Microstates of AdS black holes and supersymmetric localization

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IPM, Tehran, May 8-11, 2017

Recent Trends in String Theory and Related Topics

in collaboration with

K. Hristov-A. Zaffaroni; arXiv: 1705.xxxxx

A. Nedelin-A. Zaffaroni; arXiv: 1611.09374

A. Zaffaroni; arXiv: 1604.03122

Black hole microstates

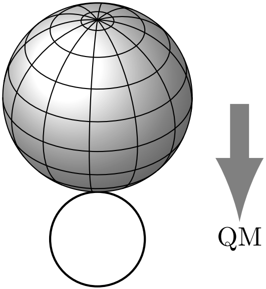
Static AdS_4 black holes

Horizon: $AdS_2 \times S^2$

Conformal boundary: $S^2 \times S^1$

Entropy $\sim N^{3/2}$ in M-theory

$\sim N^{5/3}$ in massive IIA



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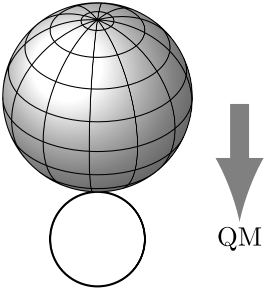
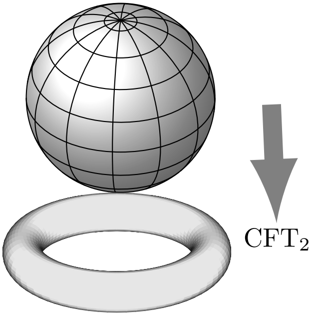


AdS_5 black strings

Horizon: $AdS_3 \times S^2$

Conformal boundary: $S^2 \times T^2$

Central charge $\sim N^2$ in type IIB



Rotating AdS_5 black holes

Horizon: $\mathrm{AdS}_2 \times_w S^3$

Conformal boundary: $S^3 \times S^1$

Entropy $\sim N^2$ in type IIB

$$S_{\mathrm{BH}} = \frac{2\pi}{g} \sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_1 Q_3 - \frac{\pi}{4 G_{\mathrm{N}}^{(5)} g} \left(J_\phi + J_\psi \right)} \, .$$

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Deriving $S_{\text{BH}} = \dfrac{\text{Area}}{4G_{\text{N}}}$.

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Tools

Holography + supersymmetric localization

Attractor mechanism

In four-dimensional gauged supergravity with $n_{\mathrm{V}}+1$ vectors, the black hole entropy can be obtained by extremizing

$$\mathcal{R}\left(X^{\Lambda}\right) = X^{\Lambda} q_{\Lambda} - \mathfrak{n}^{\Lambda} \mathcal{F}_{\Lambda} \quad \text{where} \quad \mathcal{F}_{\Lambda} \equiv \frac{\partial \mathcal{F}}{\partial X^{\Lambda}} \, .$$

with respect to X^{Λ}. [Ferrara-Kallosh, Dall'Agata-Gnecchi]

- $(q_{\Lambda}, \mathfrak{n}^{\Lambda})$: a set of electric and magnetic charges for the black hole.
- $\mathcal{F}\left(X^{\Lambda}\right)$: supergravity prepotential.
- X^{Λ}: a set of covariantly-constant homogeneous holomorphic sections.

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- X^Λ: a set of covariantly-constant homogeneous holomorphic sections.

In this talk we consider two different prepotentials,

- $\mathcal{F} = -2i\sqrt{X^0X^1X^2X^3} \, , \qquad \mathcal{F} = -\dfrac{X^1X^2X^3}{X^0} \, .$

Outline

Part I

A topologically twisted index

Part II

The topologically twisted index on $S^2 \times S^1$

Part III

The topologically twisted index on $S^2 \times T^2$

Part I

A topologically twisted index

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Topologically twisted index

Supersymmetric localization reduces the partition function on $\Sigma_g \times T^n$ with $n = 0, 1, 2$ to a contour integral

$$Z_{\Sigma_g \times T^n}(\mathfrak{n}, y) = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m} \in \Gamma_\mathfrak{h}} \oint_\mathcal{C} Z_{\text{int}}\left(\mathfrak{m}, x; \mathfrak{n}, y\right) ,$$

summed over the lattice of gauge magnetic charges $\mathfrak{m}$.

[Benini-Zaffaroni, 1504.03698]

- $Z_{\text{int}}\left(\mathfrak{m}, x; \mathfrak{n}, y\right)$: meromorphic differential form in x.
- $(\mathfrak{n}, y)$: magnetic fluxes and fugacities for the flavor symmetries.

How to solve the matrix integral?

1. Resum the integrand and consider the contour integral of the sum

$$Z_{\text{resummed}}\left(x;\mathfrak{n},y\right)=\frac{1}{|\mathcal{W}|}\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}}Z_{\text{int}}\left(\mathfrak{m},x;\mathfrak{n},y\right)\,.$$

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2. Writing a set of equations for finding the poles of Z_{resummed} in the plane x, which we call *Bethe ansatz equations*, and constructing a *Bethe functional* whose derivative reproduces the BAEs.

3. The solution to the BAEs is then used to evaluate $Z(\mathfrak{n},y)$ using the residue theorem.

[Benini-Hristov-Zaffaroni, 1511.04085]

Part II

The topologically twisted index on $S^2 \times S^1$

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Focus on quiver CS-YM-gauge theories with gauge group

$$\mathcal{G} = \prod_{a=1}^{|G|} \mathrm{U}(N)_a \,.$$

The meromorphic integrand for the background $S^2 \times S^1$ reads

$$Z_{\text{int}} = \prod_{\text{Cartan}} \left(\frac{dx}{2\pi i x} x^{k\mathfrak{m}} \right) \prod_{\alpha \in G} (1 - x^{\alpha}) \prod_{I} \prod_{\rho_I \in \mathfrak{R}_I} \left(\frac{x^{\rho_I/2}\, y_I}{1 - x^{\rho_I}\, y_I} \right)^{\rho_I(\mathfrak{m}) - \mathfrak{n}_I + 1} ,$$

$$Z^{\text{top}} = x^{\mathfrak{t}} \xi^{\mathfrak{m}} \,.$$

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Resumming $\sum_{\substack{\mathfrak{m}_a \leq M-1 \\ (k_a > 0)}}$ and $\sum_{\substack{\mathfrak{m}_a \geq 1-M \\ (k_a < 0)}}$, we find terms like

$$\prod_{i=1}^{N} \frac{\left(e^{i B_i^{(a)}} \right)^M}{e^{i B_i^{(a)}} - 1} \,,$$

M: large positive integer cut-off.

The topologically twisted index on $S^2 \times S^1$

In this way the contributions from the residues at the origin have been moved to the solutions to the BAEs,

$$e^{i\,\mathrm{sign}(k_a) B_i^{(a)}} = 1\,,$$

where we defined

$$e^{i\,\mathrm{sign}(k_a) B_i^{(a)}} = \xi^{(a)} (x_i^{(a)})^{k_a} \prod_{\substack{\text{bi-fundamentals} \\ (a,b) \text{ and } (b,a)}} \prod_{j=1}^{N} \frac{\sqrt{\frac{x_i^{(a)}}{x_j^{(b)}} y_{(a,b)}}}{1 - \frac{x_i^{(a)}}{x_j^{(b)}} y_{(a,b)}} \frac{1 - \frac{x_j^{(b)}}{x_i^{(a)}} y_{(b,a)}}{\sqrt{\frac{x_j^{(b)}}{x_i^{(a)}} y_{(b,a)}}}$$
$$\times \prod_{\substack{\text{fundamentals} \\ a}} \frac{\sqrt{x_i^{(a)} y_a}}{1 - x_i^{(a)} y_a} \prod_{\substack{\text{anti-fundamentals} \\ a}} \frac{1 - \frac{1}{x_i^{(a)}} \tilde{y}_a}{\sqrt{\frac{1}{x_i^{(a)}} \tilde{y}_a}}\,.$$

The topologically twisted index on $S^2 \times S^1$

It is convenient to use the variables $u_i^{(a)}$ and Δ_I, defined modulo 2π,

$$x_i^{(a)} = e^{iu_i^{(a)}} , \qquad y_I = e^{i\Delta_I} , \qquad \xi^{(a)} = e^{i\Delta_m^{(a)}} ,$$

and take the logarithm of the Bethe ansatz equations

$$0 = \log\left(\text{BAEs}\right) - 2\pi i n_i^{(a)} \equiv \frac{\partial \mathcal{V}}{\partial u_i^{(a)}} .$$

- Ansatz: $u_i^{(a)} = i\sqrt{N} t_i + v_i^{(a)}$. [Benini-Hristov-Zaffaroni, 1511.04085]

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- Using continuous distribution at large N, [SMH-Zaffaroni, 1604.03122]

$$\boxed{\mathcal{V}\left(\text{chemical potentials}\right) = F_{S^3}\left(\text{R-charges}\right) \, .}$$

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- The index can be derived from the Bethe potential

[SMH-Zaffaroni, 1604.03122]

$$\boxed{\mathbb{R}\text{e} \log Z(\mathfrak{n}_I, \Delta_I) = i \sum_I \mathfrak{n}_I \frac{\partial \mathcal{V}\left(\Delta_I\right)\big|_{\text{BAEs}}}{\partial \Delta_I} \,.}$$

$\mathcal{I}$-extremization principle versus attractor mechanism

$\mathcal{I}$-extremization principle

The entropy of a dyonic black hole with charges $(q_I, \mathfrak{n}_I)$ is obtained as a Legendre transform of $\log Z(\mathfrak{n}_I, \Delta_I)$,

$$S_{\mathrm{BH}}(q_I, \mathfrak{n}_I) = -\sum_I \mathfrak{n}_I \frac{\partial \overline{\mathcal{V}}(\Delta_I)}{\partial \Delta_I} - i \sum_I q_I \Delta_I \Big|_{\bar{\Delta}_I} .$$

- $\overline{\mathcal{V}}(\Delta_I) \equiv -i\mathcal{V}(\Delta_I)\Big|_{\mathrm{BAEs}}$
- $\bar{\Delta}_I$ is the extremum of $\mathcal{I}(\Delta_I) = -\sum_I \mathfrak{n}_I \frac{\partial \overline{\mathcal{V}}(\Delta_I)}{\partial \Delta_I} - i q_I \Delta_I$.

[Benini-Hristov-Zaffaroni, 1511.04085; 1608.07294]

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$$X^\Lambda \to \Delta^I , \qquad \mathcal{F} = -2i\sqrt{X^0 X^1 X^2 X^3} .$$

Part III

The topologically twisted index on $S^2 \times T^2$

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The index is given by [Benini-Zaffaroni, 1504.03698]

$$Z(\mathfrak{n}, y) = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m} \in \Gamma_{\mathfrak{h}}} \oint_{\mathcal{C}} \prod_{\text{Cartan}} \left(\frac{dx}{2\pi i x} \eta(q)^2 \right) (-1)^{\sum_{\alpha>0} \alpha(\mathfrak{m})} \prod_{\alpha \in G} \left[\frac{\theta_1(x^\alpha; q)}{i\eta(q)} \right] \times \prod_I \prod_{\rho_I \in \mathfrak{R}_I} \left[\frac{i\eta(q)}{\theta_1(x^{\rho_I} y_I; q)} \right]^{\rho_I(\mathfrak{m}) - \mathfrak{n}_I + 1} .$$

The Dedekind eta function is defined by

$$\eta(q) = \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) , \qquad q = e^{2\pi i \tau} \quad \text{and} \quad \mathbb{I}\mathrm{m}\, \tau > 0 .$$

The Jacobi theta function reads

$$\begin{aligned} \theta_1(x; q) = \theta_1(u; \tau) &= -i q^{\frac{1}{8}} x^{\frac{1}{2}} \prod_{k=1}^{\infty} \left(1 - q^k\right) \left(1 - x q^k\right) \left(1 - x^{-1} q^{k-1}\right) \\ &= -i \sum_{n \in \mathbb{Z}} (-1)^n e^{iu\left(n+\frac{1}{2}\right)} e^{\pi i \tau \left(n+\frac{1}{2}\right)^2} . \end{aligned}$$

The topologically twisted index on $S^2 \times T^2$

The modular properties of the elliptic functions are,

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)\,, \qquad \theta_1\left(\frac{u}{\tau};-\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\,e^{\frac{iu^2}{4\pi\tau}}\,\theta_1\left(u;\tau\right)\,.$$

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The asymptotic behavior of the $\eta(q)$ and $\theta_1(x;q)$ as $q \to 1$ can be derived by invoking their modular properties. [SMH-Nedelin-Zaffaroni, 1611.09374]

$$\boxed{\log\left[\eta(\tau)\right] = -\frac{1}{2}\log\left(\frac{\beta}{2\pi}\right) - \frac{\pi^2}{6\beta} + \mathcal{O}\left(e^{-1/\beta}\right)\,,}$$

$$\boxed{\log\left[\theta_1(u;\tau)\right] = -\frac{\pi^2}{2\beta} - \frac{u^2}{2\beta} - \frac{1}{2}\log\left(\frac{\beta}{2\pi}\right) + \frac{\pi}{\beta}u\,\text{sign}\left[\mathbb{R}\text{e}(u)\right] + \mathcal{O}\left(e^{-1/\beta}\right)\,.}$$

$i/(2\pi\tau) = 1/\beta$ is the formal "temperature" variable.

$\mathcal{N} = 4$ Super-Yang-Mills

It describes the dynamics of N D3-branes wrapped on S^2.

$$W = \mathrm{Tr}\left(\phi_3 \left[\phi_1, \phi_2\right]\right) \; .$$

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The topologically twisted index for the SU(N) SYM theory is given by

$$Z = \frac{\mathcal{A}}{N!} \sum_{\mathfrak{m} \in \mathbb{Z}^N} \int_{\mathcal{B}} \frac{dw}{2\pi i w} w^{\sum_{i=1}^{N} \mathfrak{m}_i} \int_{\mathcal{C}} \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \times$$
$$\times \prod_{j \neq i}^{N} \frac{\theta_1\left(\frac{x_i}{x_j}; q\right)}{i\eta(q)} \prod_{a=1}^{3} \left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{x_j} y_a; q\right)}\right]^{\mathfrak{m}_i - \mathfrak{m}_j - \mathfrak{n}_a + 1} ,$$

where we defined

$$\mathcal{A} = \eta(q)^{2(N-1)} \prod_{a=1}^{3} \left[\frac{i\eta(q)}{\theta_1\left(y_a; q\right)}\right]^{(N-1)(1-\mathfrak{n}_a)} .$$

How to solve the matrix integral?

- Take a large positive integer M and resum $\sum_{\mathfrak{m}\leq M-1}$:

$$Z = \frac{\mathcal{A}}{N!} \int_{\mathcal{B}} \frac{dw}{2\pi i w} \int_{\mathcal{C}} \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \prod_{i=1}^{N} \frac{\left(e^{iB_i}\right)^M}{e^{iB_i}-1} \times$$
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- Find the zeros of denominator

$$1 = e^{iB_i} = w \prod_{j=1}^{N} \prod_{a=1}^{3} \frac{\theta_1\left(\frac{x_j}{x_i} y_a; q\right)}{\theta_1\left(\frac{x_i}{x_j} y_a; q\right)} .$$

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- Evaluate the residues, [SMH-Nedelin-Zaffaroni, 1611.09374]

$$Z=\mathcal{A}\sum_{I\in\text{BAEs}}\frac{1}{\det\mathbb{B}}\prod_{j\neq i}^{N}\frac{\theta_1\left(\frac{x_i}{x_j};q\right)}{i\eta(q)}\prod_{a=1}^{3}\left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{x_j}y_a;q\right)}\right]^{1-\mathfrak{n}_a}.$$

The index at high temperature

- High-temperature limit of the BAEs, up to exponentially suppressed corrections:

$$0 = -2\pi i n_i + iv + \frac{1}{\beta}\sum_{j=1}^{N}\sum_{a=1}^{3}\left[F'\left(u_i - u_j + \Delta_a\right) - F'\left(u_j - u_i + \Delta_a\right)\right] ,$$

where we introduced the polynomial functions

$$F(u) = \frac{u^3}{6} - \frac{1}{2}\pi u^2 \text{sign}[\mathbb{R}\text{e}(u)] + \frac{\pi^2}{3}u \,,$$

$$F'(u) = \frac{u^2}{2} - \pi u \, \text{sign}[\mathbb{R}\text{e}(u)] + \frac{\pi^2}{3} \,.$$

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- The Bethe potential is given by [SMH-Nedelin-Zaffaroni, 1611.09374]

$$\mathcal{V}\left(\{u_i\}\right) = \sum_{i=1}^{N}\left(2\pi n_i - v\right)u_i + \frac{i(N-1)}{\beta}\sum_{a=1}^{3}F\left(\Delta_a\right) + \frac{i}{2\beta}\sum_{i\neq j}^{N}\sum_{a=1}^{3}\left[F\left(u_i - u_j + \Delta_a\right) + F\left(u_j - u_i + \Delta_a\right)\right] .$$

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- Solution to the BAEs:

$$u_j = -\frac{i\beta}{N}\left(n_j - \frac{1}{N}\sum_{i=1}^{N} n_i\right), \qquad \forall \quad j\,.$$

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- Note that,

$$\begin{aligned} a\left(\Delta_a\right) &= \frac{9}{32}\,\text{Tr}\,R^3\left(\Delta_a\right) \\ &= |G|\ \dim\,\text{SU}(N) + \sum_{a=1}^{3}\dim\,\mathfrak{R}_a\left(\frac{\Delta_a}{\pi}-1\right)^3 \\ &= \frac{3\left(N^2-1\right)}{\pi^3}\Delta_1\Delta_2\Delta_3\,. \end{aligned}$$

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$$\boxed{\mathcal{V}\left(\Delta_a\right)\big|_{\text{BAEs}} = \frac{i\pi^3}{6\beta}\,\mathrm{Tr}\,R^3\left(\Delta_a\right) = \frac{16i\pi^3}{27\beta}a\left(\Delta_a\right) .}$$

$a\left(\Delta_a\right)$: 4d central charge. [SMH-Nedelin-Zaffaroni, 1611.09374]

The index at high temperature

- Substituting the pole configurations back into the index, we get

$$\log Z = -\frac{N^2-1}{2\beta} \sum_{\substack{a<b \\ (\neq c)}} \Delta_a \Delta_b \mathfrak{n}_c - N \log N \,.$$

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- Note that,

$$c_r\left(\mathfrak{n}_a, \Delta_a\right) = 3\,\mathrm{Tr}\,\gamma_3 R^2\left(\Delta_a\right) = 3\left(N^2-1\right)\sum_{\substack{a<b \\ (\neq c)}} \Delta_a \Delta_b \mathfrak{n}_c\,.$$

[Benini-Bobev, 1211.4030]

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- At finite N,

$$\boxed{\log Z = -\frac{\pi^2}{6\beta} c_r \left(\mathfrak{n}_a, \Delta_a\right) = -\frac{16\pi^3}{27\beta} \sum_{a=1}^{3} \mathfrak{n}_a \frac{\partial a(\Delta_a)}{\partial \Delta_a} \,.}$$

[SMH-Nedelin-Zaffaroni, 1611.09374]

The index at high temperature

- The same is true for all quiver gauge theories satisfying

$$\operatorname{Tr} R = |G| + \sum_I \left(\frac{\Delta_I}{\pi} - 1 \right) = \mathcal{O}(1) \,,$$
$$k = \operatorname{Tr} \gamma_3 = |G| + \sum_I (\mathfrak{n}_I - 1) = \mathcal{O}(1) \,.$$

Theories of D3-branes at large N. [SMH-Nedelin-Zaffaroni, 1611.09374]

Conclusions

- Similar results for 3d quivers dual to M-theory backgrounds $\mathrm{AdS}_4 \times Y_7$ ($N^{3/2}$) and massive type IIA ones ($N^{5/3}$).

  [SMH-Zaffaroni, 1604.03122; SMH-Mekareeya, 1604.03397]

- Refinement for angular momentum on S^2. [Benini-Zaffaroni, 1504.03698]
- The Bekenstein-Hawking entropy of rotating AdS_5 black holes in terms of states in the dual $\mathcal{N} = 4$ SYM theory...

$$\mathcal{S}(Q_I, J_i) = \log Z_{\mathcal{N}=4} + 2\pi i \left(\sum_{I=1}^{3} Q_I \Delta_I - \sum_{i=1}^{2} J_i \omega_i \right) \Bigg|_{\bar{\Delta}_I, \bar{\omega}_i} ,$$

- The partition function of $\mathcal{N} = 4$ SYM theory on $S^3 \times S^1$,

$$Z_{\mathcal{N}=4} = \exp \left(-i\pi N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2} \right) I .$$

- $\Delta_1 + \Delta_2 + \Delta_3 + \omega_1 + \omega_2 = 0\,.$

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Thank you for your attention!