mathrm{MMG}} &= a_{\mathrm{TMG}} + c \cdot \frac{1}{48}\left(R + \frac{4}{9b_{\mathrm{TMG}}^2}\right)\left(R + \frac{4}{3b_{\mathrm{TMG}}^2}\right), \\
b_{\mathrm{MMG}} &= b_{\mathrm{TMG}} - c \cdot \frac{b_{\mathrm{TMG}}}{4}\left(R + \frac{4}{9b_{\mathrm{TMG}}^2}\right).
\end{aligned}
$$

For Type N solutions we have:

$$
\begin{aligned}
a_{\mathrm{MMG}} &= a_{\mathrm{TMG}} - c \cdot \frac{R\, a_{\mathrm{TMG}}}{24}\,, \\
b_{\mathrm{MMG}} &= b_{\mathrm{TMG}} - c \cdot \frac{R\, b_{\mathrm{TMG}}}{12}\,.
\end{aligned}
$$

Group 2: These solutions exist in TMG, but only if the cosmological constant vanishes. Hence, they have $R = 0$ in TMG. But in MMG, for these solutions the cosmological constant is proportional to the MMG parameter c and therefore $R \neq 0$ is possible.

Group 3: These solutions exist only in MMG.

Solutions without the Cotton Tensor

In many of the solutions it is possible to set $b = 0$ by choosing other parameters appropriately, after which remarkably one always ends up at the merger point. Thus, these are solutions of the MMG theory without the Cotton tensor. The fundamental equation of this specific model can be obtained from MMG by taking the limit $\mu \to \infty$, $\gamma \to \infty$ while keeping γ/μ^2 constant.

$$G_{\mu\nu} + a g_{\mu\nu} + b C_{\mu\nu} + c J_{\mu\nu} = 0\,,$$

$$a = \frac{\bar{\Lambda}_0}{\bar{\sigma}},\ \ b = \frac{1}{\mu\bar{\sigma}},\ \ c = \frac{\gamma}{\mu^2\bar{\sigma}}$$

This limit was considered before in:

M. Alishahiha, M.M. Qaemmaqami, A. Naseh and A. Shirzad, *On 3D Minimal Massive Gravity*, JHEP 1412 (2014) 033, arXiv:1409.6146.

G. Giribet and Y. Vasquez, *Minimal Log Gravity*, Phys.Rev. D91 (2015) 024026, arXiv:1411.6957.

For this limit to be consistent, one should make sure that Bianchi identity is satisfied, i.e.

$$V^{\mu} = \epsilon^{\mu\rho\sigma} S_{\rho}{}^{\tau} C_{\sigma\tau} = 0\,.$$

For our solutions it turns out that V^{μ} is either identically zero or becomes zero for the corresponding solution.

Therefore, we see that for all our solutions, whenever $b = 0$ is possible, the merger point condition $ac = 1$ is satisfied. Exceptions appear only when the Cotton tensor identically vanishes. Hence, we reach the following conclusion:

When the Cotton tensor is absent in the MMG field equation, simply transitive homogeneous solutions exist only at the merger point, provided that they are not conformally flat. They satisfy the Bianchi condition.

It is known that conformally flat spacetime solutions of MMG are locally maximally symmetric away from the merger point:

A.S. Arvanitakis, *On Solutions of Minimal Massive 3D Gravity*, Class.Quant.Grav. 32 (2015) 115010, arXiv:1501.01808.

This helps us identifying some of our solutions which have a vanishing Cotton tensor but not at the merger point.

On the other hand at the merger point, a conformally flat solution is not necessarily maximally symmetric.

Some Future Directions

- Understand the model without the Cotton tensor better.

- Cotton tensor and rotation

- More work is required to identify some of our solutions.

- Three of the solutions that exist only in MMG appear when $ac = 1/81$ and $9b^2 = -8c$. Whether this particular point in the parameter space of MMG has any physical significance like chiral and merger points remains to be seen.

• We focused on simply transitive homogeneous spacetimes. A natural generalization would be to allow a non-trivial isotropy group.

• The method we used can be applied to other 3-dimensional models.

• Do asymptotically Lifshitz black holes exist in MMG?

• About MMG model: CFT description, Supersymmetry, Chiral gravity, Log gravity, ...

Thank you